THE UNIVERSITY OF SYDNEY

Rings, Fields and Galois Theory

Semester 1 **Assignment 1** 2009

This assignment is due at or before the beginning of the Thursday lecture on 21st May

It can be handed to me in room 526 or at the *beginning* of the lecture.

1. The Second Isomorphism Theorem

Let I be an ideal in a ring R and let S be a subring of R.

(i) (a) Prove $S + I = \{s + r : s \in S, r \in I\}$ is a subring of R.

(b) Prove that I is an ideal of $S + I$.

(ii) Show that $s \mapsto s + I$ defines a surjective ring homomorphism $\phi : S \to (S + I)/I$.

(iii) Use the First Isomorphism Theorem to give an isomorphism between $S/(S \cap I)$ and $(S + I)/I$.

2. Let $R = F[[x]]$, the ring of formal power series over a field F in the indeterminate x.

(i) Prove that R is an integral domain.

(ii) Let $a = \sum_{i=0}^{\infty} a_i x^i$ be an element of R. Show that if $b = \sum_{j=0}^{\infty} b_j x^j \in R$ satisfies $ab = 1$ then $a_0 b_0 = 1$ and $\sum_{i=0}^{k} a_i b_{k-i} = 0$ for all $k > 0$. Hence show that a has an inverse if and only if $a_0 \neq 0$. (Find an inductive formula for coefficients b_j with the required properties.)

(iii) Let L be the ring of *formal Laurent series* over F in the indeterminate x. It consists of all elements of the form $\sum_{i=-k}^{\infty} a_i x^i$, with coefficients from F and where $k \in \mathbb{Z}$. Addition and multiplication are as for R. Thus elements of L are much like elements of R, the difference being that an element of L may involve a finite number of negative powers of x. It contains R as the subring of Laurent series with no terms of negative degree.

Use Part (ii) to show that all nonzero elements of L have inverses.

(iv) Deduce L is the *field of fractions* of R.

3. (i) Show that the irreducible elements in a unique factorisation domain are prime.

(ii) For each prime number p give an example of a unique factorisation domain such that for every irreducible element of $\pi \in R$, $R/\pi R$ is a field of characteristic $p > 0$.

(iii) Give an example of a unique factorisation domain R and an irreducible element of $\pi \in R$ such that $R/\pi R$ is not a field.

(iv) Prove that if R is unique factorisation domain the product of primitive polynomials in $R[x]$ is primitive.