Mathematical Computing and Nonlinear Systems
MATH 2063/2963

Tutorial 10 Solutions

1(a) $$N(x_n) = x_n + \frac{(c - x_n^2)}{2x_n} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

so $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

fixed points are $X = \frac{1}{2}\left(X + \frac{c}{X}\right)$

or $2X^2 = X^2 + c$

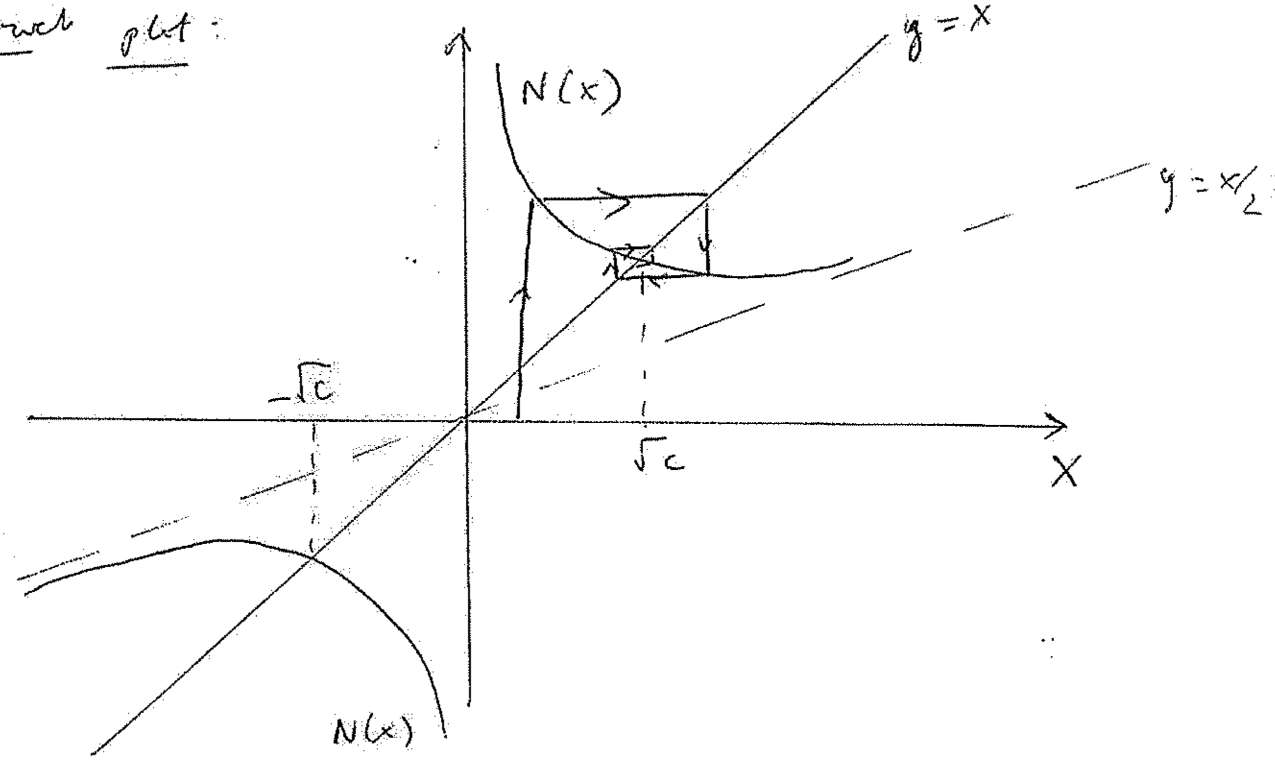
so $X^2 = c$, i.e. $X = \pm\sqrt{c}$

$$N'(x) = \frac{1}{2}\left(1 - \frac{c}{x^2}\right)$$

Then $N'(\sqrt{c}) = \frac{1}{2}(1-1) = 0$ and the fixed point is stable

$N'(-\sqrt{c}) = \frac{1}{2}(1-1) = 0$ and again the fixed point is stable

Cobweb plot:



We can see from the cobweb plot that the domain of attraction for $X = \sqrt{c}$ is $(0, \infty)$. By symmetry, the domain of attraction for $X = -\sqrt{c}$ is $(-\infty, 0)$. If $c = 0$, $x_{n+1} = \frac{1}{2} x_n$, so domain of attraction is all real x. (2)

