# Lecture 8 (Ring Homomorphisms)

Last times we defined products of rings and subrings.
We also looked at rings of functions. For example let $I$ be an interval in $\mathbb{R}$. Then

$$\mathbb{R}^I = \mathrm{Map}(I, \mathbb{R}),$$

the set of all functions from $I$ to $\mathbb{R}$ is a ring under usual (point-wise) addition and multiplication. Its zero is the constant 0 function and it has identity the constant function 1.
Then $C(I)$, all continuous functions from $I$ to $\mathbb{R}$, and $C^1(I)$, all differentiable functions from $I$ to $\mathbb{R}$ are each subrings of $\mathbb{R}^I = \mathrm{Map}(I, \mathbb{R})$. These subrings all share the same identity, the constant function 1.

**Subrings and Identity Elements** Suppose $R$ is a ring and $S$ is a subring.

(1) If $R$ has an identity element must $S$ necessarily have one?

(2) If $R$ and $S$ each have identity elements are they necessarily equal?

(3) If $S$ has an identity need $R$ have one?

In cases the answer in all cases is no!
Examples of different possibilities:

**Examples**

(1) $S = 2\mathbb{Z}$ is a subring of $R = \mathbb{Z}$, but $2\mathbb{Z}$ does not have an identity element.

(2) $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Q} \right\}$ is a subring of $R = \mathrm{M}_2(\mathbb{Q})$.
Then $1_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $1_S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so that $1_R \neq 1_S$.

(3) The zero subring $S = \{O\}$ of $2\mathbb{Z}$ has identity $O$, but $2\mathbb{Z}$ has no identity element.
$S = \{(x, 0) \mid x \in \mathbf{R}\}$ is subring of $R \times R$, (component-wise operations) $S$ has unit

(4) Let $R = \mathrm{M}_n(\mathbb{Q})$ and $S = \{(m_{ij}) \in \mathbb{Q} \mid m_{ij} = 0 \quad \text{if} \quad i > j\}$, the upper triangular matrices with rational entries. Then $S$ is a subring of $R$ and the $n \times n$ identity matrix is the identity element of both rings.

**Definition 8.1 (Ring Homomorphism)** A ring homomorphism is a map $\varphi : R \to S$ from a ring $R$ to a ring $S$ which preserves addition and multiplication, viz for all $a, b \in R$, both

$$\begin{aligned} \varphi(a+b) &= \varphi(a) + \varphi(b) \\ \text{and} \quad \varphi(ab) &= \varphi(a)\varphi(b) \end{aligned}$$

**Note** On the left hand side operation are in $R$ and on the right hand side in $S$. If we want to be ultra pedantic we should write,

$$\varphi(a +_R b) = \varphi(a) +_S \varphi(b) \quad \text{and} \quad \varphi(a ._R b) = \varphi(a) ._S \varphi(b)$$

**Examples**

1. For any ring $R$ the identity map $R \to R$, $r \to r$, $r \in R$.
2. If $S$ is subring of $R$, the inclusion map $S \hookrightarrow R$ is homomorphism, e.g. $2\mathbb{Z} \hookrightarrow \mathbb{Z}$.
3. The map $x \mapsto 2x, : \mathbb{Z} \to 2\mathbb{Z}$ is not a ring homomorphism.
4. The evaluation map,

   $$\varphi : C(\mathbb{R}) \to \mathbb{R}, \quad f \mapsto f(1)$$

5. For $A \in \mathrm{Gl}_n(\mathbb{Q})$ , invertible $n \times n$ matrices with coefficients in $\mathbb{Q}$

   $$\varphi_A : \mathrm{M}_n(\mathbb{Q}) \to \mathrm{M}_n(\mathbb{Q}); \quad M \mapsto AMA^{-1}$$

   is a homomorphism.

**Proposition 8.2** Suppose $\varphi : R \to S$ is a ring homomorphism. Then

(1) $\varphi(0_R) = 0_S$ and $\varphi(-a) = -\varphi(a)$ for all $a \in R$.

(2) $\operatorname{im}\varphi = \{\varphi(r) \mid r \in R\}$ is a subring of $S$.

(3) If $R$ has an identity element 1, $\operatorname{im}\varphi$ has identity element $\varphi(1)$.

(4) If $R$ is commutative so is $\operatorname{im}\varphi$.

**Proof**

(1) From $0_R + 0_R = 0_R$ in $R$ we deduce $\varphi(0_R) + \varphi(0_R) = \varphi(0_R)$ in $S$. Subtracting $\varphi(0_R)$ from each side gives $\varphi(0_R) = 0_S$.

