

Quiz 2: practice questions

The quiz will be worth 5% of the total mark for the unit of study: it will be held in the lecture on Wednesday 27th May, 2009. The number of questions may vary from that shown here. Material from pages 42–71 of the printed notes will be covered (i.e. tutorials 6–9), omitting pp 57–62 on Julia and Mandelbrot sets. There will be no questions on MATLAB or numerical solution of ODEs. Questions will be short answer.

1. Find all fixed points of the mapping

$$\begin{aligned}x_{n+1} &= x_n^3 - 3y_n + 3x_n \\y_{n+1} &= x_n^2 + 2y_n\end{aligned}$$

2. For each fixed point in the previous question, find the eigenvalues. Determine whether each fixed point is a sink, a source or a saddle point.
3. Find the fixed points of period one for the Hénon map given by

$$x_{n+1} = \frac{3}{50} + \frac{9}{10}y_n - x_n^2, \quad y_{n+1} = x_n.$$

[Question 2.7.7, *Dynamical Systems with Applications using MATLAB*, by Stephen Lynch.]

4. An inflation-unemployment model is given by

$$u_{n+1} = u_n - b(m - i_n), \quad i_{n+1} = i_n - (1 - c)f(u_n) + f(u_n - b(m - i_n)),$$

where $f(u) = \beta_1 + \beta_2 e^{-u}$, u_n and i_n are measures of unemployment and inflation at time n , respectively and b, c, β_1, β_2 and m are constants. Show that there is a unique fixed point at

$$(U, I) = \left(\ln\left(\frac{-\beta_2}{\beta_1}\right), m \right),$$

and that the eigenvalues of the Jacobian matrix at the fixed point are given by

$$\lambda_{1,2} = 1 - \frac{b\beta_1}{2} \pm \frac{\sqrt{\beta_1^2 b^2 + 4\beta_1 bc}}{2}.$$

[Question 2.7.10, *Dynamical Systems with Applications using MATLAB*, by Stephen Lynch, referring to the model of E. Ahmed, A. El-Misiery and H.N. Agiza, On controlling chaos in an inflation-unemployment dynamical system, *Chaos Solitons Fractals*, **10**–9 (1999), 1567–1570.]

5. For the following linear ODEs, classify the fixed point at the origin by determining the eigenvalues and eigenvectors of the Jacobian matrix:

$$\dot{x} = 2x + y, \quad \dot{y} = x + 2y.$$

Sketch the phase portrait in the (x, y) plane.

6. Show that the origin is the only fixed point for the following system and that it is a stable focus. Find the behaviour on the isoclines given by $\dot{x} = 0$ (where $dy/dx \rightarrow \infty$) and $\dot{y} = 0$ (where $dy/dx = 0$) and hence sketch the phase portrait.

$$\dot{x} = -x - y, \quad \dot{y} = x - y.$$

7. Rewrite the following ODE and the associated initial conditions as a system of three first order ODEs and appropriate initial conditions:

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + \cos\left(y \frac{dy}{dt}\right) = \exp(y^2 + t - 1), \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) + y'(0) = 3.$$

8. Consider the system

$$\dot{x} = x \left(1 - \frac{x}{2} - y\right), \quad \dot{y} = y \left(x - 1 - \frac{y}{2}\right).$$

Show that there are four equilibrium points and that $(0, 0)$ and $(2, 0)$ are saddle points, $(0, -2)$ is an unstable node and $(\frac{6}{5}, \frac{2}{5})$ is a stable focus.

9. Consider the system

$$\dot{x} = y, \quad \dot{y} = x(1 - x^2) + y.$$

Show that there are three fixed points, classify them and sketch the phase portrait. You will need to consider the special cases $\dot{x} = 0$, $\dot{y} = 0$ and $x = 0$. (If you have trouble sketching the phase portrait, use the MATLAB routines provided in laboratory 9 to check your answer.)

10. Tutorial 6, Q2; Tutorial 8, Q1, Q2; Tutorial 9, Q1, Q2.

Answers

- 1, 2. Fixed points: $(0, 0)$ (source or repeller), $(-1, -1)$ (saddle point), $(-2, -4)$ (source).
3. Fixed points: $(X, Y) = (-\frac{3}{10}, -\frac{3}{10}), (-\frac{1}{5}, -\frac{1}{5})$.
5. Eigenvalues $\lambda_1 = 1, \lambda_2 = 3$ (unstable node) with corresponding eigenvectors $(1, -1)$ and $(1, 1)$. Trajectories align with the direction $(1, 1)$ as $t \rightarrow \infty$.
6. Anti-clockwise spiral.
7. $\dot{y}_1 = y_2, \dot{y}_2 = y_3, \dot{y}_3 = y_3 - \cos(y_1 y_2) + \exp(y_1^2 + t - 1)$ with initial conditions $y_1(0) = 1, y_2(0) = 2, y_3(0) = 1$.
9. $(0, 0)$ is a saddle point and $(1, 0), (-1, 0)$ are unstable foci (with trajectories going clockwise).