THE UNIVERSITY OF SYDNEY
FACULTIES OF ARTS AND SCIENCE
SCHOOL OF MATHEMATICS AND STATISTICS

Metric Spaces

(MATH3901)

June, 1999 — Time allowed: Two hours

Lecturer: R. B. Howlett

The marks allocated to the parts of each question are shown. The total number of marks available is 100. To obtain full marks, candidates must answer all questions.

1. Let (X, d) be a metric space.

(i) Give the definitions of the following terms:

(a) an *open ball* in X;

(b) an *interior point* of a subset S of X;

(c) an *open subset* of X.

(6 marks)

(ii) Consider the set $\mathbb{R}$ of all real numbers as a metric space via its usual metric. Working directly from the definition you gave in (i) (c) above, prove that $\{ x \in \mathbb{R} \mid x > 0 \}$ is an open subset of $\mathbb{R}$. *(6 marks)*

2. Let X and Y be topological spaces, and $f: X \to Y$ any function.

(i) Define the concept of the *preimage* of a subset of Y, and give a criterion for the continuity of f in terms of this concept. *(4 marks)*

(ii) Suppose that $Y = \mathbb{R}$, with its usual topology, and suppose that $f: X \to \mathbb{R}$ is continuous. Prove that the set $\{ x \in X \mid f(x) > 0 \}$ is open. *(4 marks)*

3. (i) What does it mean to say that a topological space is *connected*? *(4 marks)*

(ii) What does it mean to say that a topological space is *Hausdorff*? *(4 marks)*

(iii) Prove that every metric space is Hausdorff. *(6 marks)*

4. Let X be a topological space.

(i) Suppose that $A \subseteq X$. What does it mean to say that a point $x \in X$ is not a point of accumulation of A? *(4 marks)*

(ii) Suppose that A and B are subsets of X, and suppose that the point $x \in X$ is not an accumulation point of A and not an accumulation point of B. Working directly from the definition, prove that x is not an accumulation point of $A \cup B$. *(8 marks)*

5. Let $\mathcal{C}$ be the set of all continuous real-valued functions on the interval $[0, \frac{1}{2}]$.

(i) Give the definition of the *uniform metric* d (also called the *sup metric*) on $\mathcal{C}$. *(3 marks)*

(ii) Prove that d as you have defined it in (i) satisfies the triangle inequality. *(5 marks)*

(iii) For each $f \in \mathcal{C}$ let $Tf\colon [0, \frac{1}{2}] \to \mathbb{R}$ be defined by the formula

$$(Tf)(t) = t(f(t) + 1) \qquad (\text{for all } t \in [0, \tfrac{1}{2}]).$$

Show that $T\colon \mathcal{C} \to \mathcal{C}$ defined by $f \mapsto Tf$ (for all $f \in \mathcal{C}$) is a contraction mapping on $(\mathcal{C}, d)$. *(8 marks)*

(iv) In view of Part (iii), the Contraction Mapping Theorem can be applied to deduce that there is a unique $f \in \mathcal{C}$ such that $Tf = f$. What property of the space $(\mathcal{C}, d)$ ensures that the Contraction Mapping Theorem can be applied, given that T is a contraction? (You are not asked to show that $\mathcal{C}$ has this property.) *(2 marks)*

6. Let X be a topological space.

(i) What does it mean to say that a subset A of X is *compact*? *(4 marks)*

(ii) Suppose that A and B are subsets of X such that A is compact and B is closed. Prove that $A \cap B$ is compact. *(8 marks)*

7. Let $\mathbb{Z}$ be the set of all integers, and let $\mathcal{T}$ be the set of all subsets U of $\mathbb{Z}$ such that

either $0 \notin U$,

or $\mathbb{Z} \setminus U$ is finite.

(i) Show that $\mathcal{T}$ constitutes a topology on $\mathbb{Z}$. *(4 marks)*

(ii) Show that $\mathbb{Z}$ is compact in the topology $\mathcal{T}$. *(8 marks)*

8. Suppose that X is a Hausdorff space and A is a compact subset of X. Prove that A is closed. *(8 marks)*

— END OF PAPER —