The University of Sydney

MATH3962: Rings, Fields and Galois Theory

(http://www.maths.usyd.edu.au/u/UG/SM/MATH3962/)

Semester 1, 2009 — Lecturer: Adrian Nelson

## Tutorial 1

**1.** Let $A$, $B$, $C$ and $D$ be sets, and let $f: A \to B$, $g: B \to C$ and $h: C \to D$ be functions. Show that the composites $(hg)f$ and $h(gf)$ are equal.

**2.** Let $G$ be a group and $g \in G$, and consider the sequence $g$, $g^2$, $g^3$, $g^4$, $\ldots$ of positive powers of $g$. Show that either

(a) all the terms of this sequence are different from one another, or

(b) the identity element 1 appears somewhere in the sequence.

In case (b), show also that if $n$ is the least positive integer such that $g^n = 1$ then the sequence is periodic of period $n$, and the first $n$ terms are different from one another.

**3.** The set of all permutations of $\{1, 2, \ldots, n\}$ is a group under permutation multiplication. This group is called the *symmetric group* of degree $n$, and we denote it by $\mathrm{Sym}(n)$.

($i$) Let $\begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix}$ be an arbitrary permutation of $\{1, 2, 3, 4\}$. Show that

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix} (1,2) \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix}^{-1} = (i, j),$$

and deduce that if a normal subgroup of $\mathrm{Sym}(4)$ contains one transposition $(i, j)$ then it contains them all.

($ii$) Prove results similar to Part ($i$), with $(1, 2)$ replaced by $(1, 2, 3)$, $(1, 2, 3, 4)$, and $(1, 2)(3, 4)$.

($iii$) How many permutations of each "cycle type" are there in $\mathrm{Sym}(4)$? (That is, how many transpositions, how many 3-cycles, etc..)

($iv$) Find all the normal subgroups of $\mathrm{Sym}(4)$.

**4.** Let $G$ be a group of permutations of $\{1, 2, 3, 4, 5\}$. Suppose that

(a) for each $i \in \{1, 2, 3, 4, 5\}$ there exists an $\alpha_i \in G$ such that $\alpha_i 5 = i$,

(b) $(4, 5) \in G$.

Show that $G = \mathrm{Sym}(5)$.

(Hint: By considering $\alpha_i(4, 5)\alpha_i^{-1}$ show that $G$ contains transpositions $(a, 3)$, $(b, 2)$ and $(c, 1)$ for some $a$, $b$, $c$. Then show that there exist $i$, $j$, $k$ such that $G$ contains all 6 permutations of $\{i, j, k\}$. Eventually, show that $G$ contains all ten transpositions.)