Definition. Let $F$ be a field. An $F$-algebra is a vector space $A$ over $F$ equipped with an operation $A \times A \to A$ which is bilinear. In other words, for a vector space $A$ to be an $F$-algebra there must be a vector multiplication operation $(v, w) \to vw$ (defined for all $v, w \in A$) such that

$$(\lambda v + \mu w)u = \lambda(vu) + \mu(wu)$$

and

$$u(\lambda v + \mu w) = \lambda(uv) + \mu(ww)$$

for all $u, v, w \in A$ and $\lambda, \mu \in F$.

For example, $\mathbb{R}^3$ with the usual vector product (cross product) is an $\mathbb{R}$-algebra. Note that this algebra is not associative: it is not true that $(u \times v) \times w = u \times (v \times w)$ for all $u, v$ and $w$ in $\mathbb{R}^3$. In fact this algebra is an example of what are known as Lie algebras, which are the second most important kind of algebras. The most important kind, and the only kind that we will be concerned with in this course, are associative algebras: those algebras $A$ satisfying $(uv)w = u(vw)$ for all $u, v, w \in A$. The best example of an associative $F$-algebra is $\text{Mat}_n(F)$, the set of all $n \times n$ matrices over $F$. It is well known that this is an $n^2$ dimensional vector space over $F$, and that matrix multiplication (defined in the usual way) is bilinear and associative. Another example is $\mathcal{P}(F)$, the set of all polynomials over $F$. This is an infinite dimensional vector space, and multiplication of polynomials, defined in the usual manner, is a bilinear associative operation. Unlike the algebra of $n \times n$ matrices over $F$, the algebra of polynomials over $F$ is commutative: it satisfies $pq = qp$ for all $p, q \in \mathcal{P}(F)$.

Suppose that $A$ is an $F$-algebra which is finite-dimensional as a vector space over $F$. Let $v_1, v_2, \ldots, v_n$ be a vector space basis for $A$. Then there exist scalars $\alpha_{ijk} \in F$ such that

$$v_iv_j = \sum_{k=1}^{n} \alpha_{ijk}v_k \quad \text{for all } i, j \in \{1, 2, \ldots, n\}.$$ 

These scalars $\alpha_{ijk}$ are called the structure constants of $A$ for the given basis. Note that the structure constants determine the multiplication completely, since if $u, v \in A$ are arbitrary then there exist scalars $\lambda_i, \mu_j$ with $u = \sum_i \lambda_i v_i$ and $v = \sum_j \mu_j v_j$, and this gives

$$uv = \left(\sum_i \lambda_i v_i\right)\left(\sum_j \mu_j v_j\right) = \sum_{i,j} \lambda_i \mu_j \sum_k \alpha_{ijk}v_k = \sum_k \left(\sum_{i,j} \lambda_i \mu_j \alpha_{ijk}\right)v_k. \quad (1)$$

Conversely, given a vector space $A$ with basis $v_1, v_2, \ldots, v_n$, if we choose scalars $\alpha_{ijk}$ arbitrarily and use Eq. (1) to define a multiplication operation on $A$, then the resulting operation is associative, and hence gives $A$ the structure of an $F$-algebra.

Examples

(i) A 2-dimensional vector space over $\mathbb{R}$ with basis $v_1, v_2$ can be given an $\mathbb{R}$-algebra structure by defining $v_1 v_2 = v_2 v_1 = v_1$ (for both values of $j$) and $v_2^2 = -v_1$. The resulting algebra is easily seen to be isomorphic to $\mathbb{C}$ via $\lambda v_1 + \mu v_2 \mapsto \lambda + i\mu$.

(ii) The is a 4-dimensional associative $\mathbb{R}$-algebra with basis $1, i, j, k$ and multiplication defined by

$$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad ik = -j, \quad kj = -i, \quad 1i = 1 = i, \quad 1j = j1 = j, \quad 1k = k1 = k, \quad i^2 = j^2 = k^2 = -1.$$
This algebra is known as the \textit{algebra of quaternions} over \( \mathbb{R} \). The elements \( \pm 1, \pm i, \pm j, \pm k \) of the quaternion algebra form a group called the \textit{quaternion group} of order 8 (obtained in a different guise in Lectures 2 and 3).

(iii) If we let \( X_{ij} \) be the \( n \times n \) matrix whose \((r,s)\)-entry is \( \delta_{ir}\delta_{js} \) then the \( n^2 \) matrices \( X_{ij} \) for \( i, j \in \{1, 2, \ldots, n\} \) form a basis for the algebra of all \( n \times n \) matrices. The structure constants for this basis are all either 0 or 1. Specifically, \( X_{ij}X_{kl} = \delta_{jk}X_{il} \).

If \( A, B \) are \( F \)-algebras then their \textit{direct sum} is

\[
A \oplus B = \{ (a, b) \mid a \in A, b \in B \}
\]

made into a vector space in the usual way, with multiplication given by

\[
(a, b)(a', b') = (aa', bb') \quad \text{for all } a, a' \in A \text{ and } b, b' \in B.
\]

Note that if \( a_1, a_2, \ldots, a_n \) form a basis for \( A \) and \( b_1, b_2, \ldots, b_m \) a basis for \( B \) then \( A \oplus B \) has a basis

\[
(a_1, 0), (a_2, 0), \ldots, (a_n, 0), (0, b_1), (0, b_2), \ldots, (0, b_m);
\]

furthermore, the first \( n \) of these span a subalgebra \( A' \) of \( A \oplus B \) which is isomorphic to \( A \), the remaining \( m \) basis vectors span a subalgebra \( B' \) of \( A \oplus B \) isomorphic to \( B \), and \( A' \) and \( B' \) annihilate each other (meaning \( ab = 0 \) whenever \( a \in A' \) and \( b \in B' \)).

