Mathematical Computing and Nonlinear Systems MATH 2963/2063

Tutorial 6 Solutions

1. Assume the result is true for $p = P$.

Then
$$A^{P+1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} F_{2P-2} & F_{2P-1} \\ F_{2P-1} & F_{2P} \end{pmatrix}$$

$$= \begin{pmatrix} F_{2P-2} + F_{2P-1} & F_{2P-1} + F_{2P} \\ F_{2P-2} + 2F_{2P-1} & F_{2P-1} + 2F_{2P} \end{pmatrix}$$

$$= \begin{pmatrix} F_{2P} & F_{2P+1} \\ F_{2P} + F_{2P-1} & F_{2P+1} + F_{2P} \end{pmatrix}$$

on using the definition of the Fibonacci numbers.

$$= \begin{pmatrix} F_{2P} & F_{2P+1} \\ F_{2P+1} & F_{2P+2} \end{pmatrix}$$ on using it again.

$$= \begin{pmatrix} F_{2P} & F_{2(P+1)-1} \\ F_{2(P+1)-1} & F_{2(P+1)} \end{pmatrix}$$ which is the result obtained by substituting $P \to P+1$ in the original assumed formula.

$\therefore$ True for $(P+1)$ if true for P.

But when $P = 1$, the formula gives $\begin{pmatrix} F_0 & F_1 \\ F_1 & F_2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ which is correct.

So by induction the result is true for all p.

2. Fixed points (X, Y) satisfy

$$X = X^2 - 5X + Y \;, \quad Y = X^2$$

ie $X = 2X^2 - 5X$, ie $X = 0$ or $X = 3$.

So the fixed points are $(0,0)$ and $(3,9)$.

The Jacobian Matrix is $\begin{pmatrix} 2X-5 & 1 \\ 2X & 0 \end{pmatrix}$

with eigenvalues λ satisfying $(2X-5-\lambda)(-\lambda) - 2X = 0$

When $X=0$, this gives $\lambda(\lambda+5) = 0$

$\Rightarrow \lambda = 0$ or $\lambda = -5$.

$\lambda = 0$ is between -1 and 1, so stable, but $\lambda = -5$ is unstable. The associated eigenvectors are $\lambda = 0$: $-5x + y = 0 \Rightarrow \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ works

$\lambda = -5$: $y = 0 \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ works.

$\Rightarrow (0,0)$ is an unstable saddle, expanding along the direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and contracting along $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

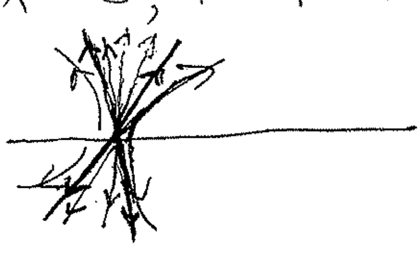
When $X=3$, $\lambda^2 - \lambda - 6 = 0$

$\Rightarrow (\lambda-3)(\lambda+2) = 0 \Rightarrow \lambda = 3$ or -2, both outside $-1 < \lambda < 1$ so both unstable.

$\lambda = 3$ has eigenvector with $-2x + y = 0$, $\Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ works

$\lambda = -2$ has eigenvector with $3x + y = 0 \Rightarrow \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ works.

$\Rightarrow X = 3, Y = 9$ is an unstable repeller.



3. (a) The Jacobian matrix is

$$\begin{pmatrix} 1 & 1 \\ a\cos(x_n+y_n) & 1+a\cos(x_n+y_n) \end{pmatrix}$$

with determinant $= 1 + a\cos(x_n+y_n) - a\cos(x_n+y_n) = 1$

so the mapping is indeed area-preserving.

(b) When $a = 0$, $x_{n+1} = x_n + y_n$

$y_{n+1} = y_n$

So $y_n = y_0$ for all n, and then

$x_n = x_0 + n y_0$; this is the explicit solution, to be evaluated mod 2π as specified.

If $y_0 = (p/q) \cdot 2\pi$ p, q integers, ie is a rational multiple of 2π, then after q iterations x_n will come back to x_0 when evaluated mod 2π, so there will be q-cycle. If y_0 is an irrational multiple of 2π, iterates will never return to x_0 and will countably fill $[0, 2\pi]$.