Now suppose $a \in R$. Then from $a + (-a) = 0_R$ in $R$ we deduce $\varphi(a) + \varphi(-a) = \varphi(0_R) = 0_S$. Hence $\varphi(-a) = -\varphi(a)$.

(2) From the first part $\operatorname{im}\varphi \neq \emptyset$ because it contains $0_S = \varphi(0_R)$, and is closed under taking inverses.

The definition of homomorphism shows $\operatorname{im}\varphi$ is closed under addition and multiplication.

(3) For any $a \in R$, $\varphi(1a) = \varphi(a) = \varphi(a1)$,

$$\varphi(1)\varphi(a) = \varphi(1a) = \varphi(a) = \varphi(a1) = \varphi(a)\varphi(1).$$

(4) Exercise. □

**Definition 8.3 (Kernels)** If $\varphi : R \to S$ is a homomorphism of rings the *kernel* of $\varphi$, $\ker\varphi$ is the subset of $R$ of elements mapped to 0.

$$\ker\varphi = \{x \in R \,|\, \varphi(x) = 0\}.$$

**Observation** Kernels are subrings with extra important closure properties.

**Theorem 8.4** Let $\varphi : R \to S$ be homomorphism of rings, Then $\ker\varphi$ is subring of $R$. Moreover it is closed under left and right multiplication from elements of $R$, i.e. $ax, xa \in \ker\varphi$ for any $a \in R$ if $x \in \ker\varphi$.

**Proof** We check the subring conditions.
First note $\ker\varphi \neq \emptyset$, because from above $\varphi(0) = 0$ shows $0 \in \ker\varphi$.
Now we check closure under ring operations in $R$.
Suppose $x, y \in \ker\varphi$ So $\varphi(x) = \varphi(y) = 0$. Then

$$\begin{aligned} \varphi(x+y) &= \varphi(x) + \varphi(y) = 0 + 0 = 0, \\ \varphi(xy) &= \varphi(x)\varphi(y) = 0 \times 0 = 0 \\ \text{and} \quad \varphi(-x) &= -\varphi(x) = -0 = 0 \end{aligned}$$

Hence $x+y$, $xy$ and $-x$ are in $\ker\varphi$.
Further for $x \in \ker\varphi$ and any $r \in R$,

$$\begin{aligned} \varphi(rx) &= \varphi(r)\varphi(x) = \varphi(r)0 = 0 \\ \text{and} \quad \varphi(xr) &= \varphi(x)\varphi(r) = 0\varphi(r) = 0, \end{aligned}$$

show $rx, xr \in \ker\varphi$.

**Proposition 8.5** Let $\varphi : R \to S$ be a homomorphism of rings. Then $\varphi$ is injective if and only if $\ker\varphi = \{0\}$.

**Proof** Suppose $\varphi$ is injective.
Let $x \in \ker\varphi$. Then $\varphi(x) = 0 = \varphi(0)$ implies $x = 0$. Also from above $0 \in \ker\varphi$. Hence $\ker\varphi = \{0\}$.
Suppose $\ker\varphi = \{0\}$.
Let $x, y \in R$ satisfy $\varphi(x) = \varphi(y)$. Then

$$\varphi(x-y) = \varphi(x) + \varphi(-y) = \varphi(x) - \varphi(y) = 0.$$

So $x - y \in \ker\varphi = \{0\}$. Hence $x - y = 0$, and thus $x = y$.
This shows $\varphi$ is injective.

**Note** Not every subring can be kernel.

1. $\mathbb{Q}$ is subring of $\mathbb{R}$, but $1 \in \mathbb{Q}$, $\sqrt{2} \in R$, $1 \times \sqrt{2} = \sqrt{} \not\in \mathbb{Q}$, shows it is not the kernel of any ring homomorphism.

2. $\mathbb{Z}$ subring of $\mathbb{Q}$, but $1 \in \mathbb{Z}$, $1/2 \in Q$, $1 \times 1/2 = 1/2 \notin \mathbb{Z}$, shows it is not the kernel of any ring homomorphism.

**Definition 8.6 (Ideals)** A subring $I$ of a ring $S$ such that $rx, xr \in I$ for all $x \in I$, $r \in R$ is called an *ideal*.

**Sufficient Conditions** Note if $I$ closed under multiplication on the left and right by elements of $R$ then it is certainly closed under multipliction. So a non-empty subset of $\mathbb{R}$ is an ideal if for all $a, b \in I$ and $r \in R$

$$(1) \quad a+b \in I, \quad (2) \quad -a \in I, \quad (3) \quad ra, ar \in I.$$

**NB** If the ring has an identity taking $r = -1$, we find (3) implies (2).