Group representation theory    Lecture 18, 8/10/97

Modules and representations of algebras

Definition. Let $F$ be a field and $A$ an $F$-algebra. A left $F$-module is a vector space $M$ over the field $F$ together with a function $(a,m) \mapsto am$ from $A \times M$ to $M$ which is bilinear and satisfies $(ab)m = a(bm)$ for all $a, b \in A$ and $m \in M$. The module $M$ is said to be unital if in addition $1m = m$ for all $m \in M$, where 1 is the identity element of $A$.

Right modules are defined analogously, the function $A \times M \rightarrow M$ being replaced by a function $M \times A \rightarrow M$. We shall adopt the convention (which is universal) that all $A$-modules are assumed to be unital unless it is explicitly stated otherwise.

The connection between modules and representations works for in the same way for algebras as it does for groups. We have already defined a matrix representation of an $F$-algebra $A$ to be a linear map $\phi : A \rightarrow \text{Mat}_d(F)$ (for some $d$) which preserves multiplication and takes the identity element of $A$ to the identity matrix. In other words, a representation of $A$ is an algebra homomorphism $\phi : A \rightarrow \text{Mat}_d(F)$ such that $\phi 1 = I$. In view of the relationship between matrices and linear transformations the following is virtually a reformulation of this definition.

Definition. A representation of an $F$-algebra $A$ is a homomorphism $\phi$ from $A$ to the algebra of all linear transformations $V \rightarrow V$, where $V$ is a vector space over $F$, such that $\phi 1 = \text{id}$.

We neglected to define the concept of a homomorphism of $F$-algebras, but the definition is obvious: an $F$-algebra homomorphism is a map from one $F$-algebra to another which preserves addition, multiplication and scalar multiplication. It is clear that the set of all linear transformations on a vector space $V$ is an algebra if addition, multiplication and scalar multiplication of linear maps are defined in the usual way (so that multiplication of linear maps is composition). This algebra is usually denoted by $\text{End}_F(V)$, since linear maps $V \rightarrow V$ are also called $F$-endomorphisms of $V$. The student is invited to write out for her/himself a proof of the following proposition, imitating the proof for groups given in Lecture 3.

Proposition. Suppose that $A$ is an $F$-algebra. If $V$ is an $A$-module, and for each $a \in A$ we define $\phi_a : V \rightarrow V$ by $(\phi_a)v = av$ for all $v \in V$, then $\phi_a \in \text{End}_F(V)$; furthermore, the map $\phi : A \rightarrow \text{End}_F(V)$, given by $a \mapsto \phi_a$ for all $a \in A$, is a representation of $A$. Conversely, given a representation $\phi : A \rightarrow \text{End}_F(V)$, the vector space $V$ becomes an $A$-module if the required map $A \times V \rightarrow V$ is defined by the rule that $av = (\phi a)v$ for all $a \in A$ and $v \in V$.

Thus if $G$ is a group then an $FG$-module gives rise to a representation of the group algebra $FG$, and hence gives rise to a representation of $G$, in view of the correspondence between representations of $G$ and representations of $FG$ that we described in Lecture 16. So we have another item to add to the already long list of concepts that are more or less equivalent to the concept of a representation of a group!

Definition. Let $A$ be an $F$-algebra. The left regular $A$-module is the vector space $A$ made into an $A$-module via the map $A \times A \rightarrow A$ given by $(a,b) \mapsto ab$.

Note that this definition is in agreement with the definition, given in Lecture 9, of the regular representation of a group. The left multiplication action of $G$ on itself can be regarded as a permutation representation of $G$, which in turn yields a linear representation of $G$ on a vector
space which has the elements of $G$ as a basis. Since the group algebra $FG$ is such a vector space, this amounts to saying that $FG$ is a $G$-module, the action of $G$ on $FG$ being given by

$$x \left( \sum_{g \in G} \lambda_g g \right) = \sum_{g \in G} \lambda_g x g$$

for all $x \in G$ and scalars $\lambda_g \in F$. In accordance with the theorem from Lecture 16 and the discussion above, a $G$ module is the same thing as an $FG$-module, and by Eq. (1) we see that in this case the map $FG \times FG \to FG$ that makes $FG$ into an $FG$-module is given by

$$\left( \sum_{x \in G} \mu_x x, \sum_{g \in G} \lambda_g g \right) \mapsto \sum_{x \in G} \mu_x \left( \sum_{g \in G} \lambda_g x g \right) = \left( \sum_{x \in G} \mu_x x \right) \left( \sum_{g \in G} \lambda_g g \right).$$

In other words, this is the left regular $FG$-module.

