1 Lecture 1 (Solving Quadratics and Cubics)

Quadratics can be solved by taking square roots.

Observe

$$(2ax + b)^2 = 4a^2x^2 + 4abx + b^2.$$

To solve a general quadratic

$$ax^2 + bx + c = 0, \quad a \neq 0$$

multiply by $4a$ and complete the square.

$$\begin{aligned} 4a^2x^2 + 4abx^2 + 4ac &= 0 \\ 4a^2x^2 + 4abx + b^2 &= b^2 - 4ac \\ (2ax + b)^2 &= b^2 - 4ac \end{aligned}$$

This reduces to finding a square root of $b^2 - 4ac$.

Let $(b^2 - 4ac)^{1/2}$ stand for a square root. Then we derive

$$\begin{aligned} (2ax + b) &= \pm(b^2 - 4ac)^{1/2} \\ x &= \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a} \end{aligned}$$

the standard formula.

Solving Cubics

Recall

$$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 \tag{1}$$

This gives

$$(3ax + b)^3 = [27a^3x^3 + 27a^2bx] + [9(3ax)b^2 + b^3].$$

To solve a general quadratic

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0$$

multiply by $27a^2$ and complete the cube

$$27a^2x^2 + 27abx^2 + 9ac(3a)x + 27a^2d = 0$$

$$\begin{aligned} 27a^3x^3 + 27a^2bx + 3(3ax)b^2 + b^3 &= 3(3ax)b^2 + b^3 - 9ac(3a) - 27ad^2 \\ (3ax + b)^3 &= 3(b^2 - ac)(3ax) + b^3 - 27ad^2 + b^3 \\ (3ax + b)^3 &= 3(b^2 - 3ac)(3ax + b) - 3(b^2 - 3ac)b + b^3 - 27ad^2 \\ (3ax + b)^3 &= 3(b^2 - 3ac)(3ax + b) - 2b^3 + 9abc - 27a^2d \end{aligned}$$

If we put $y = 3ax + b$ we reduce to solving

$$y^3 = 3Py + Q$$

where $P = b^2 - 3ac$, and $Q = 9abc - 2b^3 - 27a^2d$.

Solution Cardano (1545).

To solve a cubic equation

$$y^3 = 3Py + Q$$

write $y = u + v$.

Then

$$\begin{aligned} y^3 = (u+v)^3 &= u^3 + 3u^2v + 3uv^2 + v^3 \\ &= 3uv(u+v) + u^3 + v^3 \\ &= 3Py + Q \end{aligned}$$

provided

$$P = uv \qquad Q = u^3 + v^3.$$

Then $u^3 + v^3 = Q$ and $u^3v^3 = P^3$. So u^3 and v^3 are a pair of roots of the quadratic

$$z^2 - Qz + P = 0.$$

Hence if pick a pair u, v of cube roots of

$$\frac{Q + (Q^2 - 4P^3)^{1/2}}{2}, \quad \frac{Q - (Q^2 - 4P^3)^{1/2}}{2}$$

respectively, such that $uv = P$ we get a solution Cardano's cubic.

Example Solve $x^3 - x^2 + x - 1 = 0$.

Complete the cube and put into standard form for Cardano's method

$$\begin{aligned} x^3 - x^2 + x - 1 &= 0 \\ 27x^3 - 27x^2 &= -9(3x) + 27 \\ (3x)^3 - 3(3x)^2 + 3(3x) - 1 &= 3(3x) - 9(3x) + 27 - 1 \\ (3x-1)^3 &= -6(3x) + 26 \\ (3x-1)^3 &= -6(3x-1) + 20 \end{aligned}$$

Putting $y = 3x - 1$ we are reduced to solving

$$y^3 = -6y + 20$$

Set $y = u + v$. Then

$$y^3 = 3uv(u + v) + u^3 + v^3 = 3uvy + u^3 + v^3 = -6y + 20$$

if and only $uv = -2$ and $u^3 + v^3 = 20$.

Hence u^3 and v^3 must satisfy $u^3v^3 = -8$ and $u^3 + v^3 = 20$. Thus they must be the roots roots of the quadratic

$$z^2 + 20z - 8 = (z - 10)^2 - 108$$

So

$$u^3, v^3 = 10 \pm \sqrt{108}, \quad \text{(both real)}.$$

If we let $u, v = \sqrt[3]{10 \pm \sqrt{108}}$ stand for their respective real cube roots then

$$uv = \sqrt[3]{10 + \sqrt{108}} \times \sqrt[3]{10 - \sqrt{108}}$$

is a real cube root of $10 + \sqrt{108} \times 10 + \sqrt{108} = -8$.

