1 Lecture 4 (Left and Right Cosets)

Definition 4.1 (Left Cosets) Suppose $H \leq G$. Then for $x \in G$

$$xH = \{xh : h \in H\}$$

is the left coset of H in G generated by x.
The set of left cosets of $H \leq G$ is denoted G/H.

As $x \in xH$ for each $x \in G$, the left cosets are non-empty and have union G. Distinct left cosets do not intersect because

$$xH \cap yH \neq \emptyset \Leftrightarrow y^{-1}x \in H \Leftrightarrow xH = yH$$

Thus the distinct left cosets partition G. So the left cosets of H are the equivalence classes of an equivalence relation $\sim_l$, and from above the relation is given by

$$x \sim_l y \quad \text{if} \quad y^{-1}x \in H.$$

We note

$$H = \bar{1} = \{x \in G : x \sim_l 1\}.$$

Definition 4.2 (Right Cosets) Similarly

$$Hx = \{hx : h \in H\}$$

is the right coset of H in G generated by x.
The set of right cosets of $H \leq G$ is denoted $H \backslash G$.

Since $x \in Hx$, for each x in G the right cosets are non-empty and have union G. Distinct right cosets do not intersect because

$$Hx \cap Hy \neq \emptyset \Longleftrightarrow y^{-1}x \in H \Longleftrightarrow Hx = Hy$$

So the right cosets of H are the equivalence classes of an equivalence relation $\sim_r$ and the equivalence relation is given by

$$x \sim_r y \quad \text{if} \quad xy^{-1} \in H.$$

We note

$$H = \bar{1} = \{x \in G : x \sim_r 1\}.$$

Note that $\sim_l$ has the property

$$x \sim_l y \Longrightarrow gx \sim_l gy \quad \text{for all } g \in G,$$

because if $y^{-1}x \in H$,

$$(gy)^{-1}gx = y^{-1}g^{-1}gx = y^{-1}x \in H.$$

Similarly $\sim_r$ has the property

$$x \sim_r y \Longrightarrow xg \sim_l yg \quad \text{for all } g \in G.$$

Definition 4.3 (Left and Right Invariance) Let $\sim$ be an equivalence relation on a group G.
1. Call $\sim$ left invariant if $x \sim y \Longrightarrow gx \sim gy$ for all $g \in G$.
2. Call $\sim$ right invariant if $x \sim y \Longrightarrow xg \sim yg$ for all $g \in G$.

Observations When $\sim$ is left invariant, $x \sim y \Longleftrightarrow gx \sim gy$ for all $g \in G$.
When $\sim$ is right invariant, $x \sim y \Longleftrightarrow xg \sim yg$ for all $g \in G$.
Hint: For the converse take $g = 1$.

Proposition 4.5 Let $\sim$ be an equivalence relation on G and $H = \bar{1} = \{h \in G : h \sim 1\}$.
1. The equivalence relation $\sim$ is left invariant if and only of H is a subgroup *and* $\bar{x} = xH$ for all $x \in G$.
2. The equivalence relation $\sim$ is right invariant if and only of H is a subgroup *and* $\bar{x} = Hx$ for all $x \in G$.

Proof The $\Leftarrow$ implication follows from the discussion above. It shows left cosets by a subgroup H are equivalence classes of a left invariant equivalence relation for which $\bar{1} = H$ is a subgroup and similarly right cosets by a subgroup H are equivalence classes of a right invariant equivalence relation for which $\bar{1} = H$ is a subgroup .
It remains only to prove the $\Rightarrow$ implications.
Case 1. Suppose $\sim$ is a left invariant equivalence relation.
First we show first H is a subgroup by showing, (a) it is non-empty, (b) it is closed under taking inverses and (c) it is closed under multiplication.

(a) The subset H is non-empty because $1 \in \bar{1} = H$

(b) Suppose $x \in H$. Then

$$\begin{aligned} x \sim 1 &\quad \text{(definition)} \\ x^{-1}x \sim x^{-1} &\quad \text{(left invariance)} \\ 1 \sim x^{-1} &\quad (x^{-1}x = 1) \\ x^{-1} \sim 1 &\quad \text{(by} \sim \text{symmetric)} \end{aligned}$$

So $x^{-1} \in H$.

(c) Suppose $x, y \in H$. Then $x \sim 1$ and $y \sim 1$.

From $y \sim 1$ and left invariance we have $xy \sim x$.

From $xy \sim x$ and $x \sim 1$ we have $xy \sim 1$, by transitivity of $\sim$.

Hence $xy \in H$.

Second we show $\bar{x} = xH$, i.e. $x \sim y$ if and only $xH = yH$.

$$\begin{aligned} x \sim y &\quad \text{definition)} \\ \Leftrightarrow y^{-1}x \sim y^{-1}y &\quad \text{(left invariance of} \sim) \\ \Leftrightarrow y^{-1}x \sim 1 &\quad \text{(inverse property)} \\ \Leftrightarrow y^{-1}x \in H &\quad \text{(definition of } H) \\ xH = yH &\quad \text{(left coset condition)} \end{aligned}$$

Case 2. This follows similarly. $\square$