(c) Fixed points satisfy $X = X + Y$ mod 2π

$\Rightarrow Y = 0$ and $Y = Y + \sin(X+Y) \Rightarrow X = 0, \pi,$ 2π etc. The Jacobian eigenvalues satisfy

$(1-\lambda)(1 + a\cos(X+Y) - \lambda) - a\cos(X+Y) = 0$

$X=0, Y=0$: $(1-\lambda)(1+a-\lambda) - a = 0$

$\Rightarrow \lambda^2 - (2+a)\lambda + 1 = 0 \Rightarrow \lambda = \dfrac{2+a \pm \sqrt{(2+a)^2 - 4}}{2}$

$= \dfrac{2+a \pm \sqrt{a^2+4a}}{2}$ so for $a > 0$ one eigenvalue is greater than 1, and one is between 0 and 1. So this is an unstable saddle.

$X = \pi,\ Y = 0$:

Eigenvalues satisfy $(1-\lambda)(1-a-\lambda) + a = 0$

$$\Rightarrow \lambda^2 - (2-a)\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{2-a \pm \sqrt{(2-a)^2 - 4}}{2} = \frac{2-a \pm \sqrt{a^2-4a}}{2}$$

For $0 < a < 4$, these are complex, with modulus $\frac{\sqrt{(2-a)^2 + 4 - (2-a)^2}}{2} = 1$.

In this case, the solution is neutrally stable since $|\lambda| = 1$ for both eigenvalues.

For $a > 4$, the negative-square-root λ is less than -1, so this case is unstable (the fixed point is in fact then a saddle)

4. $$x_{n+1} = x_n \cos\alpha - (y_n - x_n^2)\sin\alpha$$
$$y_{n+1} = x_n \sin\alpha + (y_n - x_n^2)\cos\alpha$$

(a) The Jacobian is $\begin{pmatrix} \cos\alpha + 2x_n\sin\alpha & -\sin\alpha \\ \sin\alpha - 2x_n\cos\alpha & \cos\alpha \end{pmatrix}$

with determinant $\cos^2\alpha + 2x_n\sin\alpha\cos\alpha + \sin^2\alpha - 2x_n\sin\alpha\cos\alpha$

$= 1$. So the map is area-preserving.

(b) If $\alpha = 0$, $$x_{n+1} = x_n$$
$$y_{n+1} = y_n - x_n^2.$$

So $x_n = x_0$ for all n and

$y_n = y_0 - n x_0^2$.

If $\alpha = \pi$, $$x_{n+1} = -x_n$$
$$y_{n+1} = -(y_n - x_n^2)$$

$\Rightarrow$ $x_n = (-1)^n x_0$ for all n and

$y_1 = -(y_0 - x_0^2) = x_0^2 - y_0$

$y_2 = -(y_1 - x_1^2) = y_0$

etc etc

$\Rightarrow$ $x_n = (-1)^n x_0$, $y_n = \begin{cases} y_0 & (n \text{ even}) \\ x_0^2 - y_0 & (n \text{ odd}) \end{cases}$

(c) For the inverse map, we want x_n and y_n as a function of x_{n+1}, y_{n+1}. (6)

Multiply the first equation by $\cos\alpha$, the second by $\sin\alpha$, and add:

$$x_n = x_{n+1}\cos\alpha + y_{n+1}\sin\alpha$$

Multiply the second equation by $\cos\alpha$, the first by $\sin\alpha$, and subtract

$$\Rightarrow y_n - x_n^2 = y_{n+1}\cos\alpha - x_{n+1}\sin\alpha$$

$$\Rightarrow y_n = y_{n+1}\cos\alpha - x_{n+1}\sin\alpha + (x_{n+1}\cos\alpha + y_{n+1}\sin\alpha)^2$$

or, transferring the rôles of n and $n+1$ to get a forward map,

$$x_{n+1} = x_n\cos\alpha + y_n\sin\alpha$$

$$y_{n+1} = -x_n\sin\alpha + y_n\cos\alpha + (x_n\cos\alpha + y_n\sin\alpha)^2$$

is the inverse map $\underline{F}^{-1}$.