So $uv = -2$.

Hence we have real solution to our cubic

$$x = \frac{\sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} + 1}{3}.$$

In fact $x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1)$ has roots 1 and $\pm i$.

We deduce

$$\frac{\sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} + 1}{3} = 1.$$

Set $\omega = \exp\left(\frac{2\pi i}{3}\right) = \frac{-1+i\sqrt{3}}{2}$. Then 1, ω and $\omega^2 = \exp\left(\frac{4\pi i}{3}\right) = \frac{-1-\imath\sqrt{3}}{2}$ are the three cube roots of 1.

The other two roots of the cubic are

$$\frac{\omega\sqrt[3]{10 + \sqrt{108}} + \omega^2\sqrt[3]{10 - \sqrt{108}} + 1}{3} = i$$

and

$$\frac{\omega^2\sqrt[3]{10 + \sqrt{108}} + \omega\sqrt[3]{10 - \sqrt{108}} + 1}{3} = -i$$

End of lecture 1

2 Appendix, (Solving Quartics)

Consider a quartic Polynomial

$$x^4 - S_1 x^3 + S_2 x^2 - S_3 x + S_4.$$

If the roots of the quartic are t_1, t_2, t_3, t_4 then the S_i are elementary functions on these roots

The six signed root sums with two pluses and two minuses fall into 3 pairs $\pm\lambda$, $\pm\mu$, $\pm\nu$,

$$\begin{aligned}
\lambda &= t_1 - t_2 + t_3 - t_4 \\
\mu &= t_1 + t_2 - t_3 - t_4 \\
\nu &= t_1 - t_2 - t_3 + t_4
\end{aligned}$$

Any permutation of the roots permutes λ^2, μ^2, ν^2 amongst thenselves. Hence the elementary symmetric polynomials in them are fixed by all permutations of the roots. Hence they are roots of a cubic whose coefficients are polynomials in the S_i.

$$\begin{aligned}
\lambda^2 &= t_i^2 - 2t_1t_2 + 2t_1t_3 - 2t_1t_4 - 2t_2t_3 + 2t_2t_4 - 2t_3t_4 \\
&= S_1^2 - 4(t_1t_2 + t_1t_4 + t_2t_3 + t_3t_4) \\
&= S_1^2 - 4[S_2 - (t_1t_3 + t_2t_4)] \\
&= S_1^2 - 4S_2 + 4(t_1t_3 + t_2t_4)
\end{aligned}$$

and similarly

$$\mu^2 = S_1^2 - 4S_2 + 4(t_1t_2 + t_3t_4), \quad \nu^2 = S_1^2 - 4S_2 + 4(t_1t_4 + t_2t_3).$$

So

$$\lambda^2 + \mu^2 + \nu^2 = 3S_1^2 - 12S_2 + 4S_2 = 3S_1^2 - 8S_2$$

Set

$$\rho = t_1t_3 + t_2t_4, \quad \sigma = t_1t_2 + t_3t_4, \quad \tau = t_1t_4 + t_2t_3$$

Then

$$\rho + \sigma + \tau = S_2.$$

and

$$\rho\sigma + \rho\tau + \sigma\tau \quad = \sum t_i^2 t_j t_k \quad = S_1S_3 - 4S_4$$

Hence we find

$$\begin{aligned}\lambda^2\mu^2+\lambda^2\nu^2+\mu^2\nu^2 &= 3(S_1^2-4S_2)^2+8(S_1^2-4S_2)(\rho+\sigma+\tau)+16(\rho\sigma+\rho\tau+\sigma\tau)\\ &= 3(S_1^2-4S_2)^2+8(S_1^2-4S_2)S_2+16(S_1S_3-4S_4).\end{aligned}$$

Interchanging t_1 and t_2 sends λ to $-\nu$, and ν to$-\lambda$ and fixes μ. Similarly for any other transposition.

Hence $\lambda\mu\nu$ is fixed by all transposition, and hence by all permutations of the roots.

Armed with knowledge we find

$$\lambda\mu\nu = \sum t_i^3 - \sum t_i^2 t_j + 2S_3$$

Using

$$\sum t_i^3 = S_1^3 - 3\sum t_i^2 t_j - 6S_3$$

gives

$$\lambda\mu\nu = S_1^3 - 4\sum t_i^2 t_j - 4S_3$$

Now

$$\sum t_i^2 t_j = S_1S_2 - 3S_3$$

Hence

$$\lambda\mu\nu = S_1^3 - 4S_1S_2 + 8S_3.$$

Thus

$$\lambda^2\mu^2\nu^2 = (S_1^3 - 4S_1S_2 + 8S_3)^2.$$