Suppose that a group $G$ has an action on a set $S$. For variety, we shall assume that this is a right action, but totally analogous statements are also valid for left actions. For each $s \in G$ the subset of $G$

$$\text{Stab}_G(s) = \{ g \in G \mid sg = s \}$$

is called the stabilizer of $s$ in $G$. It is quite straightforward to observe that $1 \in \text{Stab}_G(s)$, that $g^{-1} \in \text{Stab}_G(s)$ whenever $x \in \text{Stab}_G(s)$, and that $xy \in \text{Stab}_G(s)$ whenever $x, y \in \text{Stab}_G(s)$. Hence the stabilizer of $S$ is a subgroup of $G$. The subset of $S$

$$\mathcal{O} = \{ sg \mid g \in G \}$$

is called the orbit of $s$ under the action of $G$. If $\mathcal{O} = S$ then the action of $G$ on $S$ is said to be transitive.

As a temporary notation, for $s, t \in S$ let us write $s \sim t$ if there exists $g \in G$ such that $sg = t$. Since $s1 = s$ we have that $s \sim s$, for all $s \in S$; so the relation $\sim$ is reflexive. If $sg = t$ then $tg^{-1} = s$; thus if $s \sim t$ then $t \sim s$, and so $\sim$ is symmetric. And $\sim$ is also transitive, since if $s, t, u \in S$ with $s \sim t$ and $t \sim u$ then there exist $g, h \in G$ with $sg = t$ and $th = u$, and this yields $s \sim u$ since $s(gh) = (sg)h = th = u$. Thus $\sim$ is an equivalence relation, and in consequence the set $S$ is the disjoint union of $\sim$-equivalence classes. The equivalence class containing $s$ is the set

$$\{ t \in S \mid s \sim t \} = \{ sg \mid g \in G \},$$

which is precisely the orbit of $s$. The orbits of $G$ on $S$ are the equivalence classes for the relation $\sim$ as defined above.

One can see that if the stabilizer of an element $s$ is large then the orbit of $s$ is small, and vice versa. The two extreme cases are as follows: if the stabilizer of $s$ is the whole group $G$ then the orbit is the singleton set $\{s\}$; if the stabilizer is the trivial subgroup consisting of the identity element alone, then the elements of the orbit of $s$ are in one to one correspondence with the elements of $G$ (since if $g, h \in G$ and $sg = sh$ then $s(gh^{-1}) = s$, which means that $gh^{-1} \in \text{Stab}_G(s) = \{1\}$, and hence $g = h$). In the general case, if we write $L = \text{Stab}_G(s)$ then $sg = sh$ if and only if $gh^{-1} \in L$, which is equivalent to $g \in Lh$, and this in turn is equivalent to equality of the right cosets $Lg$ and $Lh$. (If we had started with a left action we would have obtained left cosets at this point: $gs = hs$ if and only if $gL = hL$.) So we conclude that there is a well defined bijective mapping

$$sg \mapsto Lg$$

from the orbit $\mathcal{O} = \{ sg \mid g \in G \}$ to the set $\{ Lg \mid g \in G \}$ (whose elements are the right cosets in $G$ of the stabilizer of $s$). Thus if $g_1, g_2, \ldots, g_m$ is a right transversal for $L$, so that

$$G = Lg_1 \cup Lg_2 \cup \cdots \cup Lg_m$$

(where “$\cup$” indicates disjoint union) then

$$\mathcal{O} = \{ sg_1, sg_2, \ldots, sg_m \},$$

and the $sg_i$ are pairwise distinct.

There are two different ways to define right actions of a group $G$ on $G$ itself. Firstly, the group’s multiplication operation $G \times G \to G$ can be interpreted as a function $S \times G \rightarrow S$, where the set $S$ is equal to $G$. The group axioms immediately imply that this function satisfies the defining properties of a right action. We shall call this the right multiplication action of $G$ on itself. It is a transitive action—there is only one orbit—since if $s, t \in G$ are arbitrary then the element $g = s^{-1}t$ satisfies $sg = t$. Furthermore, the stabilizer of any element is trivial, since $sg = g$ implies $g = 1$. The other standard action of $G$ on itself is the conjugacy action. To avoid confusion with the right multiplication action we use an exponential notation for the conjugacy action, and define
$x^g = g^{-1}xg$ for all $x, g \in G$. Note that whereas the right multiplication action is an action of $G$ on $G$ considered only as a set, the conjugacy action is an action of $G$ on $G$ considered as a group. For not only do we have $x^1 = 1^{-1}x1 = x$ and

