Revision questions 2004

The examination will be worth 60% of the final mark for the CSM part of the course. It will be based closely on your work on Chapters 4–7 and section 9.2 of the notes and Chapter 8 (the MATLAB summary), with particular emphasis on assignment work and on lecture material. Chapter 9 (other than 9.2) is not directly examinable, but will be useful background reading. In addition, you can expect some questions on MATLAB coding based on the codes you have written for the assignments.

1. See the questions in the example quiz.

2. Write down the equation satisfying Newton's law of cooling which states that “the rate of change in the temperature of a body is proportional to the difference in temperature between the body and the surrounding medium”, where $\theta$ is the temperature of the body at time $t$, $\theta_m$ is the temperature of the surrounding medium, and $k$ is a proportionality constant.

   Write a MATLAB program to use the routine `ode45.m` to compute $\theta$ for $0 \leq t \leq 1.0$ at timesteps of 0.1 for different initial temperatures $\theta_i$.

   Assume $\theta_m = 100$ and $k = 0.693$. Make sure you include the rhs file `newton_rhs.m`, say, which must have a first line like

   ```matlab
   function rhs=newton_rhs(t,theta)
   ```

3. Identify all errors in the following MATLAB codes. The following MATLAB script is called `newton.m`:

   ```matlab
   k=0.693;
   thetam=100;
   thetai=input('enter initial temperature: ');
   tfinal=input(' enter final time: ');
   t0=0;
   tstep=(tfinal-t0)/100;
   ```
tspan=t0:tfinal:tstep;
theta=ODE45('newton', tspan, thetai);
thetaex=thetam-(thetai-thetam)exp(-kt);

plot(t,theta,'w')
hold off
plot(t,thetaex,'r+')
hold on
title('newton cooling')
ylabel('temperature')
xlabel('time')

The associated right hand side function is called newton_rhs.m:

global k, thetam
function rhs=test(t,thetai)
% the right hand side
% for the cooling problem

rhs=k(theta-thetam)

4. If the array c has elements $c(i, j) = i + j$, $i = 1, \ldots, 5$, $j = 1, \ldots, 3$, write down the output expected from the following fragment of MATLAB code:

$[m,n]=\text{size}(c);
[\ c(m-2:m,1) \ c(m-2:m,n)]$

5. Given the equation

$$\frac{dy}{dt} = ty^2, \quad y(0) = 1,$$

find $y(0.5)$ using the Euler method with a time-step $\Delta = 0.25$. Now solve the equation analytically, and compare the result at $t = 0.5$ with your estimate using the Euler method, by expanding the analytic solution for small $t$. How could you improve the Euler approximation result?

6. The temperature in a room is assumed to be governed by Newton’s law of cooling, i.e.

$$\frac{d\theta}{dt} = -k(\theta - \theta_m(t))$$

where $\theta$ is the temperature in the room, $k$ is a (positive) proportionality constant and $\theta_m(t)$ the external temperature, given (in degrees C) by

$$\theta_m(t) = 20 + 5e^{-t}\sin t$$
(a) Determine the steady-state solution for this problem, i.e. find the temperature in
the room after a long time period.

(b) If the initial temperature of the room is $30^\circ$ and the external temperature is now
given (in degrees) by

$$\theta_m(t) = 20 + 5 \sin t$$

provide a sketch of the external temperature and (on the same axes) an approximate
graph of the behaviour of $\theta$ (the temperature in the room). Discuss the main features
of the plot for $\theta$.

7. Consider an extension of the model in the previous question to include an air-conditioner,
where the temperature equation is now

$$\frac{d\theta}{dt} = -k(\theta - \theta_m) - k_a(\theta - \theta_a).$$

Here $k_a = 0.1$ is a rate constant associated with the air-conditioner, $k = 0.03$ and $\theta_a$ is
the coil temperature ($5^\circ$) of the air-conditioner. Assume that the external temperature
is fixed at $35^\circ$. The initial temperature in the room is $20^\circ$.

(a) With $k_a$ set to zero find the solution for $\theta$ that holds when the air-conditioner is
always switched off.

