Subspace topology

Let \((X, \mathcal{U})\) be a topological space. That is, \(\mathcal{U}\) is a collection of subsets of \(X\) satisfying

- (T1) \(X, \emptyset \in \mathcal{U}\),
- (T2) whenever \((A_i)_{i \in I}\) is a family of sets in \(\mathcal{U}\), then \(\bigcup_{i \in I} A_i \in \mathcal{U}\), and
- (T3) whenever \(A, B \in \mathcal{U}\), then \(A \cap B \in \mathcal{U}\).

(Note that (T3) is equivalent to the condition that the intersection of any finite collection of elements of \(\mathcal{U}\) is in \(\mathcal{U}\), as can easily be proved by induction.)

Suppose now that \(S\) is any subset of \(X\), and put \(\mathcal{V} = \{ S \cap A \mid A \in \mathcal{U} \}\). It is not hard to prove that \(\mathcal{V}\) is a topology on \(S\). Firstly, since \(S \subseteq X\) we have \(S \cap X = S\), while it is trivial that \(S \cap \emptyset = \emptyset\). Since \(X, \emptyset \in \mathcal{U}\) (since \(\mathcal{U}\) satisfies (T1)), it follows that \(S, \emptyset \in \mathcal{V}\).

So (T1) holds for \(\mathcal{V}\). Next, suppose that \((B_i)_{i \in I}\) is a family of sets in \(\mathcal{V}\). For each \(i \in I\) there is an \(A_i \in \mathcal{U}\) with \(B_i = S \cap A_i\). Now \(\bigcup_{i \in I} A_i \in \mathcal{U}\), by (T2) for \(\mathcal{U}\), and since

\[
S \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} S \cap A_i = \bigcup_{i \in I} B_i
\]

it follows that \(\bigcup_{i \in I} B_i \in \mathcal{V}\). Hence \(\mathcal{V}\) satisfies (T3). Finally, if \(P, Q\) are arbitrary elements of \(\mathcal{V}\) then \(P = S \cap A\) and \(Q = S \cap B\) for some \(A, B \in \mathcal{U}\), and we see that

\[
P \cap Q = (S \cap A) \cap (S \cap B) = S \cap (A \cap B) \in \mathcal{V}
\]

since \(A \cap B \in \mathcal{U}\). So \(\mathcal{V}\) also satisfies (T3), and thus is a topology on \(S\).

**Definition.** Let \((X, \mathcal{U})\) be a topological space and \(S\) a subset of \(X\). The topology \(\mathcal{V}\) on the set \(S\) defined by \(\mathcal{V} = \{ S \cap A \mid A \in \mathcal{U} \}\) (as above) is called the topology on \(S\) induced by the topology \(\mathcal{U}\) on \(X\). A topological subspace of \((X, \mathcal{U})\) is a topological space of the form \((S, \mathcal{V})\), where \(S\) is a subset of \(X\) and \(\mathcal{V}\) the induced topology. The induced topology is also sometimes called the relative topology, or the subspace topology. A subset of \(X\) is said to be open relative to \(S\) if it is an open set of the subspace topology (so that it is of the form \(S \cap A\) for some \(A \in \mathcal{U}\)).

Recall that if \((X, d)\) is a metric space then there is a standard topology on \(X\) derived from the metric: it consists of those subsets \(U\) of \(X\) such that for all \(a \in U\) there is an \(\varepsilon > 0\) such that \(B_d(a, \varepsilon) \subseteq U\). Furthermore, if \(S\) is any subset of \(X\) and \(d'\) the restriction of \(d\) to \(S\), then \((S, d')\) is a metric space. (We call \(d'\) the metric induced by \(d\).) Now we can obtain a topology on \(S\) in either of two ways: the topology on \(X\) derived from the metric \(d\) induces a topology \(\mathcal{V}_\infty\) on \(S\), and there is a topology \(\mathcal{V}_\varepsilon\) on \(S\) derived from the induced metric \(d'\). One would hope that \(\mathcal{V}_1 = \mathcal{V}_2\), and this is indeed true. On the one hand, suppose that \(A \in \mathcal{V}_2\). This means that for all \(a \in A\) there is a positive number \(\mu_a\) such that \(B_{d'}(a, \mu_a) \subseteq A\). Now

\[
B_{d'}(a, \mu_a) = \{ x \in S \mid d'(x, a) < \mu_a \} = \{ x \in S \mid d(x, a) < \mu_a \} = S \cap \{ x \in X \mid d(x, a) < \mu_a \} = S \cap B_d(a, \mu_a);
\]

Moreover, \(A \subseteq \bigcup_{a \in A} B_{d'}(a, \mu_a)\) (since \(a \in B_{d'}(a, \mu_a)\) for each \(a\)), and \(\bigcup_{a \in A} B_{d'}(a, \mu_a) \subseteq A\) (since \(B_{d'}(a, \mu_a) \subseteq A\) for each \(a\), by the choice of \(\mu_a\)). Thus

\[
A = \bigcup_{a \in A} B_{d'}(a, \mu_a) = \bigcup_{a \in A} S \cap B_d(a, \mu_a) = S \cap \bigcup_{a \in A} B_d(a, \mu_a),
\]
which is an open set of the induced topology $\mathcal{V}_1$, since $\bigcup_{a \in A} B_d(a, \mu_a)$ is an open subset of $X$. On the other hand, suppose that $A \in \mathcal{V}_1$, so that $A = S \cap U$ for some open subset $U$ of $X$. Since $U$ is open, for each $a \in U$ there is a $\mu > 0$ such that $B_d(a, \mu) \subseteq U$; in particular, such a $\mu$ exists for each $a \in A$ (since $A \subseteq U$), and we find that

$$B_d^*(a, \mu) = S \cap B_d(a, \mu) \subseteq S \cap U = A,$$

which shows that $A \in \mathcal{V}_2$.

