Lecture 5 (Normal Subgroups and Congruences)

Normal Subgroups Recall a subgroup H of G is called normal if $xH = Hx$ for all $x \in G$ or equivalently $xHx^{-1} = \{xhx^{-1} : h \in H\} = H$ for all $x \in G$. When a subgroup is normal we can form the quotient group,

$$G/N = \{xH : x \in G\}, \quad (xH).(yH) = xyH$$

Suppose we have a an equivalence relation $\approx$ on a group. We could try to make the equivalence classes $G/\approx$ into a group as follows.
If $\alpha = \bar{x}$ and $\beta = \bar{y}$ set $\alpha\beta = \bar{xy}$. This will be well defined if it does not depend on the choice of equivalence class representatives, i.e. if for any other x' and y' with $\alpha = \bar{x'}$ and $\beta = \bar{y'}$ $\bar{x'y'} = \bar{xy}$, i.e. if

$$x \approx y \quad \text{and} \quad x' \approx y' \quad \Longrightarrow xx' \approx yy', \text{ for all } x, x', y, y' \in G. \tag{1}$$

Definition 5.1 (Congruences) An equivalence relation $\approx$ on a group G such that (1) holds is called a *congruence*.

Proposition 5.2 Suppose $\approx$ is a congruence on the group G and set $Q = G/\approx$. Then Q with multiplication $\alpha\beta = \bar{xy}$, if $\alpha = \bar{x}$, $\beta = \bar{y}$ is a group.

Proof By the discussion above this product is well defined. Checking the group properties is straight forward as they are inherited from G. □

Group Homomorphisms Recall a homomorphism from one group G to another K is map

$$\varphi : G \to K \quad \text{such that} \quad \varphi(a)\varphi(b), \quad \text{for all } a, b \in G.$$

Note From proposition 5.2 if $\approx$ is a congruence on G, multiplication is defined on G so that the map $^{-} : G \to G/\approx$, $x \to \bar{x}$ is a homomorphism from G onto $G/\approx$.

Proposition 5.2 The following are equivalent for an equivalence relation $\approx$ on a group G.

(1) The equivalence relation $\approx$ is a congruence.

(2) The equivalence relation $\approx$ is left and right invariant.

(3) The set $H = \{h \in G : h \approx 1\}$ is a normal subgroup of G, and for all $x \in G$, $\bar{x} = xH = Hx$.

Proof

$(1) \Rightarrow (2)$ Assume (1). Suppose $x \approx y$ and $g \in G$. Because $\approx$ is an equivalence relation, $g \approx g$. From the congruence condition we deduce both $gx \approx gy$ and $xg \approx yg$. Hence (2).

$(2) \Rightarrow (3)$ Assume (2). From left or the right invariance we have H is a subgroup. For any $x \in G$, by left invariance, the equivalence class $\bar{x} = xH$ whereas by right invariance $\bar{x} = Hx$. So $\bar{x} = xH = Hx$ for all $x \in G$. Thus (3) holds.

$(3) \Rightarrow (1)$ Assume 3. Then from the equivalence class condition for left cosets, $x \approx y$ if and only if $y^{-1}x \in H$. Suppose $x \approx y$, $x' \approx y'$.

$$(yy')^{-1}xx' = y'^{-1}y^{-1}xx' = y'^{-1}y^{-1}xy'y'^{-1}x'.$$

From $x \approx y$, $y^{-1}x \in H$. Hence by H normal $h = y'^{-1}(y^{-1}x)y' \in H$. But by $x' \approx y'$, $h' = y'^{-1}x' \in H$. Hence

$$(yy')^{-1}xx = hh' \in H.$$

So $xx' \approx yy'$. □

Corollary When the conditions above hold the group $G/\approx$ is the group G/H.

□

Kernel and Image Associated to a homomorphism $\varphi : G \to K$ of groups are two groups, a subgroup of K,

$$\operatorname{im}\varphi = \{\varphi(g) : g \in G\}$$

and a *normal* subgroup of G,

$$\ker\varphi = \{g : \varphi(g) = 1\}\,.$$

Forgetting for a moment that G is a group and φ is homomorphism we know from lecture 3 that if define $x \sim_\varphi y$ if $\varphi(x) = \varphi(y)$ this is an equivalence relation, and that the natural map $\bar{x} \mapsto \varphi(x)$ defines a natural bijection from $G/\sim_\varphi$ to $\operatorname{im}\varphi$.

Lemma 5.3 Suppose $\varphi : G \to K$ is a group homomorphism. Then the following hold.

(1) Then the equivalence relation $x \sim_\varphi y$ if $\varphi(x) = \varphi(y)$, is a congruence.

(2) The subgroup
$$H = \{x \in G : x \sim_\varphi 1\} = \ker \varphi.$$
is normal in G.

(3) The natural bijection $\bar{x} \mapsto \varphi x$ is a homomorphism.

Proof

(1) If $\varphi(x) = \varphi(y)$ and $\varphi(x') = \varphi(y')$, then
$$\varphi(xx') = \varphi(x)\varphi(x') = \varphi(y)\varphi(y') = \varphi(yy').$$
hence the congruence condition holds.

(2) The equality $H = \ker \varphi$ is immediate.

(3) Its a homomorphism because
$$\bar{\varphi}(\bar{x}.\bar{y}) = \bar{\varphi}\bar{xy} = \varphi(xy) = \varphi(x)\varphi(y) = \bar{\varphi}\bar{x}.\bar{\varphi}\bar{y}\square$$

Summary

Theorem 5.4 (First Isomorphism Theorem) Given a group homomorphism $\varphi : G \to K$, the natural map
$$\pi : G/\ker \varphi \to \operatorname{im} \varphi, \quad x \ker \varphi \mapsto \varphi(x),$$
is an isomorphism.