(b) $$E(x) = x + h f(x)$$
$$= x + h(c - x^2)$$

so $x_{n+1} = E(x_n) = x_n + h(c - x_n^2)$

fixed points satisfy $X = X + h(c - X^2)$

i.e. $X^2 = c$, $X = \pm\sqrt{c}$

$$E'(x) = 1 - 2hx$$

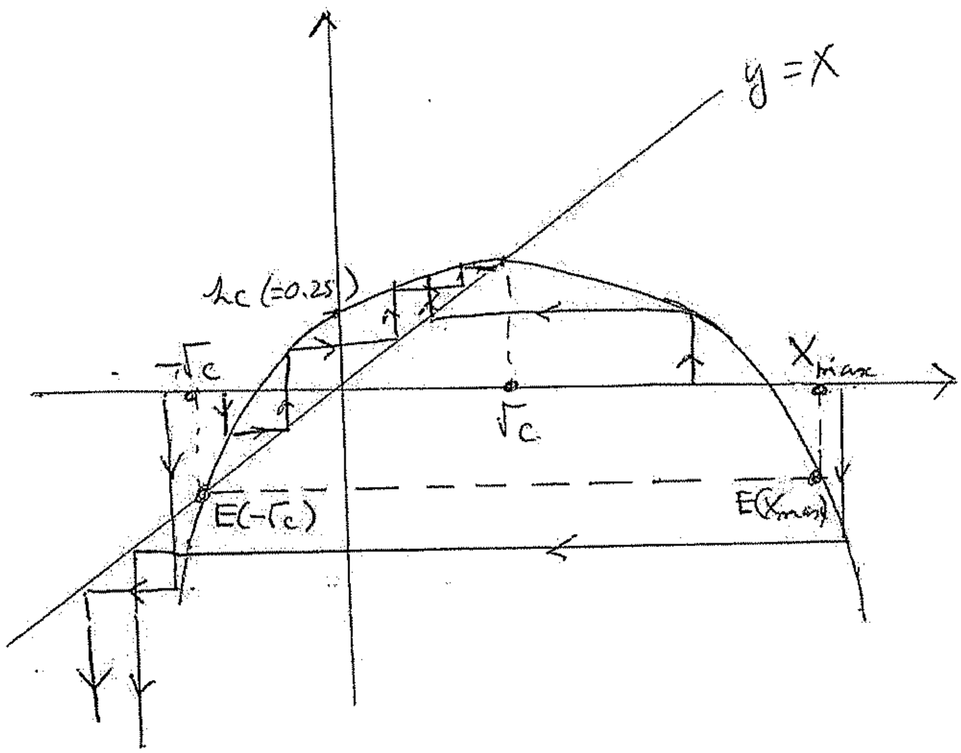
$$= \begin{cases} 1 - 2h\sqrt{c}, & X = \sqrt{c} \\ \text{i.e. stable if } & 2h\sqrt{c} < 2,\ h < \frac{1}{\sqrt{c}} \\ 1 + 2h\sqrt{c}, & X = -\sqrt{c} \\ \text{i.e. unstable for all } h\ (>0) & \end{cases}$$

If we consider the special case
$$h = \frac{1}{2\sqrt{c}}$$
then the fixed point $X = \sqrt{c}$ is stable

$$E(x) = x + h(c - x^2)$$
$$= -h\left(x^2 - \frac{x}{h} - c\right)$$
$$= -h\left(\left(x - \frac{1}{2h}\right)^2 - \frac{1}{4h^2} - c\right)$$

so that at $X = 1/2h$, $E(x)$ takes its maximum value $E(1/2h) = \frac{1}{4h} + ch$
$$= \frac{\sqrt{c}}{2} + \frac{c}{2\sqrt{c}} = \sqrt{c}$$
(as expected)

$h = \frac{1}{2\sqrt{c}}$



from the cobweb plot if $x_0 < -\sqrt{c}$, the iterates will diverge. If $x_0 = -\sqrt{c}$, then any small perturbation will lead to divergence or convergence to $X = \sqrt{c}$

For $x_0 > -\sqrt{c}$ and $x_0 < X_{max}$, we get convergence to $X = \sqrt{c}$, so the domain of attraction is $(-\sqrt{c}, X_{max})$