**Definition.** Let $A$ be an $F$-algebra. A left ideal in $A$ is a submodule of the left regular module. In other words, a left ideal is a nonempty subset $I$ of $A$ such that

(i) $x + y \in I$ whenever $x, y \in I$,
(ii) $\lambda x \in I$ whenever $x \in I$ and $\lambda \in F$, and
(iii) $ax \in I$ whenever $a \in A$ and $x \in I$.

A left ideal $I$ is minimal if $I \neq \{0\}$ and there are no nonzero left ideals $J$ with $J \subsetneq I$. That is, the left ideal $I$ is minimal if and only if it is an irreducible left $A$-module.

By Maschke’s Theorem, if $G$ is a finite group then the left regular module $\mathbb{C}G$ can be decomposed as a direct sum of minimal left ideals. To attempt to find explicitly such a direct decomposition is one possible approach to the problem of describing the irreducible representations of a finite group. We have already seen how knowledge of a full set of irreducible complex representations of $G$ enables one to write $\mathbb{C}G$ explicitly as a direct sum of complete matrix algebras, and it is a small step from this to decompose $\mathbb{C}G$ explicitly into minimal left ideals. We shall not go into the details of this, since it is fairly straightforward, and of much less importance than the question of how to find irreducible representations. So we simply state the following result without proof, and leave it to the reader to pursue the matter or not as (s)he chooses.

**Proposition.** The space $F^d$ of $d$-component column vectors over the field $F$ is an irreducible left module for the complete matrix algebra $\text{Mat}_d(F)$, the map $\text{Mat}_d(F) \times F^d \to F^d$ being the usual multiplication of matrices and column vectors. Furthermore, the left regular module for $\text{Mat}_d(F)$ can be expressed as the direct sum of $d$ minimal left ideals $C_1, C_2, \ldots, C_d$ which are all isomorphic to $F^d$. Specifically, we may take $C_j$ to consist of those matrices whose entries in columns other than the $j$th column are all zero.

A finite-dimensional $F$-algebra is said to be semisimple if the left regular module can be expressed as a direct sum of irreducible modules. The above proposition thus says that complete matrix algebras are semisimple, and it follows easily that any algebra which is a direct sum of complete matrix algebras must also be semisimple. It turns out that the converse of this is also true: a finite-dimensional algebra is which is semisimple is necessarily isomorphic to a direct sum of complete matrix algebras. We omit the proof of this.

If $A$ is an $F$-algebra and $b \in A$ an arbitrary element then the set $Ab = \{ ab \mid a \in A \}$ is a left ideal. It is obvious that $Ab \neq \emptyset$ (since $0b \in Ab$). Closure under addition and scalar multiplication is also clear: if $x, x' \in Ab$ and $\lambda \in F$ then there exist $a, a' \in A$ with $x = ab$ and $x' = a'b$, and this
yields $x + x' = (a + a')b \in Ab$ and $\lambda x = (\lambda a)b \in Ab$. Similarly, if $x = ab \in Ab$ then for all $t \in A$ we have $tx = (ta)b \in Ab$, showing that $Ab$ is also closed under left multiplication by elements of $A$.

