Lecture 7 (New Rings From Old)

Definition 7.1 (Matrix Rings) Let R be a ring. Let $M_n(R)$ be the set of $n \times n$ matrices with entries from R. with the usual matrix addition and matrix multiplication. That is, denoting the (i, j)th entry of a matrix A by A_{ij}, for $n \times n$ matrices A and B

$$(A + B)_{ij} = A_{ij} + B_{ij}$$
$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Then is it can shown that $M_n(R)$ is a ring. Verification involves showing that $M_n(R)$ satisfies the ring axioms because R does.
See RH notes for the associative law.
Left Distributive Law:
Let $A, B, C \in M_n(R)$. Then by the definition of matrix operations the (i, j) entry of $A(B + C))$

$$\sum_{k=1}^{n} A_{ik}(B_{kj} + C_{kj}) = \sum_{k=1}^{n} A_{ik} B_{kj} + \sum_{k=1}^{n} A_{ik} C_{kj}$$

by the left distributive law in R, and this is definition of matrix operations the of (i, j) entry of $AB + AC$.

Direct Products of a Family of Rings Recall that the product of two sets X_1, X_2

$$X_1 \times X_2 = \{\mathbf{x} = (x_1, x_2) \,|\, x_1 \in X_1, x_2 \in X_2\}$$

If $X_1 = X_2 = X$, then $X \times X = X^2$ can also viewed as all functions from $\{1, 2\}$ by identifying the ordered pair $\mathbf{x} = (x_1, x_2)$, with the function from $\{1, 2\} \to X$ such that $1 \to x_1$, $2 \mapsto x_2$.

Proposition 7.2 Let R_1 and R_2 be rings. Then

$$R_1 \times R_2 = \{(r_1, r_2) \,|\, r_1 \in R_1,\ r_2 \in R_2\}$$

is a ring if we define addition and multiplication component-wise

$$\begin{gathered}(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2),\\ (a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2).\end{gathered}$$

for all $a_1, b_1 \in R$, $a_2, b_2 \in R_2$.
For $R_1 = R_2 = R$, viewing $R \times R = R^2$ as functions from $\{1, 2\}$ the ring operations become in function notation,

$$\begin{gathered}(f + g)(i) = f(i) + g(i), \quad i = 1, 2,\\ (fg)(i) = f(i)g(i), \quad i = 1, 2.\end{gathered}$$

for all $f, g : \{1, 2\} \to R$.
More generally we can put a ring structure on the product of an arbitrary family of rings by defining the ring operations point-wise.

Proposition 7.3 Let $(R_i)_{i\in I}$ be an indexed family of rings. Set

$$P = \prod_{i\in I} R_i = \{(r_i)_{i\in I} \,|\, r_i \in R_1, \forall i \in I\}$$

The P is a ring, under component-wise addition and multiplication

$$\begin{gathered}(a_i)_{i\in I} + (b_i)_{i\in I} = (a_i + b_i)_{i\in I},\\ (a_i)_{i\in I}(b_i)_{i\in I} = (a_ib_i)_{i\in I}.\end{gathered}$$

Proof Verification of the ring operations is a routine exercise.
Illustration: Left Distributive Law.
Suppose $\mathbf{a} = (a_i)_{i\in I}, \mathbf{b} = (b_i)_{i\in I}, \mathbf{c} = (c_i)_{i\in I} \in P$.
For $i \in I$ the ith component of P of $\mathbf{a}(\mathbf{b} + \mathbf{c})$ is by the definition of sum and product, $a_i(b_i + c_i)$ By the left distributive law in R_i

$$a_i(b_i + c_i) = a_ib_i + a_ic_i,$$

and by the definition of sum and product in P this is the ith component of $\mathbf{ab} + \mathbf{ac}$. Hence

$$\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}.\square$$

In the special case R is a ring and $R_i \in R$ for all $i \in I$, $\prod_{i\in I} = R^I$ is identified with the set of functions from I to R. The element $\mathbf{a} = (a)_{i\in I}$ represents the function $I \to I$ which maps $i \in I$ to a_i. Then R^I is a ring under point-wise addition and multiplication:

$$
\begin{aligned}
(f+g)(i) &= f(i) + g(i)\\
(fg)(i) &= f(i)g(i)
\end{aligned}
$$

Definition 7.4 (Subrings) A subset S of a ring R is a subring if it is ring with the same addition and multiplication as R.

Note Since each ring has zero element a subring is necessarily non-empty,

Observation 7.5 Subring Conditions A non-empty subset S is of a ring R is a subring if and only if the following hold.

(1) S is closed under addition: $a + b \in S, \forall a, b \in S$.

(2) S is closed under taking negatives: $-a \in S, \forall a \in S$

(3) S is closed under multiplication: $ab \in S, \forall a, b \in S$.

The zero element of a subring S is the zero element of the over ring R.

Example

1. The zero ring is a subring of every ring.
2. We have inclusions of subrings

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

3. There are corresponding subring inclusions

$$\mathrm{M}_n\,\mathbb{Z} \subset \mathrm{M}_n\,\mathbb{Q} \subset \mathrm{M}_n\,\mathbb{R} \subset \mathrm{M}_n\,\mathbb{C}.$$

4. Also for any set I, there subring inclusions,

$$\mathbb{Z}^I \subset \mathbb{Q}^I \subset \mathbb{R}^I \subset \mathbb{C}^I.$$

Subrings and Identity Elements Suppose a ring R is a and S is a subring.

(1) If R has an identity element must S necessarily have one?

(2) If R and S each have identity elements are they necessarily equal?

(3) If S has an identity need R have one?

In cases the answer in all cases is no!

Examples

(1) $S = 2\mathbb{Z}$ is a subring of $R = \mathbb{Z}$, but $2\mathbb{Z}$ does not have an identity element.

(2) $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Q} \right\}$ is a subring of $R = \mathrm{M}_2(\mathbb{Q})$.

Then $1_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $1_S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so that $1_R \neq 1_S$.

(3) The zero subring $S = \{O\}$ of $2\mathbb{Z}$ has identity O, but $2\mathbb{Z}$ has no identity element.

$S = \{(x, 0) \mid x \in \mathbf{R}\}$ is subring of $R \times R$, (component-wise operations) S has unit

(4) Let $R = \mathrm{M}_n(\mathbb{Q})$ and $S = \{(m_{ij}) \in \mathbb{Q} \mid m_{ij} = 0 \quad \text{if} \quad i > j\}$, the upper triangular matrices with rational entries. Then S is a subring of R and the $n \times n$ identity matrix is the identity element of both rings.