(d) Fixed points (X, Y) satisfy

$$X = X\cos\alpha - (Y - X^2)\sin\alpha$$

$$Y = X\sin\alpha + (Y - X^2)\cos\alpha$$

$$\Rightarrow X(1-\cos\alpha) = -(Y-X^2)\sin\alpha \qquad \times\cos\alpha$$

$$Y - X\sin\alpha = (Y - X^2)\cos\alpha \qquad \times\sin\alpha \text{ and add}$$

$$\Rightarrow X(\cos\alpha - \cos^2\alpha) + Y\sin\alpha - X\sin^2\alpha = 0$$

$$\Rightarrow Y = \frac{X(1-\cos\alpha)}{\sin\alpha} \qquad (\sin\alpha \neq 0, \text{ ie } \alpha \neq 0, \pi)$$

Let $t = \tan\frac{\alpha}{2}$; then $\cos\alpha = \frac{1-t^2}{1+t^2}$, $\sin\alpha = \frac{2t}{1+t^2}$

$\Rightarrow \quad Y = tX$. $\qquad 1-\cos\alpha = \frac{2t^2}{1+t^2}$

$\Rightarrow X(1-\cos\alpha) = -(tX - X^2)\sin\alpha$

$\Rightarrow X = 0$ or $X = \frac{1-\cos\alpha}{\sin\alpha} + t = 2t$.

So there are 2 solutions,
$X=0, Y=0$ and
$X = 2t, Y = 2t^2$, where $t = \tan\frac{\alpha}{2}$.

(e) The Jacobian matrix is

$$\begin{pmatrix} \cos\alpha + 2x\sin\alpha & -\sin\alpha \\ \sin\alpha - 2x\cos\alpha & \cos\alpha \end{pmatrix}$$

with eigenvalues satisfying

$$(\cos\alpha + 2x\sin\alpha - \lambda)(\cos\alpha - \lambda) + \sin^2\alpha - 2x\sin\alpha\cos\alpha = 0$$

ie $\lambda^2 - 2(\cos\alpha + x\sin\alpha)\lambda + 1 = 0$

The fixed point at the origin has $\lambda = \cos\alpha \pm \sqrt{\cos^2\alpha - 1}$
ie $\lambda = \cos\alpha \pm i\sin\alpha$: complex, with modulus 1. So the origin is neutrally stable (the numerical study in Lab 6 shows that in fact it is a feature called a <u>centre</u>.

The fixed point at $X = 2t$ has

$$\lambda^2 - 2\left(\frac{1-t^2}{1+t^2} + \frac{2t.2t}{1+t^2}\right)\lambda + 1 = 0$$

ie $\lambda^2 - 2\frac{(1+3t^2)}{1+t^2}\lambda + 1 = 0$.

$$\lambda = \frac{1+3t^2}{1+t^2} \pm \sqrt{\left\{\frac{1+3t^2}{1+t^2}\right\}^2 - 1} \;, = \frac{1+3t^2 \pm \sqrt{4t^2+8t^4}}{1+t^2}$$

For $t > 0$, the + root always has modulus greater than 1, so this fixed point is unstable (and in fact is hyperbolic).

(f) For a 2-cycle, $F^{-1}(x,y) = F(x,y)$, and this provides a concise way to find any that exist. Writing the fixed point as (X,Y), if it exists,

$$X\cos\alpha - (Y - X^2)\sin\alpha = X\cos\alpha + Y\sin\alpha$$

$$X\sin\alpha + (Y - X^2)\cos\alpha = -X\sin\alpha + Y\cos\alpha + (X\cos\alpha + Y\sin\alpha)^2.$$

The first of these says $Y = \frac{X^2}{2}$; the second then gives

$$X\sin\alpha - \tfrac{1}{2}X^2\cos\alpha = -X\sin\alpha + \frac{X^2}{2}\cos\alpha + X^2\cos^2\alpha + X^3\sin\alpha\cos\alpha + \frac{X^4}{4}\sin^2\alpha.$$

$X = 0$ and $X = 2\tan\frac{1}{2}\alpha$ have to be roots of this, since fixed points are also period 2 points.