For example, suppose that $A = \text{Mat}_d(F)$ and $b \in A$ is the matrix whose entries are all zero apart from the $k$th diagonal entry, which is 1. That is, the $(i, j)$-entry of $b$ is $\delta_{ij}b_{jk}$. Then if $a \in A$ is arbitrary, the $(i, j)$-entry of $ab$ is $\sum_{k=1}^d a_{ik}\delta_{jk} = a_{ij}\delta_{jk}$, which shows that $ab$ has zero entries in all columns but the $k$th column, while the $k$th column of $ab$ is the same as the $k$th column of $a$. So the left ideal $Ab$ consists of all matrices which are zero in all columns but the $k$th. So $Ab$ is the left ideal $C_k$ of the proposition above. Note also that the element $b$ is idempotent: $b^2 = b$.

As we shall see, idempotent elements are of fundamental importance in representation theory. In particular, by investigating left ideals generated by idempotent elements in the group algebra of the symmetric group $S_n$, we shall (in the course of the next few lectures) describe a full set of irreducible $CS_n$-modules.

**Definition.** Nonzero elements $e_1, e_2, \ldots, e_k$ in an $F$-algebra $A$ form a set of orthogonal idempotents if $e_i^2 = e_i$ for all $i \in \{1, 2, \ldots, k\}$ and $e_ie_j = 0$ for all $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$.

**Definition.** An idempotent $e$ in an $F$-algebra $A$ is said to be primitive if it is not possible to find two orthogonal idempotents $e_1, e_2$ with $e_1 + e_2 = e$.

The importance of primitive idempotents derives from the following proposition.

**Proposition.** Let $A$ be an $F$-algebra and $e \in A$ an idempotent element. Then $e$ is primitive if and only if the left ideal $Ae$ is indecomposable.

**Proof.** Suppose first that $e$ is not primitive; we shall show that $Ae$ is decomposable. By our assumption there exist idempotents $e_1, e_2$ with $e = e_1 + e_2$ and $e_1e_2 = 0$. Now

$$e_1 = e_1 + 0 = e_1^2 + e_1e_2 = e_1(e_1 + e_2) = e_1e,$$

and it follows that $Ae_1 = Ae_1e \subseteq Ae$. Furthermore, we also have

$$e_2 = e - e_1 = e^2 - e_1e = (e - e_1)e = e_2e,$$

whence $Ae_2 = Ae_2e \subseteq Ae$. So $Ae_1 + Ae_2 \subseteq Ae$. On the other hand, if $a \in A$ then

$$ae = a(e_1 + e_2) = ae_1 + ae_2 \in Ae_1 + Ae_2,$$

and so $Ae \subseteq Ae_1 + Ae_2$.

We have shown that $Ae = Ae_1 + Ae_2$, and since $e_i \in Ae_i$ both summands are nonzero. If we can show that $Ae_1 \cap Ae_2 = \{0\}$ then it will follow that $Ae = Ae_1 \oplus Ae_2$, and hence that $Ae$ is decomposable. But if $x \in Ae_1$ then we have $x = ae_1$ for some $a \in A$, and therefore $x = ae_1 = ae_1^2 = (ae_1)e_1 = x e_1$. So if $x \in Ae_1 \cap Ae_2$ then $x = xe_1$ and $x = xe_2$. But substituting $x = xe_1$ into the right hand side of $x = xe_2$ gives $x = (xe_1)e_2 = x(e_1e_2) = x0 = 0$. So $Ae_1 \cap Ae_2 = \{0\}$, as required.

Conversely, suppose that $Ae$ is decomposable. Then $Ae = I_1 \oplus I_2$ for some nonzero $A$-submodules $I_1, I_2$ of $Ae$. Now $e \in Ae$ and so there exist unique $x \in I_1$ and $y \in I_2$ with $e = x + y$. If $x = 0$ then $e = y \in I_2$, which implies that $Ae = Ay \subseteq I_2$ (since $I_2$ is a left ideal), whence $e_2e_1 = 0$, as can be seen by expanding $(e_1 + e_2)^2$.
Suppose that \( I_1 \oplus I_2 \subseteq I_2 \), contradicting the assumption that \( I_1 \neq \{0\} \). Similarly \( y \neq 0 \). Now since \( x \in A e \) we have \( x = xe \), and thus

\[
(1 - x)x = x - x^2 = xe - x^2 = x(x + y) - x^2 = xy.
\]

But \( xy \in I_2 \) since \( y \in I_2 \), and \( (1 - x)x \in I_1 \) since \( x \in I_1 \), and since \( I_1 \cap I_2 = \{0\} \) we conclude that \( x - x^2 = xy = 0 \). So \( x = x^2 \) and \( xy = 0 \). Exactly similar reasoning with \( x \) and \( y \) interchanged gives \( y^2 = y \) and \( yx = 0 \). So \( x \) and \( y \) are orthogonal idempotents whose sum is \( e \); so \( e \) is not primitive.

