We noted last time that a space that is homeomorphic to a connected space is also connected. In fact, it is very easy to establish a stronger result: continuous images of connected sets are connected.

**Theorem.** Let \( f : X \rightarrow Y \) be continuous, and suppose that \( A \) is a connected subset of \( X \). Then \( f(A) \) is a connected subset of \( Y \).

**Proof.** If \( f(A) \) is disconnected then there exist open subsets \( V_1, V_2 \) of \( Y \) such that

\[
\begin{align*}
\text{f(A)} & \subseteq V_1 \cup V_2, \\
f(A) \cap V_1 & \neq \emptyset \quad \text{and} \quad f(A) \cap V_2 \neq \emptyset, \\
f(A) \cap V_1 \cap V_2 & = \emptyset.
\end{align*}
\]

Since \( f \) is continuous, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are open subsets of \( X \). By (1) above, for each \( a \in A \) we have either \( f(a) \in V_1 \) or \( f(a) \in V_2 \); that is, either \( a \in f^{-1}(V_1) \) or \( a \in f^{-1}(V_2) \). So \( A \subseteq f^{-1}(V_1) \cup f^{-1}(V_2) \). By (2) there is an \( a \in A \) such that \( f(a) \in V_1 \), giving \( a \in f^{-1}(V_1) \), and similarly there is an \( a' \in A \) with \( a' \in f^{-1}(V_2) \). So \( A \cap f^{-1}(V_1) \neq \emptyset \) and \( A \cap f^{-1}(V_2) \neq \emptyset \). Finally, \( A \cap f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \), since if there were some element \( a \) in this set it would follow that \( f(a) \in f(A) \cap V_1 \cap V_2 \), contradicting (3). We have shown that

\[
\begin{align*}
A & \subseteq f^{-1}(V_1) \cup f^{-1}(V_2), \\
A \cap f^{-1}(V_1) & \neq \emptyset \quad \text{and} \quad A \cap f^{-1}(V_2) \neq \emptyset, \\
A \cap f^{-1}(V_1) \cap f^{-1}(V_2) & = \emptyset,
\end{align*}
\]

contradicting the fact that \( A \) is connected. So the assumption that \( f(A) \) is disconnected has led to a contradiction; so \( f(A) \) is connected. \( \square \)

There is any even shorter proof using the fact that a set is disconnected if and only if there is a continuous surjective function from the set to the discrete space \( \{0, 1\} \). If \( f(A) \) is disconnected then there is a surjective continuous function \( g : f(A) \rightarrow \{0, 1\} \), and then the function from \( A \) to \( \{0, 1\} \) given by \( a \mapsto g(f(a)) \) is continuous (since composites of continuous functions are continuous) and surjective (since \( g \) is surjective). So \( A \) is disconnected.

As we shall see, it is not clear that the definition of connectedness that we have given really captures the everyday concept of connectedness, which is perhaps more to do with path-connectedness, a concept that we shall define in due course, and that is stronger than connectedness. However, if intuition suggests that a set is connected, then it ought to be true that the set is indeed connected in the technical sense. In particular, intervals in \( \mathbb{R} \) are connected sets.

There are nine different kinds of intervals: \((a, b)\), \([a, b]\), \([a, b)\), for any \( a, b \in \mathbb{R} \) with \( a < b \), \([a, b]\), for any \( a, b \in \mathbb{R} \) with \( a \leq b \), \((−\infty, a)\), \((a, \infty)\), \([a, \infty)\), \((−\infty, \infty)\) (the whole real line). \( \dagger \) Intervals can be characterized as follows: a subset \( I \) of \( \mathbb{R} \) is an interval if and only if \( I \) is nonempty, and for all \( a, b \in I \) and \( x \in \mathbb{R} \), if \( a \leq x \leq b \)

\( \dagger \) We have deviated from the convention adopted in Choo’s notes by permitting one-element subsets of \( \mathbb{R} \) to be counted as closed intervals.

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then $x \in I$. That is to say, if $I \neq \emptyset$ then $I$ is an interval if and only if every point of $\mathbb{R}$ that lies between two points of $I$ is also in $I$.

**Lemma.** Let $a, b \in \mathbb{R}$ with $a < b$, and $S$ a subset of $\mathbb{R}$ such that $a \in S$ and $b \notin S$, and let $p = \sup(S \cap [a, b])$.

(i) If $S$ is closed in $\mathbb{R}$ then $p \in S$.

(ii) If $S$ is open in $\mathbb{R}$ then $p \notin S$.

