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# Abstract evolution equations, periodic problems and applications

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# Contents

# Introduction

Parabolic equations are one of the types of partial differential equations which have important applications to the natural sciences – e.g. in biomathematics or chemical combustion theory – and have long since enjoyed great popularity among researchers in pure and applied mathematics. Within this class of equations, periodic equations seem to be of particular interest since they can take into account seasonal fluctuations occurring in the phenomena they are modelling. The interest in this kind of problems is reflected, for instance, by the recent publication of the monograph [**67**]. Concrete examples are provided by periodic Volterra-Lotka population models with diffusion or the periodic Fisher equation of population genetics (see e.g. [**67**]).

A very simple example of a one-dimensional periodic-parabolic equation is the following initial-boundary value problem:

$$(1)\qquad \begin{cases} \partial_t u(t,x) - k(t,x)\partial_x^2 u(t,x) = f\big(t,u(t,x)\big) & \text{for } (t,x)\in(0,\infty)\times(0,1) \\ u(t,0)=u(t,1)=0 & \text{for } t\in(0,\infty) \\ u(0,x)=u_0(x) & \text{for } x\in[0,1] \end{cases}$$

where $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $k\colon \mathbb{R}\times[0,1]\to\mathbb{R}$, are smooth functions, $T$-periodic in the first argument, $k$ additionally being strictly positive on $\mathbb{R}\times[0,1]$. The *initial value* $u_0$ is a given function from the unit interval into $\mathbb{R}$.

A *local classical solution* is a function

$$u\in C\big([0,\varepsilon)\times[0,1],\mathbb{R}\big)\cap C^{1,2}\big((0,\varepsilon)\times[0,1],\mathbb{R}\big),$$

for some $\varepsilon>0$, which satisfies (1). The solution is called *global* if we can choose $\varepsilon=\infty$. A $T$*-periodic solution* is a global solution which is $T$-periodic in $t\in\mathbb{R}_+$.

Considering the fact that such an equation stems from the desire to model a real-world situation, the following requirements may sound reasonable:

1) Existence of unique solutions for a large class of initial values.
2) Continuous dependence of the solution on the initial data.
3) Globality of the solution.

4) Positivity of the solution whenever the initial value is positive.

The next step after establishing the above properties, is the study of qualitative behaviour of the model:

5) Do $T$-periodic solutions exist? (especially positive ones).
6) What are the stability and attractivity properties of such solutions? (ideally something like: all solutions having positive initial values converge towards a positive $T$-periodic solution as time approaches infinity).

For a long time one of the most successful methods for attacking this kind of problems has been the abstract formulation of (1) as an 'ordinary' differential equation in a suitable Banach space of functions. In order to convey the basic idea we consider equation (1), dispensing with the periodicity assumptions:

Let $u\colon [0,\varepsilon) \times [0,1] \to \mathbb{R}$ be a classical solution of (1), and set for each $t \in [0,\varepsilon)$: $v(t) := u(t,\cdot) \in C^2\big([0,1],\mathbb{R}\big)$. Then $v$ is a continuously differentiable function from $(0,\varepsilon)$ into $C\big([0,1],\mathbb{R}\big)$. Setting:

$X_0 := \{u \in C\big([0,1],\mathbb{R}\big);\ u(0) = u(1) = 0\}$ equipped with the supremum norm
$D(A(t)) := X_1 := \big\{w \in C^2\big([0,1],\mathbb{R}\big);\ w(0) = w(1) = 0\big\}$,
$A(t)w := -k(t,\cdot)\partial_x^2 w$ for $w \in X_1$,
and $F(t,w) := f\big(t,w(\cdot)\big)$ for $w \in X_0$,

we see that $v$ satisfies the *abstract initial value problem*:

$$\left\{\begin{aligned} \partial_t v(t) + A(t)v(t) &= F\big(t,v(t)\big) \qquad \text{for } t > 0 \\ v(0) &= u_0, \end{aligned}\right. \tag{2}$$

on the Banach space $X_0$.

The fundamental properties of the operator family $\big(A(t)\big)_{0\le t\le T}$ can be summarized by saying: for every $t \in [0,T]$, the operator $-A(t)$ is the generator of a strongly continuous analytic semigroup of bounded operators on the Banach space $X_0$. Furthermore, the nonlinearity $F$ is a smooth function from $X_0$ into itself. For this kind of equation one can develop a theory which in many ways resembles the theory for ordinary differential equations.

It should be emphasized that the choice of the underlying Banach space, in our case $X_0 = C\big([0,1],\mathbb{R}\big)$, is by no means unique. For example, one could equally well choose to work in $X_0 = L_p\big((0,1)\big)$, $p \ge 1$. The choice of the Banach space should be adjusted to the particular pecularities of the problem under consideration.

The strategy, then, is to prove theorems for the abstract equation, and, by means of regularity results, translate them back into the context of classical solutions of (1).