The following lemma is a useful for finding idempotents in group algebras.

**Lemma.** Suppose that \( H \) is a subgroup of the finite group \( G \), and \( \lambda: H \to \mathbb{C}^\times \) a representation of \( H \) of degree 1. Then \( e = \frac{1}{|H|} \sum_{x \in H} \lambda(x^{-1})x \) is an idempotent in \( \mathbb{C}G \). If \( H = G \) then \( e \) is primitive.

**Proof.** If \( h \in H \) is fixed, then \( y = hx \) runs through all elements of \( H \) as \( x \) does. So

\[
he = \frac{h}{|H|} \sum_{x \in H} \lambda(x^{-1})x = \frac{1}{|H|} \sum_{x \in H} \lambda((hx)^{-1}h)x = \frac{1}{|H|} \sum_{y \in H} \lambda(y^{-1})h \sum_{y \in H} \lambda(y^{-1})y = \frac{1}{|H|} \sum_{y \in H} \lambda(y^{-1})\lambda(h)y = \lambda(h)e.
\]

It follows that

\[
e^2 = \frac{1}{|H|} \sum_{h \in H} \lambda(h^{-1})he = \frac{1}{|H|} \sum_{h \in H} \lambda(h^{-1})\lambda(h)e = \frac{1}{|H|} \sum_{h \in H} \lambda(h^{-1})h e = \frac{1}{|H|} \sum_{h \in H} e = \frac{|H|}{|H|} e = e.
\]

Thus \( e \) is idempotent.

In the case \( H = G \) the above calculations show that \( ge = \lambda(g)e \) for all \( g \in G \), and so \( (\sum_{g \in G} \alpha_g g)e = (\sum_{g \in G} \alpha_g \lambda(g))e \) for all choices of scalars \( \alpha_g \). So every element of \( \mathbb{C}Ge \) is a scalar multiple of \( e \). Thus the left ideal \( \mathbb{C}Ge \) is a one-dimensional vector space over \( \mathbb{C} \), and as \( \{0\} \) is the only proper subspace of a one-dimensional space it follows that \( \mathbb{C}Ge \) cannot be nontrivially expressed as a direct sum. So \( \mathbb{C}Ge \) is indecomposable, and by the proposition above it follows that \( e \) is primitive. \( \square \)

Note that if \( B \) is a subalgebra of \( A \) then an idempotent \( e \in B \) which is primitive as an idempotent of the algebra \( B \) need not be primitive as an idempotent of \( A \). For example, if \( G = \{1, x\} \) is a cyclic group of order 2 and \( H = \{1\} \) the subgroup of \( G \) of order 1, then the group algebra \( \mathbb{C}H \) is a subalgebra of the group algebra \( \mathbb{C}G \), and the element \( 1 \in \mathbb{C}H \) is a primitive idempotent of \( \mathbb{C}H \). But it is not primitive as an idempotent of \( \mathbb{C}G \), since it is the sum of the orthogonal idempotents \( e_1 = (1/2)(1 + x) \) and \( e_2 = (1/2)(1 - x) \). The left ideal of \( \mathbb{C}H \) generated by the idempotent 1 is \( \mathbb{C}H1 = \mathbb{C}H \), which is 1-dimensional, but the ideal of \( \mathbb{C}G \) generated by 1 is \( \mathbb{C}G1 = \mathbb{C}G \), which is two-dimensional, and the direct sum of the one-dimensional left ideals \( \mathbb{C}Ge_1 \) and \( \mathbb{C}Ge_2 \).