Tutorial 12

1. Let $\omega = e^{2\pi i/7}$. It was shown in Tut. 10, Q 3., that $\mathbb{Q}(\omega)$ has an automorphism ϕ such that $\phi(\omega) = \omega^3$, and all automorphisms of $\mathbb{Q}(\omega)$ are powers of ϕ. Now consider the polynomial

$$f(x) = (x - \omega - \omega^2)(x - \omega^3 - \omega^6)(x - \omega^2 - \omega^4)(x - \omega^6 - \omega^5)(x - \omega^4 - \omega)(x - \omega^5 - \omega^3),$$

and *without calculating them* show that the coefficients of $f(x)$ are fixed by all six automorphisms of $\mathbb{Q}(\omega)$. Deduce, using the Main Theorem of Galois Theory, that the coefficients are all rational.

2. With the notation as in Question 1, show that $\mathbb{Q}(\omega + \omega^2) = \mathbb{Q}(\omega)$, and deduce that the $f(x)$ is irreducible over $\mathbb{Q}$. (Hint: Show that no element of $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\omega))$ fixes $\omega + \omega^2$, and use the Main Theorem of Galois Theory.)

3. (Not examinable.) Let F be a field and $\alpha_1, \alpha_2, \ldots, \alpha_n$ a set of n distinct automorphisms of F. Suppose that $u_1, u_2, \ldots, u_n \in F$ satisfy

$$u_1(\alpha_1 t) + u_2(\alpha_2 t) + \cdots u_n(\alpha_n t) = 0 \qquad (*)$$

for all $t \in F$. Prove that $u_i = 0$ for all i, by means of the following steps.

- (i) The proof is by induction on n. Start it off by doing the case $n = 1$.
- (ii) Suppose now that $n > 1$ and that the result holds for sets of fewer than n automorphisms. Put tv in place of t in Eq.$(*)$, and also multiply Eq.$(*)$ by $\alpha_1 v$. Using the difference between these two equations, and the inductive hypothesis, deduce that $\lambda_i(\alpha_i v - \alpha_1 v) = 0$ for all i and all $v \in F$.
- (iii) Use the fact that $\alpha_1 \neq \alpha_i$ for all i to complete the proof.

4. Let $F = \mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/5}$ (a complex 5th root of 1). Suppose also that K is a normal extension of F with $|K : F| = 5$, and let $\alpha \in \mathrm{Aut}_F(K)$ have order 5.

- (i) Let $t \in K$ such that $t \notin F$, and let $p(x) \in F[x]$ be the minimal polynomial of t. Prove that the degree of $p(x)$ is 5.
- (ii) Prove that t, $\alpha(t)$, $\alpha^2(t)$, $\alpha^3(t)$ and $\alpha^4(t)$ are the roots of $p(x)$.
- (iii) Prove that if u is an arbitrary element of K then u and $\alpha^j(u)$ have the same minimal polynomial for all j.
- (iv) Show that $t + \zeta\alpha(t) + \zeta^2\alpha^2(t) + \zeta^3\alpha^3(t) + \zeta^4\alpha^4(t) \neq 0$ for some $t \in K$. (Use Exercise 3.)
- (v) Prove that if $u = t + \zeta\alpha(t) + \zeta^2\alpha^2(t) + \zeta^3\alpha^3(t) + \zeta^4\alpha^4(t)$ and $u \neq 0$ then $\alpha^j(u) = \zeta^{-j}u$ for each j, and deduce that the minimal polynomial of u over F has degree 5.
- (vi) With u as in Part (iv), let $u^5 = r$. Show that r is fixed by α and deduce that $r \in F$. Deduce further that $x^5 - r$ is the minimal polynomial of u over F.

5. Let $A = \mathrm{Alt}(5)$, the group of all even permutations of $\{1, 2, 3, 4, 5\}$. Show that A contains 15 permutations of cycle type $(i, j)(k, l)$, forming a single conjugacy class, 20 permutations of cycle type (i, j, k), forming a single conjugacy class, and 24 permutations of cycle type (i, j, k, l, m) forming two equal sized conjugacy classes. Using Lagrange's Theorem, deduce that A has no normal subgroups apart from A itself and $\{1\}$. By a similar method, find all the normal subgroups of $\mathrm{Sym}(5)$, the group of all permutations of $\{1, 2, 3, 4, 5\}$.