Metric Spaces

Assignment 2

1. Let \((X, d)\) and \((Y, d')\) be metric spaces, and \(f: X \to Y\) a function. Prove that \(f\) is continuous at the point \(a \in X\) if and only if for all sequences \((x_n)\) in \(X\), if \(\lim_{n \to \infty} x_n = a\) then \(\lim_{n \to \infty} f(x_n) = f(a)\).

**Solution.**

Suppose that \(f\) is continuous at \(a\). Let \(\varepsilon > 0\) be arbitrary. By continuity of \(f\) at \(a\), there exists a \(\delta > 0\) such that \(d(f(x), f(a)) < \varepsilon\) whenever \(d(x, a) < \delta\). Since \(x_n \to a\) and \(n \to \infty\), there exists a positive integer \(N\) such that \(d(x_n, a) < \delta\) whenever \(n > N\). Now whenever \(n > N\) we have \(d(x_n, a) < \delta\), and hence \(d'(f(x_n), f(a)) < \varepsilon\). Since \(\varepsilon\) was arbitrary, we have shown, as required, that for all \(\varepsilon > 0\) there exists an \(\delta > 0\) such that \(d'(f(x_n), f(a)) < \varepsilon\) whenever \(n > N\).

Conversely, suppose that \(f\) is not continuous at \(a\). Then we may choose an \(\varepsilon > 0\) such that for all \(\delta > 0\) there exists \(x \in X\) with \(d(x, a) < \delta\) and \(d'(f(x), f(a)) \geq \varepsilon\). Applying this with \(\delta = 1/n\), we conclude that for each positive integer \(n\) we may choose \(x_n \in X\) with \(d(x_n, a) < 1/n\) and \(d'(f(x_n), f(a)) \geq \varepsilon\). Since \(0 \leq d(x_n, a) < 1/n \to 0\) as \(n \to \infty\), we have that \(x_n \to a\) as \(n \to \infty\); however, it is not true that \(f(x_n) \to f(a)\) as \(n \to \infty\), since there is no value of \(n\) such that \(d'(f(x_n), f(a)) < \varepsilon\).

2. Let \(X\) and \(Y\) be topological spaces and \(f: X \to Y\) a function. Show that the following two conditions are equivalent:

(a) For all \(A \subseteq X\), if \(A\) is open in \(X\) then \(f(A)\) is open in \(Y\).

(b) For all \(A \subseteq X\) the inclusion \(f(\text{Int} A) \subseteq \text{Int}(f(A))\) holds.

**Solution.**

Suppose that (a) holds, and let \(A \subseteq X\) be arbitrary. Then \(\text{Int} A\) is an open subset of \(X\), and so by (a) it follows that \(f(\text{Int} A)\) is open in \(Y\). Furthermore, \(\text{Int} A \subseteq A\), and so \(f(\text{Int} A) \subseteq f(A)\). Thus \(f(\text{Int} A)\) is an open set contained in the subset \(f(A)\) of \(Y\). Hence

\[
\text{f}(\text{Int} A) \subseteq \bigcup \{U \mid U \text{ is open and } U \subseteq f(A)\} = \text{Int}(f(A)),
\]

and since \(A\) was an arbitrary open subset of \(X\) this shows that (b) holds.

Conversely, suppose that (b) holds, and let \(A\) be an arbitrary open subset of \(X\). Then \(A = \text{Int} A\), and so by (b),

\[
f(A) = f(\text{Int} A) \subseteq \text{Int}(f(A)).
\]

The reverse inclusion, \(\text{Int} f(A) \subseteq A\), is immediate from the definition of the interior of a set. So \(f(\text{Int} A)\) is open. Since \(A\) was an arbitrary open subset of \(X\) this shows that (a) holds.

3. Show that the function \(\cos: \mathbb{R} \to \mathbb{R}\) is not a contraction mapping, but its two-fold composite \(\cos^2\) is. (The metric is understood to be the usual metric on \(\mathbb{R}\).) Use a calculator to find a solution of \(x = \cos x\) correct to 4 decimals. (No proof required for this last bit, and not many marks awarded either!)