From the plot $E(X_{max}) = E(-\sqrt{c})$

so $X_{max} + h(c - X_{max}^2) = -\sqrt{c}$

(all this applies only if $h = \frac{1}{2\sqrt{c}}$)

so $X_{max} + \sqrt{c} + h(\sqrt{c} - X_{max})(\sqrt{c} + X_{max}) = 0$

ie $(X_{max} + \sqrt{c})(1 + h(\sqrt{c} - X_{max})) = 0$

one solution is

as clearly, $X_{max} = -\sqrt{c}$ (or alternatively, view this as a way of getting the lower limit of the domain of attraction) (4)

The other solution for X_{max} is:

$$1 + h(\sqrt{c} - X_{max}) = 0$$

$$\Rightarrow \quad X_{max} = \sqrt{c} + \frac{1}{h} = \sqrt{c} + 2\sqrt{c}$$

$$= 3\sqrt{c}$$

ie domain of attraction is $(-\sqrt{c}, 3\sqrt{c})$

2/i Euler: $y_{n+1} = y_n + h\, f(t_n, y_n)$

where $f(t,y) = \frac{dy}{dt} = 2t$

so $y_{n+1} = y_n + 2t_n h$

Thus

$$y_n - \cancel{y_{n-1}} = 2t_{n-1}h$$
$$= 2(n-1)h \times h$$
$$= 2(n-1)h^2$$

Similarly $\cancel{y_{n-1}} - y_{n-2} = 2(n-2)h^2$

$$\vdots$$

$$y_2 - \cancel{y_1} = 2h^2$$
$$\cancel{y_1} - y_0 = 0$$

Adding these equations we are left with

$$y_n - y_0 = 2h^2 \sum_{k=1}^{n-1} k = \cancel{2}h^2 \frac{n(n-1)}{\cancel{2}}$$

so $y_n = y_0 + h^2 n(n-1)$

But $t_n = nh$

so $y_n = y_0 + nh(n-1)h$

$= y_0 + t_n(t_n - h)$ (A)

The exact solution of the ODE is

$\frac{dy}{dt} = 2t \implies y = t^2 + c$

subject to $y(0) = y_0$, so $y = t^2 + y_0$

Thus <u>exactly</u> $y_n = t_n^2 + y_0$ (B)

So the error in the Euler method ((A) − (B))

is $-ht_n$.

3. $$\dot{x} = ax + y - x(x^2+y^2)$$
$$\dot{y} = -x + ay - y(x^2+y^2)$$

(a) Fixed points need $ax + y - x(x^2+y^2) = 0$ (i)

$-x + ay - y(x^2+y^2) = 0$ (ii)

$(0,0)$ obviously works. To see there are no others, multiply (i) by y, (ii) by x and subtract, $\Rightarrow x^2+y^2=0$

$\Rightarrow x=0, y=0$ is the only fixed point.

Linear stability: the Jacobian at O is $\begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix}$

$\Rightarrow$ eigenvalues s satisfy $(s-a)^2+1=0$

$\Rightarrow s = a \pm i$

$\Rightarrow$ unstable spiral if $a>0$, stable spiral if $a<0$.

(b) With $x = r\cos\theta$, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$

$y = r\sin\theta$, $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

$\Rightarrow$ equations become

$\dot{r}\cos\theta - r\dot{\theta}\sin\theta = ar\cos\theta + r\sin\theta - r^3\cos\theta$ (iii)

$\dot{r}\sin\theta + r\dot{\theta}\cos\theta = -r\cos\theta + ar\sin\theta - r^3\sin\theta$ (iv)

Multiply (iii) by $\cos\theta$, (iv) by $\sin\theta$ and add

$\Rightarrow \dot{r} = ar - r^3$

Multiply (iv) by $\cos\theta$, (iii) by $\sin\theta$ and subtract

$\Rightarrow r\dot{\theta} = -r$, ie $\dot{\theta} = -1$.

So the equations are as announced in the question.

The solution $\theta = \theta_0 - t$ of the second eqn is obvious.