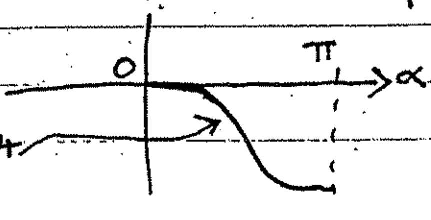
$$X^3\sin^2\alpha + 4X^2\sin\alpha\cos\alpha + 4X(\cos^2\alpha + \cos\alpha) - 8\sin\alpha = 0.$$

$X = 2\tan\frac{1}{2}\alpha = \frac{2\sin\alpha}{1+\cos\alpha}$ has to be a factor of this; in fact we see that it is equivalent to

$$[X(1+\cos\alpha) - 2\sin\alpha][X^2(1-\cos\alpha) + 2X\sin\alpha + 4] = 0.$$

So the specifically period-2 points satisfy $X^2(1-\cos\alpha) + 2X\sin\alpha + 4 = 0$ with solutions $X = \frac{-\sin\alpha \pm \sqrt{\sin^2\alpha + 4\cos\alpha - 4}}{2(1-\cos\alpha)}$

A plot of the surd shows there are no real solutions of this for $\alpha \neq 0$, and hence that there are no 2-cycles.



$f(\alpha) = \sin^2\alpha + 4\cos\alpha - 4 \leq 0.$

5. Let $\left(\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right)$ be any rational pair of (x, y);

This can be written $\left(\frac{a}{c}, \frac{b}{c}\right)$ where $c = \beta\delta$, $a = \alpha\delta$, $b = \gamma\beta$.

If the mapping is F, $F^n\left(\frac{a}{c}, \frac{b}{c}\right) = \left(\frac{g}{c}, \frac{h}{c}\right) \pmod 1$ for some integers g, h, just from the definition of the map. But there are only c^2 distinct possibilities for (g,h) before the mapping gets rounded back to some earlier value by the mod (1). Thus when n is greater than at most c^2, the mapping starts to repeat. Since the mapping is invertible (each point has one and only one pre-image), this means the original point $\left(\frac{a}{c}, \frac{b}{c}\right)$ must also have repeated. So any rational pair is a fixed point of the p-iterated map, for some integer p.

If (X, Y) is a fixed point of the p-iterated map, $nX + mY = X + k$, $qX + rY = Y + l$, for some integers n, m, q, r, k, l (again, this follows from the specification of the map), so X and Y are in fact rational. □

To show that the fixed points of the p-iterated mapping are unstable, let $\left(\frac{a}{c}, \frac{b}{c}\right)$ be the corresponding rational pair. Let A be the Jacobian matrix; the eigenvector corresponding to the unstable eigenvalue $\frac{3+\sqrt{5}}{2}$ is $\left[1, \frac{1+\sqrt{5}}{2}\right]$. Take some $\delta > 0$ and let r be some sufficiently large integer that $P \equiv \left(\frac{a}{c}, \frac{b}{c}\right) + \frac{1}{r}\left(1, \frac{1+\sqrt{5}}{2}\right)$ is in the δ-neighbourhood of $\left(\frac{a}{c}, \frac{b}{c}\right)$.

$A^n(P) = A^n\left(\frac{a}{c}, \frac{b}{c}\right) + \left(\frac{3+\sqrt{5}}{2}\right)^n \cdot \frac{1}{r}\left(1, \frac{1+\sqrt{5}}{2}\right)$, because $\left(1, \frac{1+\sqrt{5}}{2}\right)$ is the eigenvector. For sufficiently large n (typically such that $\left(\frac{3+\sqrt{5}}{2}\right)^n \sim r$, ie $n \sim \log r / \log\left(\frac{3+\sqrt{5}}{2}\right)$), the points $\left(\frac{a}{c}, \frac{b}{c}\right)$ and P will have diverged by an order 1 amount; for any δ there is such an n which demonstrates instability