Group Theory Questions

1. (Macdonald p. 15, Q. 12) A semigroup is a set equipped with an associative operation. Suppose that $S$ is a semigroup such that there is an $e \in S$ satisfying
   
   (i) $ex = x$ for all $x \in S$,
   
   (ii) for each $x \in S$ there is an element $x^{-1} \in S$ such that $xx^{-1} = e$.

   Prove that $\{ se \mid s \in S \}$ is a group.

2. (cf. Macdonald p.15, Q. 10) Let $G = \{(a, b, c) \mid a, b, c \in \mathbb{Z}\}$, and define multiplication in $G$ by the rule

   $$(a, b, c)(d, e, f) = (a + (-1)^b d, b + (-1)^c e, (-1)^d c + f).$$

   (i) Show that $G$ with this multiplication is a group.
   
   (ii) Find elements $x, y, z \in G$ with $yx = x^{-1} y$, $zy = y^{-1} z$ and $xz = z^{-1} x$.
   
   (iii) Find an abelian normal subgroup $H$ of $G$ such that the index $[G : H]$ is eight, and describe the structure of the factor group $G/H$.

3. (Macdonald p.32, Q. 5) For each invertible $2 \times 2$ matrix $M$ with complex entries let $\phi_M : \mathbb{C} \to \mathbb{C}$ be defined by

   $$\phi_M(z) = \frac{M_{11} z + M_{12}}{M_{21} z + M_{22}}$$

   where $M_{ij}$ is the $(i, j)$-entry of $M$.

   (i) Show that the set $T$ of all transformations of of the complex plane of this form is a group under composition
   
   (ii) Show that $T$ is a homomorphic image of the group of all $2 \times 2$ invertible complex matrices.
   
   (iii) Is the homomorphism from Part (ii) an isomorphism?

4. (cf. Macdonald p.33, Q. 14) Is the multiplicative group of positive rational numbers isomorphic to the additive group of all rational numbers? Is the multiplicative group of positive real numbers isomorphic to the additive group of all real numbers?

5. (Macdonald p.33, Q. 15) Show that if $A$ is any abelian group and $n$ any integer then the mapping $\phi_n$ taking each element of $A$ to its $n$th power is a homomorphism. In the following cases, discuss whether or not $\phi_n$ is an automorphism.

   (i) $A = \mathbb{Z}$,
   
   (ii) $A = \mathbb{Q}$.

6. (cf. Macdonald p.52, Q. 14) Let $G$ be a finite group $G$, and $S$ a set of complexes of $G$ that forms a group under complex multiplication. Show that the identity element of the group $S$ is a subgroup $H$ of $G$, and the other elements of $S$ are cosets of $H$. Is $H$ necessarily normal in $G$?
7. Let $G$ be a finite group of order 56, and let $S$ be a Sylow 7-subgroup of $G$. Use Sylow’s Theorems to show that $N_G(S)$, the normalizer of $S$ in $G$, either has order 7 or order 56. Show that in the latter case $G$ has a normal subgroup of order 8.

8. Let $A$ be an abelian subgroup of $G$ such that the index of $A$ in $G$ is finite, and let $G$ be the disjoint union of the cosets $x_1A, x_2A, \ldots, x_nA$ (where $x_i \in G$ for each $i$). For each $g \in G$ and $i \in \{1, 2, \ldots, n\}$ let $\eta_g(i) \in \{1, 2, \ldots, n\}$ and $a(i, g) \in A$ be such that $gx_i = x_{\eta_g(i)}a(i, g)$. Show that $\eta_g$ is a permutation of $\{1, 2, \ldots, n\}$ (for each $g \in G$). Show also that $\tau: g \mapsto a(1, g)a(2, g)\cdots a(n, g)$ defines a homomorphism $G \to A$.

Show that if $n$ is odd and $A = \{1, a\}$, where $a$ has order 2, then $\tau(a) = a$. Deduce that $A$ has a normal subgroup of order $n$.

9. (i) Show that any group of order 9 must be Abelian. (Hint: if $x$ is a fixed element of $G$ that is not in the centre of $G$ then the set of all elements of the form $x^i z$, where $i$ is an integer and $z$ is in the centre, forms an Abelian subgroup.)

(ii) Let $G$ be a finite group of order 36. Show that $G$ has a nontrivial proper normal subgroup. (Note that if a subgroup $H$ of order 3 is contained in two distinct Sylow 3-subgroups $S_1$ and $S_2$, then $N_G(H)$ contains both $S_1$ and $S_2$.)