$$x^gh = (gh)^{-1}x(gh) = h^{-1}(g^{-1}xg)h = (g^{-1}xg)^h = (x^g)^h,$$

for all $x, g, h \in G$, but also

$$(xy)^g = g^{-1}(xy)g = (g^{-1}xg)(g^{-1}yg) = x^g y^g$$

for all $x, y, g \in G$. The orbits of $G$ under the conjugacy action of $G$ are of course the conjugacy classes, as defined in Lecture 4.

**Intertwining matrices**

Let $U$ and $V$ be vector spaces over the complex field which are modules for the group $G$, and let $f: U \to V$ be a $G$-homomorphism. That is, $f$ is a linear map which satisfies $g(fu) = f(gu)$ for all $u \in U$ and $g \in G$. Let $\rho: G \to \text{GL}(V)$ and $\sigma: G \to \text{GL}(U)$ be the representations of $G$ on $V$ and $U$ respectively. That is, if $g \in G$ then $\rho g$ is the linear transformation of $V$ given by $v \mapsto gv$ for all $v \in V$, and $\sigma g$ is the linear transformation of $U$ given by $u \mapsto gu$ for all $u \in U$. For all $u \in U$ we have

$$(\rho g)f)(u) = (g(fu)) = f(gu) = f((\sigma g)u) = f(\sigma g)u,$$

and so $(\rho g)f = f(\sigma g)$. This holds for all $g \in G$. A function $f$ which satisfies $(\rho g)f = f(\sigma g)$ is said to intertwine the representations $\rho$ and $\sigma$. So here again we have two words being used to describe the same concept: an intertwining function is the same thing as a $G$-homomorphism.

Suppose that $u_1, u_2, \ldots, u_n$ is a basis for $U$ and $v_1, v_2, \ldots, v_m$ is a basis for $V$, and let $A$ be the matrix of $f$ relative to these two bases. Thus $A$ is the $m \times n$ matrix with $(i, j)$-entry $a_{ij}$ satisfying $fu_j = \sum_{i=1}^{m} a_{ij}v_i$. For each $g \in G$ let $Rg \in \text{GL}(m, \mathbb{C})$ be the matrix relative to the basis $v_1, v_2, \ldots, v_m$ of the transformation $v \mapsto gv$ of the space $V$, and let $Sg \in \text{GL}(n, \mathbb{C})$ be the matrix relative to the basis $u_1, u_2, \ldots, u_m$ of the transformation $u \mapsto gu$ of the space $U$. So $R$ and $S$ are matrix versions of the representations $\rho$ and $\sigma$. And the matrix version of the equation $(\rho g)f = f(\sigma g)$ is $(Rg)A = A(Sg)$.

**Definition.** If $R$ and $S$ are matrix representations of the group $G$ of degrees $m$ and $n$ respectively then an $m \times n$ matrix $A$ is said to intertwine $R$ and $S$ if $(Rg)A = A(Sg)$ for all $g \in G$.

So an intertwining matrix is the matrix version of a $G$-homomorphism.

Recall that a linear map is invertible if and only if its matrix (relative to any bases) is invertible. Of course, a matrix $A$ can only be invertible if it is square, and this corresponds to the fact that a linear map $U \to V$ can only be invertible if $U$ and $V$ have the same dimension. A $G$-homomorphism $U \to V$ is called a $G$-isomorphism if it is invertible. The matrix version of this is an intertwining matrix which is invertible. Now if $A$ is invertible then the equation $(Rg)A = A(Sg)$ can be rewritten as $Rg = A(Sg)A^{-1}$, and, by a definition from Lecture 3, this means that the representations $R$ and $S$ are equivalent. Conversely, if $R$ and $S$ are equivalent, so that there exists an invertible intertwining matrix $A$, then the linear map $f: U \to V$ whose matrix relative to our two fixed bases is $A$ is a $G$-isomorphism. So we can say that two $G$-modules are $G$-isomorphic if and only if the corresponding matrix representations (relative to any bases) are equivalent.