**Proof.** Note that $S \cap [a, b]$ is nonempty (since $a \in S \cap [a, b]$) and bounded above (by $b$).

So, by an axiom of the real number system, $S \cap [a, b]$ has a least upper bound. So the definition of $p$ in the statement of the lemma is meaningful. Observe that $a \leq p$ (since $a \in S \cap [a, b]$ and $p$ is an upper bound for $S \cap [a, b]$) and $p \leq b$ (since $b$ is an upper bound for $S \cap [a, b]$ and $p$ is the least upper bound for $S \cap [a, b]$).

Suppose that $S$ is closed, and suppose that $p \notin S$. Then $p \in \mathbb{R} \setminus S$, which is an open set since $S$ is closed, and so there exists an $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq \mathbb{R} \setminus S$. Of course, since we are discussing $\mathbb{R}$ with its usual metric, $B(p, \varepsilon) = (p - \varepsilon, p + \varepsilon)$. Now let $x \in S \cap [a, b]$ be arbitrary. Since $p$ is an upper bound for $S \cap [a, b]$ we have $x \leq p$, and so either $x \leq p - \varepsilon$ or $p - \varepsilon < x \leq p$. The latter alternative gives $x \in (p - \varepsilon, p] \subseteq (p - \varepsilon, p + \varepsilon) \subseteq \mathbb{R} \setminus S$, contradicting $x \in S$, and so we must have $x \leq p - \varepsilon$. Since this holds for all $x \in S \cap [a, b]$, it follows that $p - \varepsilon$ is an upper bound for $S \cap [a, b]$. But $p - \varepsilon < p$, and so this contradicts the fact that $p$ is the least upper bound for $S$.

For the second part, suppose that $S$ is open and $p \in S$. Since $b \notin S$ and $p \leq b$ it follows that $p < b$. Thus $p \in (\infty, b) \cap S$, an open set since both $(\infty, b)$ and $S$ are open, and so there exists an $\varepsilon > 0$ such that $(p - \varepsilon, p) \subseteq (\infty, b) \cap S$. In particular, $p + (\varepsilon/2) \in S$ and $p + (\varepsilon/2) < b$, and since $a \leq p < p + (\varepsilon/2)$ it follows that $p + (\varepsilon/2) \in S \cap [a, b]$. But since $p + (\varepsilon/2) > p$, this contradicts the fact that $p$ is an upper bound for $S \cap [a, b]$.

**Example.** Let $I \subseteq \mathbb{R}$ be an interval. Then $I$ is connected.

**Proof.** Suppose that $I$ is not connected. Then there exist open subsets $U_1$, $U_2$ of $\mathbb{R}$ with $I \cap U_1$ and $I \cap U_2$ nonempty, $I \cap U_1 \cap U_2 = \emptyset$, and $I \subseteq U_1 \cup U_2$. We can choose $a \in I \cap U_1$ and $b \in I \cap U_2$ (since these sets are nonempty), and then $a \neq b$ (since $I \cap U_1 \cap U_2 = \emptyset$). Swapping the names of $U_1$ and $U_2$ if necessary, we may assume that $a < b$.

Since $a, b \in I$, and $I$ is an interval, it follows from our characterization of intervals that $[a, b] \subseteq I$. Now put $A = [a, b] \cap U_1$ and $B = [a, b] \cap U_2$. Then

$$A \cup B = [a, b] \cap (U_1 \cup U_2) = [a, b],$$

since $[a, b] \subseteq I \subseteq U_1 \cup U_2$, and

$$A \cap B \subseteq I \cap U_1 \cap U_2 = \emptyset;$$

so $A = [a, b] \setminus B = [a, b] \setminus U_2$. Now if we define $p = \sup A$ then it follows from the first part of the lemma that $p \in A$, since $A = [a, b] \cap (\mathbb{R} \setminus U_2)$ and $\mathbb{R} \setminus U_2$ is closed. However, $A = [a, b] \cap U_1$ and $U_1$ is open; so it follows from the second part of the lemma that $p \notin A$. Thus we have obtained the desired contradiction.

Our characterization of intervals also yields the following converse to the above result.

**Proposition.** If $A \subseteq \mathbb{R}$ is connected and nonempty then $A$ is an interval.

**Proof.** Suppose that $A$ is connected and $A \neq \emptyset$, and suppose that $A$ is not an interval. By the characterization, of intervals there exist $a, b \in A$ and $x \notin A$ with $a \leq x \leq b$. Put $U_1 = (\infty, x)$ and $U_2 = (x, \infty)$. Then $U_1$ and $U_2$ are open subsets of $\mathbb{R}$ with $a \in A \cap U_1$ and $b \in A \cap U_2$ (showing that $A \cap U_1$ and $A \cap U_2$ are both nonempty), $A \subseteq \mathbb{R} \setminus \{x\} = U_1 \cup U_2$, and $A \cap U_1 \cap U_2 = \emptyset$ (since $U_1 \cap U_2 = \emptyset$). This shows that $A$ is not connected.