THE UNIVERSITY OF SYDNEY
MATH3962: Rings, Fields and Galois Theory
(http://www.maths.usyd.edu.au/u/UG/SM/MATH3962/)
Semester 1, 2009 Lecturer: Adrian Nelson

Tutorial 3

1. Show that for any a in a ring R, $-(-a) = a$.
Show that if the ring has an identity element 1, $(-1)a = a(-1) = -a$.

2. (*The Cancellation Law for Integral Domains*). Let R be an integral domain and $a, b, c \in R$. Suppose $ab = ac$ and $a \neq 0$. Prove that $b = c$.

3. Show that if R and S are integral domains, and $\phi: R \to S$ is a ring homomorphism, then either $\phi(x) = 0$ for all $x \in R$, or else ϕ takes the identity element of R to the identity element of S.

4. Suppose $\phi: R \to S$ is a ring homomorphism. Show $\ker \phi = \{0\}$ if and only ϕ is injective.

5. Let I and J be ideals in the ring R.

(*i*) Define $I + J = \{ x + y \mid x \in I \text{ and } y \in J \}$. Prove that $I + J$ is an ideal in R.

(*ii*) Prove that $I \cap J$ is an ideal in R.

(*iii*) Define $IJ = \{ \sum_{k=1}^{n} a_k b_k \mid n \in \mathbb{N}, \text{ and } a_k \in I, b_k \in J \text{ for all } k \}$. Prove that IJ is an ideal in R, and that $IJ \subseteq I \cap J$.

6. Determine $I + J$, $I \cap J$, IJ for the ideals $I = 84\mathbb{Z}$ and $J = 90\mathbb{Z}$ in the ring $\mathbb{Z}$.

7. (*i*) Let R be a ring with 1. An element $u \in R$ is called a *unit* if there exists a $v \in R$ such that $uv = vu = 1$. Show that if I is an ideal of R and some element $x \in I$ is a unit then $I = R$.

(*ii*) Let F be a field and I an ideal in F. Prove that either $I = \{0\}$ or $I = F$.

(*iii*) Show that a commutative ring F with identity which has exactly two ideals $\{0\}$ and F is a field.

8. Let I be a nonzero ideal in the ring $M_2(\mathbb{Q})$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a nonzero element of I.

(*i*) Prove that I contains the elements $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}$.

(*ii*) Prove that I contains $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(*iii*) Prove that $I = M_2(\mathbb{Q})$.

Using a similar procedure, can you find all the ideals in the ring $M_2(\mathbb{Z})$?