

Lecture 3

Review of equivalence relation.

Relations A relation \sim is defined on set S if for each pair of elements we have some criteria to decide if $a \sim b$ or not.

For example on the reals \mathbb{R} we have the relation $a < b$ if $b - a > 0$. Relations correspond to subsets of $S \times S$. To relation we can associate the sub-set $R = \{(a, b) \mid a \sim b\}$, and to subset of R of $S \times S$ we can associate the relation $a \sim b$ if $(a, b) \in R$.

Definition 3.1 (Relations) A relation on S is called

- *reflective* if $x \sim a$ for all $x \in S$;
- *symmetric* if for $x, y \in S$, $x \sim y$ implies $y \sim x$;
- *transitive* if for $x, y, z \in S$, $x \sim y$ and $y \sim z$ together imply $z \sim z$.

For example on \mathbb{R} , $<$ is transitive but not reflective or symmetric whereas \leq is reflective and transitive but not symmetric.

Definition 3.2 (Equivalence Relations) A relation \sim on a set S is called an *equivalence relation* if is reflective, symmetric and transitive.

For example equality $=$ is an equivalence relation.

Definition 3.3 (Classes) If \sim is an equivalence relation on S and $s \in S$ the equivalence class of is

$$\bar{x} = \{y \in S \mid x \sim y\}.$$

The equivalence classes partition S in to mutually disjoint subsets. They are non-empty because $x \in \bar{x}$. If $x, y \in S$ then either $x \sim y$ and $\bar{x} = \bar{y}$, or $x \not\sim y$ and $\bar{x} \cap \bar{y} = \emptyset$.

Definition 3.4 The set of equivalence classes $\{\bar{x} \mid x \in S\}$ is denoted S / \sim .

Definition 3.5 (The Canonical Map) The surjective map $S \rightarrow S / \sim$ such that $x \mapsto \bar{x}$ is called the *canonical map*.

Exercise/Example Let $\varphi : S \rightarrow X$ be a map from S to a set X . Then $a \sim_f b$ if $f(a) = f(b)$ defines an equivalence relation on S , with equivalence classes the inverse images of the elements of $\text{im } f$: for $a \in S$, $\bar{a} = f^{-1}f(a)$. Thus the natural map

$$\pi : S / \sim_f \rightarrow \text{im } f, \quad \bar{a} \mapsto f(a).$$

is a bijection from S / \sim to $\text{im } f$.