**Solution.**

Suppose that \(\cos\) is a contraction mapping. Then there is a \(K < 1\) such that

\[
\frac{\cos x - \cos y}{x - y} \leq K < 1
\]

whenever \(x \neq y\). But if we keep \(y\) fixed and let \(x\) approach \(y\) then the ratio \((\cos x - \cos y)/(x - y)\) approaches \(-\sin y\), the derivative of \(\cos\) at the point \(y\). So the above inequality gives \(|\sin y| \leq K < 1\), which is false for some values of \(y\). So \(\cos\) is not a contraction mapping.

Since \(\frac{d}{dx}(\cos x) = (\sin x)\sin x\), the Mean Value Theorem tells us that for all \(a, b \in \mathbb{R}\) there is a \(c \in [a, b]\) (or \([b, a]\) if \(b < a\)) such that

\[
\cos^2 a - \cos^2 b = (a - b)(\cos c)\sin c.
\]

Now \(|\sin c| \leq 1\), and since \(\sin\) is increasing on the interval \([-1, 1]\) (since \(1 < \pi/2\)) we deduce that \(|\sin(\cos c)| \leq \sin 1\), and so

\[
|\sin(\cos c)\sin c| \leq (\sin 1)|\sin c| \leq \sin 1,
\]

irrespective of the values of \(a\) and \(b\). So for all \(a, b \in \mathbb{R}\),

\[
|\cos^2 a - \cos^2 b| \leq (\sin 1)|a - b|,
\]

which shows that \(\cos^2\) is a contraction mapping, since \(\sin 1 < 1\).

Since \(\cos x\) takes the value 1 at \(x = 0\) and 0 at \(x = \pi/2\), it seems that the graphs of \(y = x\) and \(y = \cos x\) must cross reasonably near to \(x = 0.7\). Putting \(x_0 = 0.7\) and \(x_i = \cos(x_{i-1})\) for all positive integers \(i\), we find after a few iterations that 0.7391 is a good approximation to the fixed point.

4. Find metric spaces \((X, d_X)\) and \((Y, d_Y)\) and a function \(f: X \to Y\) such that \(f\) is uniformly continuous and bijective, \((X, d_X)\) is complete and \((Y, d_Y)\) is not complete. (Modify an example from one of the tutorial sheets.)

**Solution.**

Let \(X = \mathbb{R}\) and \(Y = (-\pi/2, \pi/2)\), a subspace of \(\mathbb{R}\). (The metrics \(d_X\) and \(d_Y\) are the usual ones.) The function \(\arctan\) is a uniformly continuous bijection...
from \( \mathbb{R} \) to \((-\pi/2, \pi/2)\). Indeed, since the derivative of \( \arctan x \) is \( 1/(1 + x^2) \), the Mean Value Theorem tells us that for all \( x, y \in \mathbb{R} \) there is a \( c \in \mathbb{R} \) such that

\[
\arctan x - \arctan y = (x - y)(1 + c^2)^{-1},
\]

and it follows that for all \( \varepsilon > 0 \) if \( |x - y| < \varepsilon \) then \( |\arctan x - \arctan y| < \varepsilon \). (So the definition of uniform continuity holds with \( \delta \) chosen to equal \( \varepsilon \).) Since \( \arctan x \) is strictly increasing it is injective, and since it is continuous and approaches \( \pi/2 \) as \( x \to \infty \) and \( -\pi/2 \) as \( x \to -\infty \), it maps \( \mathbb{R} \) to \((-\pi/2, \pi/2)\) suitably. We know from lectures that \( X = \mathbb{R} \) is complete, whereas \( Y \) is not, since \((-\pi/2, \pi/2)\) is not closed as a subset of \( \mathbb{R} \).

5. Let \( (X, d) \) be a complete metric space and \( f: X \to X \) a function. Suppose that for some positive integer \( r \) the \( r \)-fold composite function \( f^{(r)} \) (defined by \( f^{(r)}(x) = f(f(...f(x))) \), where there are \( r \) \( f \)'s on the right-hand side) is a contraction mapping. Let \( x \) be any point of \( X \), and let \( (x_n)_{n=1}^\infty \) be the sequence defined by \( x_0 = x \) and \( x_i = f(x_{i-1}) \) for all positive integers \( i \). Prove that \( (x_n)_{n=1}^\infty \) converges in \( X \). (You may use the fact, proved in lectures, that this is true in the case \( r = 1 \), or use the \( r = 1 \) proof as a guide to the construction of a general proof.)

