Comments on Tutorial 8 Exercises and Solutions

2. Other methods:

High School (Square to eliminate square roots).

$$\begin{aligned}
&x = \sqrt{2} + \sqrt{3}\\
&(x - \sqrt{2})^2 = 3\\
&x^2 - 2\sqrt{2}x + 2 = 3\\
&x^2 - 1 = 2\sqrt{2}x\\
&(x^2 - 1)^2 = 8x^2\\
&x^4 - 10x^2 + 1 = 0.
\end{aligned}$$

Linear Algebra (Looking Ahead to **3.**)

Multiplication by $\sqrt{3}+\sqrt{2}$ is a linear transformations of the $\mathbb{Q}$-subspace of $\mathbb{R}$ generated by $\{1, \sqrt{3}, \sqrt{2}, \sqrt{3}\sqrt{2}\}$. Easy to derive

$$\begin{aligned}
\sqrt{3}+\sqrt{2}\begin{pmatrix}1\\ \sqrt{3}\\ \sqrt{2}\\ \sqrt{3}\sqrt{2}\end{pmatrix} &= \begin{pmatrix}\sqrt{3}+\sqrt{2}\\ 3+\sqrt{3}\sqrt{2}\\ 2+\sqrt{3}\sqrt{2}\\ 2\sqrt{3}+3\sqrt{2}\end{pmatrix}\\
&= \begin{pmatrix}0&1&1&0\\ 3&0&0&1\\ 2&0&0&1\\ 0&2&3&0\end{pmatrix}\begin{pmatrix}1\\ \sqrt{3}\\ \sqrt{2}\\ \sqrt{3}\sqrt{2}\end{pmatrix}
\end{aligned}$$

So $\mathbf{u} = \begin{pmatrix}1\\ \sqrt{3}\\ \sqrt{2}\\ \sqrt{3}\sqrt{2}\end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix}0&1&1&0\\ 3&0&0&1\\ 2&0&0&1\\ 0&2&3&0\end{pmatrix}$ with eigenvalue $\sqrt{3}+\sqrt{2}$. So $\sqrt{3}+\sqrt{2}$ is root of the characteristic polynomial of A. Calculate

$$\det(xI - A) = x^4 - 10x^2 + 1.$$

3. $E = \mathbb{Q}(\alpha, \beta)$. Find the Theorem from lectures that tells you $[E : \mathbb{Q}]$ is a multiple 3 and less than 9.

5./6. These cover definitions and results we have dealt with in lectures. Specifically the definition of algebraic over a field, minimum polynomial, and the fact that minimum polynomials are irreducible. Note all of these terms are relative to some field.

Question 6 falls just short of making very explicit the following.

Observation Let $K : F$ be a field extension.

Suppose $\alpha \in K$ is a root of a *monic* polynomial $p(x) \in F[x]$ and $p(x)$ is irreducible in $F[x]$. Then α is algebraic over F and $p(x) = m_{\alpha,F(x)}$, the minimal polynomial of α over F.