

The University of Sydney
SCHOOL OF MATHEMATICS & STATISTICS

MATHEMATICAL COMPUTING & NONLINEAR SYSTEMS
MATH2063/MATH2963 (NORMAL/ADVANCED)

**PLEASE FILL IN THE DETAILS BELOW AND RETURN THIS EXAMINATION PAPER
WITH YOUR ANSWER BOOK(S).**

Name:

SID:

Seat Number:

Time Allowed: 2 hours

Lecturer: *C. Macaskill*

All questions may be attempted by any student (whether enrolled in
2063 or 2963).

Advanced questions are starred.

University-approved non-programmable calculators may be used.

No books or notes are allowed in the exam.

THIS EXAMINATION PAPER CONSISTS OF 8 QUESTIONS AND IS WORTH 70 MARKS.

**For the various parts of questions 1 and 2, fill in just one box, corresponding to the
statement which is always true.**

1. *8 marks*

Multiple choice

2. *8 marks*

Multiple choice

For the various parts of questions 3, 4 and 5 all answers should be included in the relevant spaces on this examination paper.

3. 8 marks

Consider the Duffing equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y + \varepsilon y^3 = \cos t$$

with initial conditions $y(0) = 2$, $dy/dt(0) = 1$, and ε a parameter.

- (a) Write the Duffing equation above as a system of two first order equations with associated initial conditions.

- (b) Complete the MATLAB codes, outlined below, for the integration of the Duffing equation, and for plotting of the phase portrait, using the initial conditions given above. (Dots indicate missing code.)

duffing.m

```
%
% program to integrate the Duffing equation
% d^2y/dt^2 + dy/dt + y + epsilon y^3 = cos t
%

global .....

yin=[ ... ; ... ];

epsilon=input('enter epsilon: ');
tspan=0:0.01:100;

[t,yout]=ode45( ..... , ..... , ..... );

y1=yout( ... , ... );

y2=yout( ... , ... );

figure(1)
plot( ... , ... )
```

```

set(gca,'fontsize',14)
xlabel('....')

ylabel('....')

title('phase plane: Duffing equation')

```

duffing_rhs.m

```

function ... = duffing_rhs( ... , ... )

% function returns the rhs function
% for the Duffing equation.

global .....

f(1)= ..... ;

f(2)=-..... -..... -epsilon* ..... + cos(t);

f=f';

```

4. 9 marks

Consider the exponential map

$$x_{n+1} = f(x_n) = ax_n e^{-x_n}, \quad x_n \geq 0, \quad a > 0.$$

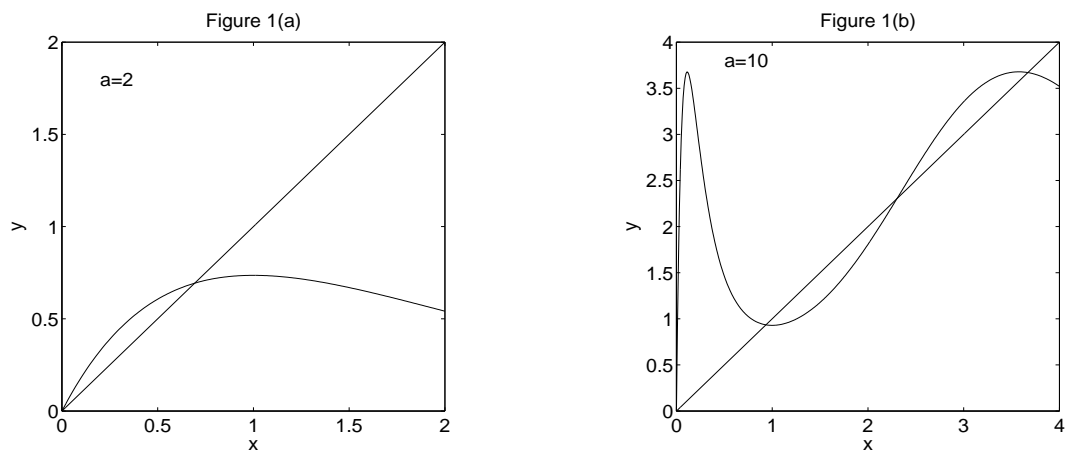


Figure 1: Plots for the exponential map

- (a) Show that there are two fixed points for the exponential map and demonstrate that one of these is stable for $a < 1$, while the other is stable for $1 < a < e^2$. On figure 1(a),

which shows plots of $y = x$ and $y = f(x)$ with $a = 2$, use cobweb plots to illustrate this behaviour.

