Last lecture

- Recalled the definition of a group, subgroups, ...
- Recalled the definition of an equivalence relations, ...

The quotient of a set by an equivalence relation

Let $\sim$ be an equivalence relation on a set S.

Definition 2.1

The quotient of S by $\sim$ is the set $S/{\sim} = \{\,\overline{x} : x \in S\,\}$ of equivalence classes in S.

Motivating example

Suppose that N is a normal subgroup of G and let $\sim$ be the equivalence relation given by $x \sim y$ if $xN = yN$.
Then the equivalence classes are the cosets of N in G, so the quotient of G by $\sim$ is the set

$$G/{\sim} = \{\,xN : x \in G\,\} = G/N$$

of the cosets of N in G.

Since N is a normal subgroup $G/N = G/{\sim}$ is itself a group.

We are going to use this quotient construction to construct rings and fields. Before we do this we revisit the construction of G/N.

Congruences

If $f : S \longrightarrow T$ is any function then we can define an equivalence relation $\sim$ on S by declaring that $x \sim y$ if and only if $f(x) = f(y)$.

The equivalence classes are in one–to–one correspondence with the elements in the image of f: $t \in \text{image}(f) \quad \leftrightsquigarrow \quad \{\, x \in S : f(x) = t\,\}$.

Exercise Show that *every* equivalence relation on a set S arises from a function in this way. [Hint: Construct a function $S \longrightarrow S/{\sim}$.]

A congruence on a group G is an equivalence relation $\approx$ such that

$$g \approx h \text{ and } g' \approx h' \quad \Longrightarrow \quad gg' \approx hh', \text{ for all } g, g', h, h' \in G.$$

Lemma 2.2

Suppose that $\varphi : G \longrightarrow H$ is a group homomorphism. Then the equivalence relation on G defined by $x \approx y$ if $\varphi(x) = \varphi(y)$ is a congruence on G.

Proof We have, $\varphi(gg') = \varphi(g)\varphi(g') = \varphi(h)\varphi(h') = \varphi(hh')$.

Left and right invariant equivalence relations

Suppose that $\sim$ is an equivalence relation on a group G.

Then $\sim$ is left invariant if $x \sim y \quad \Longrightarrow \quad gx \sim gy$, for all $g \in G$

Then $\sim$ is right invariant if $x \sim y \quad \Longrightarrow \quad xg \sim yg$, for all $g \in G$

Proposition 2.3

Suppose that $\sim$ is an equivalence relation on G. Then $\sim$ is an congruence if and only if it is both left and right invariant.

Proof ($\Longrightarrow$) Suppose that $x \sim y$ and $x' \sim y'$, for $x, x', y, y' \in G$.

Then $xx' \sim xy'$ and $xy' \sim yy'$ since $\sim$ is left and right invariant.

Hence, $xx' \sim yy'$ by transitivity, showing that $\sim$ is a congruence.

($\Longleftarrow$) The converse is just the definitions (check!). □

Subgroups and invariance

Recall that a subgroup N of G is normal if $xN = Nx$ for all $x \in G$.

Proposition 2.4

Suppose $\sim$ is an equivalence relation and set $H = \{ x \in G : x \sim 1 \}$.

1. *$\sim$ is left invariant $\iff$ H is a subgroup of G and $\overline{x} = xH$, $\forall x \in G$.*
2. *$\sim$ is right invariant $\iff$ H is a subgroup of G and $\overline{x} = Hx$, $\forall x \in G$.*
3. *$\sim$ is a congruence $\iff$ H is a normal subgroup of G.*

Proof ($\Longleftarrow$) Recall (slide 1.13) that if H is a subgroup then we can define an equivalence relation $\sim$ by $x \sim y$ if $xH = yH$. This relation is left invariant and the equivalence classes are the left cosets xH.
Similarly, define $x \sim' y$ if $hX = Hy$. This is a right invariant equivalence relation with equivalence classes te right cosets (Check!).
If H is a normal subgroup then the equivalence relations $\sim$ and $\sim'$ coincide, so $\sim$ is a congruence.
Hence, all of the $\Longleftarrow$ implications now follow.

Subgroups and invariance.../2

(proof...) ($\Longrightarrow$) (1) Suppose that $\sim$ is left invariant.
Now, $x \sim 1$ and $y \sim 1 \implies xy^{-1} \sim 1 \implies xy^{-1} \in H$.
$\implies$ H is a subgroup of G by the subgroup criterion (Theorem 1.3).
Now fix $x \in G$ and consider $\overline{x} = \{ y \in G : y \sim x \}$.
Then $y \in \overline{x} \iff y \sim x \iff x^{-1}y \sim 1 \iff x^{-1}y \in H$
$\iff y = xh$ for some $h \in H$.
Hence, $\overline{x} = xH$ as claimed.

The proof of (2) is similar.

Taken together, parts 1 and 2 imply part 3. □

The sets $xH = \{ xh : h \in H \}$ and $hX = \{ hx : h \in H \}$ are the left and right cosets of H in G, respectively.

Congruences and quotients

Suppose that G is a group and that $\approx$ is a congruence on G.

Let $Q = G/\approx\, = \{ \overline{x} : x \in G \}$, the set of equivalence classes in G.

Suppose that $\alpha, \beta \in Q$ and write $\alpha = \overline{x}$ and $\beta = \overline{y}$, for $x, y \in G$.

Define a multiplication on Q by setting $\alpha\beta = \overline{xy}$.

Claim The product $\alpha\beta$ is well–defined.
That is, if $\alpha = \overline{x'}$ and $\beta = \overline{y'}$ then $\alpha\beta = \overline{x'y'}$.

Proof Since $\approx$ is a congruence, $x \approx x'$ and $y \approx y' \implies xx' \approx yy'$.

Proposition 2.5

Suppose that $\approx$ is a congruence on G and let $Q = G/\approx$.
Then Q is a group with multiplication $\alpha\beta = \overline{xy}$, if $\alpha = \overline{x}$ and $\beta = \overline{y}$.

Proof Associativity is straightforward, $1_Q = \overline{1_G}$ and $\alpha^{-1} = \overline{x^{-1}}$ if $\alpha = \overline{x}$.

The group Q is the quotient of G by $\approx$. If the congruence $\approx$ corresponds to a normal subgroup N according to Prop. 2.4 then $G/\approx\, \cong G/N$.