4(c) Alternative approach

We can write

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \left[\bar{A} R_n + \bar{B} I_n \right] \quad (*)$$

$$= (\sqrt{5})^n \left[\bar{A} \begin{pmatrix} \cos n\phi \\ \frac{\sqrt{5}\cos(n+1)\phi - \cos n\phi}{2} \end{pmatrix} + \bar{B} \begin{pmatrix} \sin n\phi \\ \frac{\sqrt{5}\sin(n+1)\phi - \sin n\phi}{2} \end{pmatrix} \right]$$

where R_n + I_n are the real and imaginary parts of one of the complex solutions.

This follows because we have complex conjugate eigenvalues (and eigenvectors) so the general solution can be written eg.

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (A+iB)(R_n + i I_n) + (C+iD)(R_n - i I_n)$$

$$= (A+C)R_n - (B+D)I_n + i\left((A+D)R_n + (B-C)I_n\right)$$

We know the iterates must be real, so choose $A=-D$, $B=C$

Then

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (A+B)R_n + (A-B)I_n$$

$$= \bar{A} R_n + \bar{B} I_n, \quad \text{as in } (*)$$

Now solving for the particular solution:

$$x_0 = 0 \implies \bar{A} = 0$$

$$y_0 = 1 \implies \bar{B}\,\frac{\sqrt{5}\sin\phi}{2} = 1 \quad \text{so} \quad \bar{B} = \frac{2}{\sqrt{5}\sin\phi}$$

+ thus

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \frac{(\sqrt{5})^{n-1}}{\sin\phi} \begin{pmatrix} 2\sin n\phi \\ \sqrt{5}\sin(n+1)\phi - \sin n\phi \end{pmatrix}, \text{ as before.}$$