- (b) Consider the iterated map $f(f(x_n))$. Show that fixed points X for the iterated map are roots of the equation

$$a^2 X e^{-X(1+ae^{-X})} = X. \quad (1)$$

Show that the two fixed points found in the previous part are both solutions of equation (1). Figure 1(b) shows plots of $y = x$ and of $y = f(f(x))$ for the case $a = 10$. Identify the two fixed points of the exponential map $x_{n+1} = f(x_n)$ on this plot, as points A and B .

- (c) Show that the non-zero fixed point in part (a) loses stability in an oscillatory fashion at $a = e^2$.

You are given that there are two further solutions X of equation (1) in the previous part (b) for the case $a = 10$. Identify these points on figure 1(b) as points C and D and explain the significance of these points for the original exponential map $f(x_n)$. Use a cobweb plot on figure 1(b) to demonstrate the behaviour of the doubly-iterated map $f(f(x_n))$ and explain your observations.

5. 10 marks

Consider the epidemic equations as treated in lectures:

$$\begin{aligned}\frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI - \gamma I \\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

where the constants β and γ are positive, and the initial conditions are $S(0) = S_0 > 0$, $I(0) = I_0 > 0$ and $R(0) = R_0 = 0$.

(a) Explain what S, I and R represent and show that $S(t) + I(t) + R(t) = N$ where $N =$ constant.

(b) Find an expression for dS/dR and hence show that $S = S_0 \exp(-\beta R(t)/\gamma)$.

(c) Show that R satisfies the first order equation

$$\frac{dR}{dt} = \gamma [N - R - S_0 \exp(-\beta R(t)/\gamma)].$$

(d) * Show that the equation in the previous part (c) can be non-dimensionalised to

$$\frac{du}{d\tau} = a - bu - e^{-u}$$

by choosing suitable definitions for u and τ . Show that $a > 1$ and $b > 0$.

(e) * Show that there is one fixed point, say $u = u^*$, for the ODE given in the previous part (d). Give a finite interval within which the fixed point lies and show that for large b , the fixed point is approximately given by

$$u^* \approx (a - 1)/(b - 1).$$

The following questions should be answered in one or more standard answer books.

6. 9 marks

Solve the following pair of first-order difference equations, using the matrix method. Give the Jacobian matrix, find its eigenvalues and eigenvectors, give the general solution, and find the particular solution for the initial conditions $x_0 = 2, y_0 = 1$.

$$\begin{aligned}x_{n+1} &= 3x_n + 2y_n \\y_{n+1} &= x_n + 4y_n\end{aligned}$$

7. 9 marks

Consider the system defined by the equations

$$\dot{x} = -1 + 4y^2, \quad \dot{y} = -x + y.$$

- Show that the system has two fixed points, and give their locations.
- Investigate the linear stability of each fixed point, and give its classification (as a stable or unstable node, saddle, stable or unstable focus, or centre). Calculate the corresponding eigenvectors for any fixed point with real eigenvalues.
- Find the isoclines where $\dot{y} = 0$, determine the behaviour on the x - and y -axes and hence sketch the phase diagram of the system, showing the direction of motion along each trajectory.

8. * 9 marks

The third-order Rössler system is governed by the three equations

$$\begin{aligned}\frac{dx}{dt} &= -y - z \\ \frac{dy}{dt} &= x + ay \\ \frac{dz}{dt} &= b + z(x - c)\end{aligned}$$

where a, b, c are real parameters. In this question we consider only the special case $b = 0$.

- Find the two fixed points of the system, and show that they are given by

$$(x, y, z) = (0, 0, 0) \quad \text{and} \quad (x, y, z) = (c, -c/a, c/a).$$

- Determine the Jacobian matrix for the Rössler system and hence show that the eigenvalues s satisfy the cubic equations

$$s^3 + (c - a)s^2 + (1 - ac)s + c = 0$$

at $(0, 0, 0)$ and

$$s^3 - as^2 + (1 + \frac{c}{a})s - c = 0.$$

at $(c, -c/a, c/a)$.

- Consider the special case $c > 0$. Determine the eigenvalues at the fixed point $(0, 0, 0)$, and hence find the values of a for which this fixed point is stable. In this case there is a simple root of the cubic which can be found by inspection.
- Now consider $c > 0$ and small, so that $c = c_1\varepsilon$, where $0 < \varepsilon \ll 1$ and $c_1 > 0$. By assuming that one eigenvalue is of the form $s_1 = \alpha\varepsilon$, where α is to be determined, find approximate values for all three eigenvalues at the fixed point $(c, -c/a, c/a)$, and determine the stability of this fixed point in this case.