THE UNIVERSITY OF SYDNEY

MATH3962: Rings, Fields and Galois Theory

(http://www.maths.usyd.edu.au/u/UG/SM/MATH3962/)

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Tutorial 11

1. Let α be a real 7th root of 2. Using Eisenstein's Criterion, show that $F = \mathbb{Q}(\alpha)$ is a degree 7 extension of $\mathbb{Q}$. Show also that F is not a splitting field for the polynomial $x^7 - 2$.

2. Let $\omega = e^{2\pi i/7}$, a complex 7th root of 1. Determine the minimal polynomial $p(x)$ of ω over $\mathbb{Q}$, and thus show that $E = \mathbb{Q}(\omega)$ is a degree 6 extension of $\mathbb{Q}$. Show also that E is a splitting field for $x^7 - 1$, and that there are exactly six $\mathbb{Q}$-automorphisms of E. (See questions in Tutorials 9 and 10.)

3. Let E and F be as in Questions 1 and 2, and let $K = E(\alpha)$. Show that the minimal polynomial of α over E is a divisor of $x^7 - 2$, and hence show that if $d = [K : E]$ then $d \leq 7$. Show that F is a subfield of K, and hence that $[K : \mathbb{Q}]$ is a multiple of 7. Use this to prove that $d = 7$, hence that $x^7 - 2$ is irreducible over E and is the minimal polynomial of α over E. Is K a splitting field for $x^7 - 2$ over E?

4. Continuing with the notation as in Questions 1, 2 and 3, show that $p(x)$ is irreducible as an element of $F[x]$ (by considering the degree $[F(\omega) : F]$, noting that $F(\omega) = K$). Show that the identity map $F \to F$ extends to an automorphism $g\colon K \to K$ such that $\omega \mapsto \omega^3$. Show that the identity map $E \to E$ extends to an automorphism $f\colon K \to K$ such that $\alpha \mapsto \omega\alpha$.

5. Let $f,\ g \in \mathrm{Aut}_{\mathbb{Q}}(K)$ be as in Q4 above. Show that g^6 is the identity automorphism of K, and that the order of g is 6. Show also that the order of f is 7. Show that if $0 \leq j \leq 6$ and $0 \leq k \leq 5$ then $(f^j g^k)\omega = \omega^{3^k}$ and $(f^j g^k)\alpha = \alpha\omega^j$, and that these 42 automorphisms $f^j g^k$ are the only automorphisms of K. Show that $(gfg^{-1})(\alpha) = f^3(\alpha)$.

6. Let $K\colon F$ and $E\colon K$ be field extensions that are both algebraic. Assume that the overall extension $E\colon F$ is separable. Show that $K\colon F$ and $E\colon K$ are both separable. (Recall that $E\colon F$ separable means that for every $t \in E$ the minimal polynomial of t over F is a separable polynomial, which in turn means that there is no extension of F over which this polynomial has a repeated factor.)