Lecture 13 (Fields Of Fractions)

Polynomial Functions

Let R be a ring and x an indeterminate. Then if t is an element of R, or of some larger ring S containing R as subring, we can evaluate any $f(x) \in R[x]$ at t. If $f(x) = a_0 + a_1 x + \cdots + a_n x^n$,

$$f(t) = a_0 + a_1 t + \cdots + a_n t^n$$

This defines a map $\mathrm{eval}_t : R[x] \to S$, $f \mapsto f(t)$. Addition of polynomial is so defined that eval_t respects addition.
If S is commutative, or more generally if t commutes with all elements of S, $(rt = tr$, for all $r \in R)$, then eval_t also respects and multiplication, because then $\mathrm{eval}_t(a_i x^i)\,\mathrm{eval}_t(b_j x^j) = a_i t^i b_j t^j = a_i b_j t^{i+j} = \mathrm{eval}_t\, a_i b_j x^{i+j}$. In summary.

Proposition 13.1 If R is subring of S and $t \in S$ commutes with all elements of R then

$$\mathrm{eval}_t : R[x] \to S[x], \quad f(x) \mapsto f(t),$$

is a homomorphism.

The Field Of Fractions of an Integral Domains

Recall how the rational numbers $\mathbb{Q} = \{a/b \mid a, b \in Z, b \neq 0\}$ is the field made up from quotients of integers.
For all $a, a', b \neq 0, b' \neq 0$ in $\mathbb{Z}$.

$$a/b = a'/b' \quad \text{if and only if} \quad ab' = ba'.$$

Addition

$$\frac{a}{b} + \frac{a'}{b'} = \frac{ab' + ba'}{bb'}.$$

Multiplication

$$\frac{a}{b}\frac{a'}{b'} = \frac{aa'}{bb'},$$

Inversion, $a/b = 0$ if and only if $a = 0$. For $a/b \neq 0$, $(a/b)^{-1} = b/a$.
Similarly from the polynomial ring $F[x]$ we build the field of rational functions, (quotients of polynomials), $F(x) = \{a/b \mid a, b \in F[x], b \neq 0\}$.

Theorem 13.2 Let R be an integral domain. Then there is field $F(R)$ with the following properties.

QF1. R is embedded as a subring of F.

QF2. Every element of $F(R)$ is of the form ab^{-1}, $a, b \in R$, $b \neq 0$.

QF3. If R is embedded in a field K then $F(R)$ is embedded in the field K, and is the smallest subfield of K containing R.

Proof (Sketch) As a set $F(R) = \{a/b \,|\, a, b \in R, b \neq 0\}$, where two formal symbols a/b, a'/b', $a, a', b, b' \in R$ with $b, b' \neq 0$, represent the same element of $F(R)$ if and only if $ab' = a'b$.
To show this is well defined, you show that $(a, b) \sim (a', b')$ if $ab' = ba$ is an equivalence relation on $R \times R \setminus \{0\}$. Then you let a/b be the equivalence class of (a, b).
Next you show that

$$\frac{a}{b} + \frac{a'}{b'} = \frac{ab' + ba'}{bb'} \quad \text{and} \quad \frac{a}{b}\frac{a'}{b'} = \frac{aa'}{bb'},$$

are well defined addition and multiplication operations on $F(R)$, which make it into a commutative ring with $0 = 0/1$ and with identity element $1 = 1/1$.
Then you verify that $a/b = 0$ if and only if $a = 0$. Also for a, b both non-zero $a/b \times b/a = ab/ab = 1$. So every $a/b \neq 0$ is invertible: $(a/b)^{-1} = b/a$.
So $F(R)$ is a field.

QF1. You verify that $\frac{a}{1} = \frac{a'}{1}$ if and only if $a = a'$ and that $\frac{a}{1} + \frac{a'}{1} = \frac{a + a'}{1}$, $\frac{a}{1}\frac{a'}{1} = \frac{aa'}{1}$. Hence you can embed R in $F(R)$ by relabelling each $\frac{a}{1} = a$.

QF2. You show that the formal symbol $\frac{a}{b} = ab^{-1}$ in the field $F(R)$:

$$\frac{a}{b} = \frac{a}{1} \times \frac{1}{b} = ab^{-1}.$$

QF3. If R is a subfield of a field K then for all $a, b \in R$, $b \neq 0$, $ab^{-1} \in K$. So $F(R)$ is a subfield of K, and further any subfield of K containing R must contain all such elements, i.e. all elements of $F(R)$. □