Solution.

There exists a positive number \( K < 1 \) such that \( d(f^{(r)}(x), f^{(r)}(y)) \leq Kd(x, y) \) for all \( x, y \in X \). Since \( f^{(r)}(x_i) = x_{i+1} \) (for each nonnegative integer \( i \)) it follows that

\[
d(x_{n+r}, x_{n+1}) = d(f^{(r)}(x_{n-r}), f^{(r)}(x_{n+1})) \leq Kd(x_{n-r}, x_{n+1}),
\]

and iterating this yields

\[
d(x_{n+r}, x_{n+1}) \leq K^r d(x_{n+1}, x_{n+1}) \leq K^r \leq \cdots \leq K^n d(x_{1}, x_{1}),
\]

where \( n \) is any positive integer. Now if \( s, t \in \mathbb{Z}^+ \) with \( s < t \), and if \( i \in \{0, 1, \ldots, r-1\} \), then, by the triangle inequality,

\[
d(x_{s+i}, x_{t+i}) \leq \sum_{j=0}^{t-s-1} d(x_{(s+j)i}, x_{(s+j+1)i}) \leq \sum_{j=1}^{t-s} K^{s+j} d(x_{j}, x_{1+i}) = K^s \frac{M}{1-K} d(x_{1+i}, x_{1+i}) \leq \frac{K^s M}{1-K}.
\]

where \( M = \max\{d(x_0, x_r), d(x_1, x_{r+1}), d(x_2, x_{r+2}), \ldots, d(x_{r-1}, x_{2r-1})\} \). We also have, for all nonnegative integers \( p, q \),

\[
d(x_{r+p}, x_{r+q}) = d(f^{(r)}(x_p), f^{(r)}(x_q)) \leq Kd(x_p, x_q),
\]

and so it follows that for all \( i, j \in \{0, 1, \ldots, r-1\} \) and all positive integers \( s, \)

\[
d(x_{sr+i}, x_{sr+j}) \leq Kd(x_{(s-1)r+i}, x_{(s-1)r+j}) \leq \cdots \leq K^{s-1} d(x_{r+i}, x_{r+j}) \leq K^s d(x_i, x_j) \leq K^P
\]

where \( P = \max\{d(x_i, x_j) \mid i, j \in \{0, 1, \ldots, r-1\} \} \).

Given \( \varepsilon > 0 \), choose \( s \) large enough so that \( K^s M/(1 - K) \) and \( K^P \) are both less than \( \varepsilon/3 \), and put \( N = sr \). Let \( n, m > N \) be arbitrary. Let \( i, j \in \{0, 1, \ldots, r-1\} \) be the remainders obtained on dividing \( m, n \) by \( r \), so that \( m = tr + i \) and \( n = ur + j \) for some integers \( t, u \geq 0 \). Then

\[
d(x_m, x_n) \leq d(x_{tr+i}, x_{sr+i}) + d(x_{sr+i}, x_{sr+j}) + d(x_{sr+j}, x_{ur+j}) \leq \frac{K^s M}{1-K} + K^s P + \frac{K^s M}{1-K} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\]

Hence \( (x_n)_{n=1}^\infty \) is a Cauchy sequence, and hence convergent since \( X \) is complete.

Alternatively, since \( f^{(r)} \) is a contraction mapping, the proof given in lectures shows that, for each \( i \in \{0, 1, \ldots, r-1\} \), the sequence \( (x_{nr+i})_{n=1}^\infty \) converges in \( X \), the limit \( x \) being the unique fixed point of the function \( f^{(r)} \). So, given \( \varepsilon > 0 \), there exists an integer \( n_i \) such that \( d(x_{nr+i}, x) < \varepsilon \) for all \( n > n_i \). Now put \( N = \max\{n_i, r+i \mid 0 \leq i < r \} \). Let \( n \) be any integer greater than \( N \). Choosing \( i \in \{0, 1, \ldots, r-1\} \) such that \( n - i \) is a multiple of \( r \), we have \( n = mr + i \) for some \( m \), and \( m > m_i \) since \( n > N \). So

\[
d(x_n, x) = d(x_{mr+i}, x) < \varepsilon. \text{ Hence } \lim_{n \to \infty} x_n = x.
\]