(b) For $k_a = 0.1$ (i.e. air-conditioner always on) solve for the temperature when the
room cools from $35^\circ$.

(c) Show how the above solutions can be used to find the solution for the case where the
air-conditioner switches on when the room temperature exceeds $25^\circ$ and switches off
when the room temperature drops below $20^\circ$.

8. A factory under construction at Tempe wants to discharge polluted water into the Cooks
River. The concentration of pollutant is given by $C(t)$, and the concentration of biological
organisms by $B(t)$. The differential equations modelling the breakdown of pollutant by
organisms are:

$$\frac{dC}{dt} = Q(C_f - C)/V - R_1 BC,$$
$$\frac{dB}{dt} = B(R_2 C - D - Q/V).$$

(a) Show that the equations are dimensionally consistent.

(b) Draw a diagram illustrating the problem that the equations are modelling, labelling
inflow, outflow etc.

(c) Are the equations linear or nonlinear? Why? Can they be solved analytically?

(d) What is a steady-state solution? Find the steady-state solutions to the above equa-
tions. Show that there is one solution with $B \to 0$ as $t \to 0$. What does this solution
correspond to physically?

(e) Show that the other steady state solution gives $C_s = (D + Q/V)/R_2$ for the steady-
state $C_s$. If the factory output concentration $C_f$ is doubled what happens to $C_s$ after
a long period of time, assuming a steady-state is re-established?
(f) Draw a graph showing a typical plot of the behaviour of the pollutant concentration and organism concentration as a function of time after a sudden increase in the factory output pollutant concentration level. Identify the transient and steady-state parts of your plots. What will happen to your plots if the volume of the tank is increased?

9. Given the equations for the single tank problem as in the previous question, write down the equations describing $C_1(t)$ and $C_2(t)$ for a two-tank system, where $C_1$ and $C_2$ are the pollutant concentrations in tanks 1 and 2 respectively. What extra term appears in the equation for the organism concentration in tank 2, $B_2(t)$, as opposed to $B_1(t)$ ?(Don’t actually derive the equations for $B_1(t)$ and $B_2(t)$.)

10. Derive the two-tank equations for pollution concentration and organism concentration, given a fixed flow rate $Q$, factory output level $C_f$ and assuming a linear death rate for organisms, and that the rate of digestion of pollutant is proportional to the product of pollutant concentration and organism concentration, while the rate of growth of organisms is also proportional to this product.

11. Chapter 9 of the notes, exercises 9.6, questions 1, 2 and 3.

12. Let $S$ and $I$ be measures of the blood sugar level and blood insulin level in a given individual. Let $F$ and $D$ be measures of the food input rate and the insulin dosage rate for that individual. Explain the physical significance of each term in the following equations.

$$\frac{dI}{dt} = a_3(S - S_{eq})H(S - S_{eq}) - a_4 I + b_2 D$$

$$\frac{dS}{dt} = -a_1 S I + a_2(S_{eq} - S) [H(S_{eq} - S)] + b_1 F$$

13. For the model equations of the previous question, give diagrams showing how the sugar level $S(t)$ and the insulin level $I(t)$ of a person without diabetes vary after a single meal, given that the sugar level is initially at equilibrium and that $I(0) = 0$.

14. In the model equations given above, consider the case of a diabetic ($a_3 = 0$), with no insulin injections ($b_2 = 0$), show that if $I(0) = 0$, then $I(t) = 0$ for all time, and find an explicit expression for $S(t)$ if $S(0) = 100$. Write down a few lines of MATLAB code to plot the function that you find for $S$.

15. Draw a simple plot of the unit step function $H(S - S_{eq})$, as a function of $S$. Write down MATLAB code for calculating this function.

16. Write down MATLAB code for the food intake function $F(t)$ for three meals, breakfast, lunch and dinner, where, for example, the breakfast intake $F_b(t)$ is given by

$$F_b(t) = r_b \exp(-k_b t)$$

if $t > 0$ and the intake is zero for $t \leq 0$. Here $r_b$ and $k_b$ are constants and there are similar functional forms for lunch and dinner.