The first is solvable using partial fractions:

$$dr\left[\frac{A}{r} + \frac{Br+C}{a-r^2}\right] = dt$$

where $A(a-r^2) + Br^2 + Cr \equiv 1$

$\Rightarrow A = \frac{1}{a}$, $B = \frac{1}{a}$, $C = 0$

$$\Rightarrow \quad \frac{dr}{a}\left[\frac{1}{r} + \frac{r}{a-r^2}\right] = dt$$

$$\Rightarrow \quad \log r - \frac{1}{2}\log(a-r^2) = at + \log K \quad \text{(integrating)}$$

$$\Rightarrow \quad \frac{r}{\sqrt{a-r^2}} = Ce^{at}$$

$$= \frac{r_0}{\sqrt{a-r_0^2}}\, e^{at} \quad \text{(applying initial cond}^{n}\text{)}$$

Hence $r^2(a-r_0^2) = r_0^2(a-r^2)e^{2at}$

$$\Rightarrow \quad r^2 = \frac{ar_0^2 e^{2at}}{a-r_0^2 + r_0^2 e^{2at}} = \frac{ar_0^2}{r_0^2 + (a-r_0^2)e^{-2at}}$$

as required.

If $a > 0$, then as $t \to \infty$, $r \to \sqrt{a}$, with the origin an unstable fixed point. $\theta = \theta_0 - t \Rightarrow$ orbits wind clockwise round origin. If they start out within $r = \sqrt{a}$, they wind out to $r = \sqrt{a}$ as $t \to \infty$; if they start outside $r = \sqrt{a}$, they wind in to it.

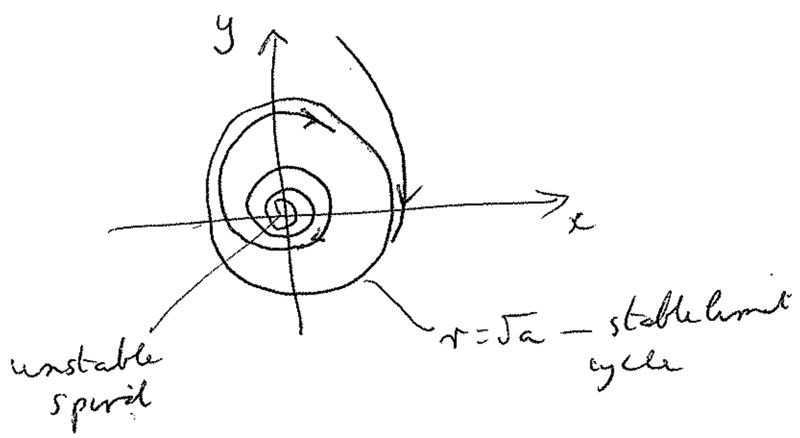
If $a < 0$, let $b = -a$, then

$$r^2 = \frac{b r_0^2}{-r_0^2 + (b + r_0^2)e^{2bt}}, \quad b > 0$$

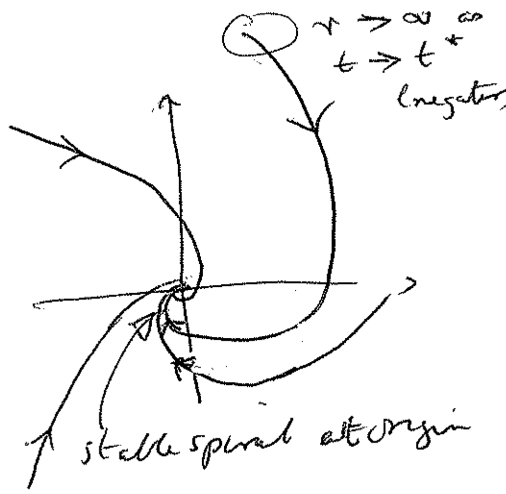
as $t \to \infty$, $r \to 0$
as $t \to -\infty$, the denominator can be zero:
i.e. $-r_0^2 + (b + r_0^2)e^{2bt} = 0$

so $\frac{1}{2b} \ln\left(\frac{r_0^2}{b + r_0^2}\right) = t^*$, & $t^* < 0$.

So solutions have $r \to \infty$, as $t \to t^*$ (from above) and $r \to 0$ (stable spiral) as $t \to \infty$ i.e. there is no limit cycle for $a < 0$.



($a > 0$)



($a < 0$)
(no limit cycle)