**Quotient modules**

If $S$ and $T$ are arbitrary subsets of the group $G$ then it is customary to define their product $ST$ by the rule that $ST = \{ st \mid s \in S, \text{ and } t \in T \}$. If $H$ is a normal subgroup of $G$, so that $gH = Hg$
for all \( g \in G \), then \((xH)(yH) = (xy)H\) for all \( x, y \in G \). This yields a well-defined multiplication operation on the set \( G/H = \{ gH \mid g \in G \} \), and it can be checked that under this operation \( G/H \) is a group. The group \( G/H \) is called the quotient of \( G \) by \( H \).

If the group \( G \) is Abelian (commutative) then every subgroup \( H \) is normal, and so the quotient group always exists. In particular, if \( V \) is a vector space over a field \( F \) then \( V \) is an abelian group under the operation of vector addition, and since any vector subspace \( U \) of \( V \) is also an abelian subgroup of \( V \) it follows that the quotient group \( V/U \) can be formed. It is clear hat \( V/U \) is Abelian. Note that since the operation on \( V \) in this case is written as +, the coset of \( U \) containing the element \( v \in V \) is written as \( v + U \) rather than \( vu \), and the group operation on \( V/U \) is also written as +. We have \( V/U = \{ v + U \mid v \in V \} \),

\[
(x + U) + (y + U) = (x + y) + U \quad \text{for all } x, y \in U.
\]

We now give \( V/U \) some extra structure, by defining a scalar multiplication operation on it. The relevant formula is as follows:

\[
\lambda(v + U) = (\lambda v) + U \quad \text{for all } v \in V \text{ and } \lambda \in F.
\]

It is necessary to check that this is well-defined, since it is possible to have \( v_1 + U = v_2 + U \) without having \( v_1 = v_2 \). But if \( v_1 + U = v_2 + U \) then \( v_1 - v_2 \in U \), and since the subspace \( U \) has to be closed under scalar multiplication it follows that \( \lambda v_1 - \lambda v_2 = \lambda(v_1 - v_2) \in U \), and hence \( \lambda v_1 + U = \lambda v_2 + U \). This shows that \( \lambda v + U \) does not depend on the choice of the representative element \( v \) in the coset \( v + U \), but only on the coset \( v + U \) itself. In other words, the formula above does give a well-defined scalar multiplication operation on \( V/U \).

Recall that a vector space over \( F \) is a set—whose elements we call “vectors”—equipped with addition and scalar multiplication operations, such that the following eight axioms are satisfied:

(i) \((u + v) + w = u + (v + w)\) for all vectors \( u, v \) and \( w \);
(ii) \(u + v = v + u\) for all vectors \( u \) and \( v \);
(iv) each vector \( v \) has a negative, which is a vector \(-v\) satisfying \(v + (-v) = 0\);
(v) \(\lambda(\mu v) = (\lambda\mu)v\) for all scalars \( \lambda \) and \( \mu \) and all vectors \( v \);
(vi) \(1v = v\) for all vectors \( v \), where \( 1 \) is the identity element of \( F \);
(vii) \(\lambda(u + v) = \lambda u + \lambda v\) for all vectors \( u \) and \( v \) and all scalars \( \lambda \);
(viii) \((\lambda + \mu)v = \lambda v + \mu v\) for all scalars \( \lambda \) and \( \mu \) and all vectors \( v \).

It is trivial to check that the addition and scalar multiplication operations we have defined on \( V/U \) satisfy these axioms. (Of course the first five of the axioms just say that a vector space is an abelian group under addition, and we had already noted above that \( V/U \) satisfies this.) It is left to the reader to check all the details. We call \( V/U \) a quotient (vector) space.

We proceed to embellish the above situation further by assuming that \( V \) and \( U \) are equipped with \( G \)-actions. More precisely, suppose that \( V \) is a \( G \)-module and \( U \) a submodule of \( V \). Then the quotient space \( V/U \) is also a \( G \)-module, with \( G \)-action satisying

\[
g(v + U) = (gv) + U \quad \text{for all } g \in G \text{ and } v \in V.
\]

As with addition and scalar multiplication, it is crucial to check that this \( G \)-action is well defined. The argument needed is totally analogous to the argument in the scalar multiplication case: if \( v_1 + U = v_2 + U \) then \( v_1 - v_2 \in U \), and since \( U \) is closed under the \( G \) action it follows that \( gv_1 - gv_2 = g(v_1 - v_2) \in U \), whence \( gv_1 + U = gv_2 + U \). It is again left to the reader to check the axioms.