One of the difficulties, when dealing with semilinear equations, which is not exemplified by (1), is the fact that the nonlinear term $F$ may not have the degree of regularity needed for the abstract existence theory or, worse, may not even be defined as a function on the whole of $X_0$. This would be the case, for instance, if we allow the nonlinearity $f$ in (1) to depend on $\partial_x u$. In this situation one would have to consider $F$ as a function from $C^1\big([0,1],\mathbb{R}\big)$ into $C\big([0,1],\mathbb{R}\big)$. In general it might be necessary to view $F$ as a function from $Z$ into $X_0$, where $Z$ is a Banach space lying between $X_1$ and $X_0$. We are thus confronted with the following question: which spaces lying between $X_1$ and $X_0$ provide the appropriate setting for our problem?

There are basically two approaches known which furnish suitable intermediate spaces:

1) Fractional power spaces associated to the operator A(0), and
2) Interpolation spaces.

Fractional power spaces are by far better known than interpolation spaces. This is the type of space used in the books [**66**], [**100**] and [**67**], to name just a few. While interpolation theory is certainly familiar to specialists working in existence theory, it is virtually unknown to mathematicians devoted to the qualitative theory of partial differential equations. This is in some respects a regrettable situation, as interpolation theory makes it possible to develop a much more elegant theory of semilinear equations and is capable of extension to problems having a much greater degree of generality (e.g. initial-boundary value problems where the boundary operator also depends on time and problems involving nonlinear boundary conditions). Another advantage is that while the definition of the fractional power spaces depends heavily on the operator $A(0)$, the corresponding interpolation spaces depend (up to equivalent norms) only on the norm isomorphic class of $X_1$ – equipped with the graph norm induced by $A(0)$ – and not on any other properties of $A(0)$.

By now the reader may have guessed what the intention of these notes is. We have set ourselves the task of giving a hopefully clear and essentially self-contained account of the theory for abstract evolution equations of the type (2), when the family $\big(A(t)\big)_{0\le t\le T}$ consists of operators having domains of definition, $D\big(A(t)\big)$, independent of $t \in [0,T]$, and the nonlinearity is defined on an interpolation space. Although these results are by no means restricted to periodic evolution equations we devote quite a deal of space to this type of problem. In order to keep the length of this volume within reasonable bounds, we have chosen to provide the basis for the qualitative analysis of parabolic equations rather than to actually carry out this analysis in concrete instances. In order to compensate this ommision we have tried to supply references where specific equations are considered.

In Section 0 we have collected a few basic functional analytic facts for easy reference.

We have also seized the opportunity to introduce some general notation which shall be used throughout the book.

In Chapter I we develop the fundamentals of the linear theory. After a brief review of the theory of analytic $C_0$-semigroups we introduce the notion of the evolution operator associated to a linear nonautonomous evolution equation and prove an existence theorem for solutions of such equations. The proof of the existence of the evolution operator is only sketched since it is very technical and easy to find in the existing literature. A quick introduction to interpolation theory from the user's perspective is given and – in order to convince the reader that interpolation spaces actually do exist – the most frequently used interpolation methods are described. Section 5 constitutes the core of the whole theory. There we prove the basic estimates for the evolution operator which will enable us to treat semilinear problems in Chapter IV.

Chapter II deals with linear periodic equations. We start by introducing the period-map associated to a linear periodic evolution equation and proceed to give some estimates for it which are related to the stability properties of the zero solution of the homogeneous equation. We also spend some time on proving estimates for the period-map involving spectral decompositions. They enable a more differentiated analysis of the asymptotics of solutions of the homogeneous equation. Finally, a theorem of 'Floquet-type' is proved which allows, for instance, to decouple the unstable part of a linear evolution equation.

Chapter III is a collection of results which are either of a very technical nature (and are therefore condemned to isolation), or are of a somewhat more specialized nature (and thus also condemned to isolation). It begins with the description of a general technique for solving Volterra integral equations. This is the technique with which the evolution operator is actually constructed. We prove that an evolution equation with unbounded principal part $A(t)$ may be approximated (in an appropriate sense) by a sequence of evolution equations whose principal parts are the Yosida-approximations of $A(t)$, and hence bounded. The question of how the evolution operator depends on a parameter is also addressed. Maximum principles for second order parabolic equations are described. They imply that the evolution operator associated with the abstract formulation of such problems is a positive irreducible operator. The last section in this chapter deals with superconvexity. This rather odd looking concept turns out to be quite useful when studying the linear stability properties of periodic-solutions to semilinear parabolic equations.