The following result (for metric spaces) appears as Theorem 3.1 on p. 52 of Choo’s notes. Note, however, that there is a misprint: the important assumption that $f$ is continuous was accidentally omitted. We prove the result here in the more general context of topological spaces.

**Proposition.** Let $X, Y$ be topological spaces and $f: X \to Y$ a continuous mapping. Let $S$ be any subspace of $X$, and $f_S: S \to Y$ the restriction of $f$. Then $f_S$ is continuous.

**Proof.** Let $U$ be an open subset of $Y$. By definition,

$$f_S^{-1}(U) = \{ x \in S \mid f_S(x) \in U \} \subseteq \{ x \in S \mid f(x) \in U \} = S \cap \{ x \in X \mid f(x) \in U \} = S \cap f^{-1}(U).$$

Now $f^{-1}(U)$ is an open subset of $X$ since $U$ is open in $Y$ and $f: X \to Y$ is continuous. So $S \cap f^{-1}(U)$ is an open subset of $S$ (in the subspace topology). Thus we have shown that $f_S^{-1}(U)$ is open in $S$ whenever $U$ is open in $Y$; hence $f_S$ is continuous.

A similarly straightforward result says that the composite of two continuous functions is always continuous.

**Proposition.** If $X, Y$ and $Z$ are topological spaces, and $f: X \to Y$ and $g: Y \to Z$ continuous functions, then the function $g \circ f: X \to Z$ (defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$) is continuous.

**Proof.** Our task is to show that $(g \circ f)^{-1}(U)$ is open in $X$ whenever $U$ is open in $Z$.

Let $U \subseteq Z$ be open. Then

$$(g \circ f)^{-1}(U) = \{ x \in X \mid (g \circ f)(x) \in U \}$$

$$= \{ x \in X \mid g(f(x)) \in U \}$$

$$= \{ x \in X \mid f(x) \in g^{-1}(U) \}$$

$$= f^{-1}(g^{-1}(U)).$$

Since $g$ is continuous and $U$ is open it follows that $g^{-1}(U)$ is open. Now since $f$ is continuous it follows that $f^{-1}(g^{-1}(U))$ is open. So we have shown that $(g \circ f)^{-1}(U)$ is open whenever $U$ is open, as required.

**Bases**

If $X$ and $Y$ are topological spaces then there is a natural way to make the Cartesian product $X \times Y = \{ (x, y) \mid x \in X \text{ and } y \in Y \}$ into a topological space. Before we can discuss this we need to introduce another concept.

**Definition.** Let $X$ be a topological space. A collection $\mathcal{B}$ of open subsets of $X$ is called a base for the topology on $X$ if every open set can be expressed as a union of sets in $\mathcal{B}$.
Example. In $\mathbb{R}$ the open intervals form a base for the topology. More generally, in any metric space the open balls form a base for the topology. To prove this one must show that every open set is expressible as a union of the open balls. The proof of this was incorporated in one of our proofs above, but it will do us no harm to repeat it!

Proposition. If $X$ is a metric space and $U \subseteq X$ is open, then $U$ is the union of the open balls it contains.

Proof. On the one hand, the union of all the open balls contained in $U$ is obviously a subset of $U$; on the other, if $x \in U$ is arbitrary then $x \in \text{Int}(U)$ (as every point of an open set is an interior point), hence $x$ lies in an open ball contained in $U$, and hence $x$ is in the union of all the open balls contained in $U$. \hfill \Box$

In many cases when it is desirable to make a set into a topological space, the most convenient way to do so is to specify a base for the topology, rather than attempt to describe all open sets directly. The situation with metric spaces illustrates this: open sets are defined in terms of open balls. One could perhaps manage to give a reasonable discussion of metric spaces without using the concept of an open set, but one could not sensibly avoid talking about open balls.

Note that a base for a topology determines the topology uniquely: there cannot be two different topologies on one set $X$ sharing a common base $B$. This is because the open sets of the topology can be characterized as those sets that are unions of sets in $B$. (The definition of the concept of a base says that all open sets are unions of sets in $B$; on the other hand, since the elements of $B$ are themselves open sets and the union of any collection of open sets is open, it is also true that every set which is a union of sets in $B$ is an open set.) However, it is not the case that every collection of subsets of an arbitrary set $X$ can serve as a base for a topology on $X$. This is because the intersection of two open sets has to be open, and it is clear that if $B$ is an arbitrary collection of subsets of $X$ then there is no guarantee that the intersection of any two elements of $B$ will be expressible as a union of elements of $B$. Provided that the collection $B$ does have this property, and provided that the elements of $B$ cover $X$, then it will be the case that the collection $B$ determines a topology on $X$. 

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