If \( A \) is an \( F \)-algebra then an \textit{identity element} for \( A \) is an element \( 1 \in A \) such that \( 1a = a1 = a \) for all \( a \in A \). Henceforth in this course we shall use the term “\( F \)-algebra” as an abbreviation for “associative \( F \)-algebra with an identity element”.

\textbf{Definition.} Let \( G \) be a finite group an \( F \) a field. The \textit{group algebra} \( FG \) is an \( F \)-algebra having the elements of \( G \) as a basis, the multiplication of basis elements coinciding with multiplication in the group \( G \).

The elements of \( FG \) are \textit{formal linear combinations} of elements of \( G \): expressions of the form \( \sum_{g \in G} \lambda_g g \). What this really means is that we choose some vector space over \( F \) whose dimension is \( |G| \)—the space of all \( |G| \)-component column vectors would do—and fix a basis of this space. Then choose (arbitrarily) a one to one correspondence between these basis elements and elements of \( G \), and use this to identify the elements of \( G \) with the basis vectors. Multiplication in \( G \) then determines a natural way to define multiplication of the basis elements, and, as we have seen, bilinearity then determines multiplication uniquely for arbitrary elements of the space.

\textbf{Examples}

(i) Let \( G = \{1, x\} \) be the group of order 2. The real group algebra \( \mathbb{R}G \) is \( \{ \lambda 1 + \mu x \mid \lambda, \mu \in \mathbb{R} \} \), with multiplication given by

\[
(\lambda 1 + \mu x)(\lambda' 1 + \mu' x) = (\lambda \lambda' + \mu \mu' 1 + (\lambda \mu' + \mu' \lambda)x).x.
\]

Now choose new basis for \( \mathbb{R}G \) consisting of the two elements \( e = \frac{1}{2}(1 + x) \) and \( f = \frac{1}{2}(1 - x) \).

It is easily seen that \( e \) and \( f \) are \textit{idempotent elements}: \( e^2 = e \) and \( f^2 = f \). Furthermore, \( ef = fe = 0 \). Thus it follows that for all \( \lambda, \lambda', \mu, \mu' \in \mathbb{R} \)

\[
(\lambda e + \mu f)(\lambda' e + \mu' f) = (\lambda \lambda' e + (\mu \mu') f),
\]
and hence $\mathbb{R}G$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ via the one to one correspondence $\lambda e + \mu f \leftrightarrow (\lambda, \mu)$.

(ii) Let $G$ be the Klein 4-group, with elements $1, a, b$ and $c$ (where $a, b$ and $c$ all have order 2 and $ab = c$). The real group algebra $\mathbb{R}G$ consists of all linear combinations $\lambda 1 + \mu a + \nu b + \xi c$, where $\lambda, \mu, \nu, \xi \in \mathbb{R}$. It can be checked that the four elements of the form $\frac{1}{4}(1 \pm a \pm b \pm c)$ where the product of the signs is $+1$ are all idempotents and annihilate one another. Moreover, they form a basis for $\mathbb{R}G$. In this way it can be shown that $\mathbb{R}G$ is isomorphic to the direct sum of four copies of $\mathbb{R}$.

(iii) The complex group algebra of $S_3$ is the six dimensional complex vector space

\[ \{ \alpha \text{id} + \beta (12) + \gamma (13) + \delta (23) + \varepsilon (123) + \zeta (132) \mid \alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{C} \} \]

with multiplication determined by the rule for multiplying permutations. It can be shown that this algebra is isomorphic to the sum of two copies of $\mathbb{C}$ and one copy of $\text{Mat}_2(\mathbb{C})$.

**Definition.** Let $A$ be an $F$-algebra. A matrix representation of $A$ of degree $d$ is a function $\phi: A \to \text{Mat}_d(F)$ satisfying $\phi 1 = I$ and

\[
\begin{align*}
\phi(a + b) &= \phi a + \phi b \\
\phi(ab) &= (\phi a)(\phi b) \\
\phi(\lambda a) &= \lambda(\phi a)
\end{align*}
\]

for all $a, b \in A$ and $\lambda \in F$.

The connection all this has with the representation theory of groups is provided by the following easy proposition.

**Proposition.** If $\phi: FG \to \text{Mat}_d(F)$ is a representation of a group algebra $FG$ then the restriction of $\phi$ to the basis $G$ of $FG$ gives a matrix representation of the group $G$. Conversely, if $\psi: G \to \text{GL}(d, F)$ is a matrix representation of $G$ then we can obtain a matrix representation of $FG$ by extending the domain of definition of $\psi$ to the whole of $FG$ by the formula

\[
\psi(\sum_{g \in G} \lambda_g g) = \sum_{g \in G} \lambda_g (\psi g).
\]

Thus, representations of $G$ are essentially the same as representations of $FG$.  

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