In Chapter IV we start the investigation of semilinear evolution equations. The concept of a mild solutions of (1) is defined and it is shown that – with an additional regularity condition on the nonlinearity – every mild solution is also a classical solution. Of course, we also give theorems on the existence of mild solutions and on their continuous dependence on the initial data. Since they are rather difficult to localize in the existing literature we have included some results on Nemitskii-operators on spaces of Hölder-continuous functions which are necessary when translating a parabolic equation into the language of abstract evolution equations. We establish a simple result on globality of solutions and give applications to the question when $L_\infty$-a priori bounds for solutions of parabolic equations imply their globality. Finally, in Section 18, we prove some results on continuous and differentiable parameter dependence of solutions to parameter dependent semilinear evolution equations.

Chapter V deals with time-periodic semilinear evolution equations. We review, as a motivation, some results from the qualitative theory for semilinear autonomous equations. We then proceed to introduce the class of time-periodic equations we shall be interested in, define the concept of the period-map for such problems, and establish the equivalence between fixed-point of this map and periodic solutions of the original equation. After defining the basic concepts from stability theory we prove the equivalence between the Ljapunov stability of a periodic solution and its stability as a fixed-point of the period-map. We also prove the principles of linearized stability and linearized instability. We base our proof on stability results for fixed-points of mappings. We conclude this chapter with some results on when stability established in a weak norm implies stability with respect to a stronger norm.

In the last chapter we show how the abstract theory may be applied to concrete equations arising from the applied sciences. Here we consider semilinear parabolic initial-boundary value problems on bounded subdomains of $\mathbb{R}^n$ and semilinear parabolic initial value problems on the whole of $\mathbb{R}^n$. In the last section we describe how to treat a model from epidemiology, for which it is not immediately clear that it fits in the parabolic context.

We assume a working knowledge in functional analysis, calculus in Banach spaces and semigroup theory. Furthermore, we will use some of the well established theory on partial differential equations in the applied chapters. We have striven to provide precise references for the material which we only quote, whenever we feel it is not standard.

Most of our results are probably known to the specialists and though they often seem optimal for the range they cover, we have not aimed at greatest generality. To our knowledge there is no presentation of this kind of theory on this level and we hope to render the more general literature (see the references throughout the text) accessible to the reader interested in applications. It is also our intention that these notes provide the abstract setting for the theory contained in [**67**]. Most of the inspiration and information for this work were drawn from the lectures and papers by H. Amann, for the abstract chapters, and by P. Hess, for the applied ones. During all our years in Zürich we were able to profit from their vast knowledge. To both of them we express our gratitude. As is probably true for every book the fact that it finally appears in published form is the product of many circumstances and the merit not only of the authors. Various people, friends and colleagues, have made at different stages and in different ways a contribution to this work. Thanking them here is more than a mere perfunctory act. We are specially indebted to: F. Baur, J. Escher, D. Guidetti, U. Hassler, M. Hieber, A. Johnston, G. Schätti, G. Simonett and A. Wyler.

Finally, we mention that the camera ready copy of this text was typset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX.

# 0. General notation

We start by fixing the notation to which we shall adhere in this notes and recalling some basic functional analytic facts which shall be frequently used. The reader is advised to skim this section to become acquainted with our terminology and use it as a reference whenever needed. For the notation on function spaces we refer to the appendix.

**A. Sets:** We shall use standard set theoretic notation without further comment. If $X$ and $Y$ are sets we denote by $Y^X$ the set of all mappings from $X$ into $Y$. The terms map, mapping and function shall be used synonymously and randomly. A mapping is called *one-to-one* if it is injective and *onto* if it is surjective.

Let $X$ and $Y$ be arbitrary sets and $f : X \to Y$ a mapping. The *graph* of $f$ is defined as

$$\operatorname{graph}(f) := \{(x, f(x)); x \in X\} \subset X \times Y.$$

Suppose now that $X_1 \subset X$ and $Y_1 \subset Y$ are given, and that $f(X_1) \subset Y_1$. Then, $f$ induces in an obvious way a mapping from $X_1$ into $Y_1$ which, by abuse of notation, shall be denoted by the same symbol $f$. Thus, if $\mathcal{F}_1(X, Y)$ and $\mathcal{F}_2(X_1, Y_1)$ are classes of $Y$ and $Y_1$-valued functions defined on $X$ and $X_1$, respectively, we shall write

$$\mathcal{F}_1(X, Y) \cap \mathcal{F}_2(X_1, Y_1)$$

for the class of $Y$-valued functions $f$ in $\mathcal{F}_1(X, Y)$ such that $[x \mapsto f(x)] \in \mathcal{F}_2(X_1, Y_1)$.

The sets of all natural, integer, rational, real and complex numbers shall be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. We set $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$. It is clear how to define $\mathbb{Q}^*$, $\mathbb{R}^*$ and $\mathbb{C}^*$. Sometimes the symbol $\mathbb{K}$ will be used to denote a fixed choice of either one of the fields $\mathbb{R}$ or $\mathbb{C}$. If $\lambda$ is a complex number we write $\operatorname{Re}(\lambda)$ for its real part, $\operatorname{Im}(\lambda)$ for its imaginary part and $\bar{\lambda}$ for its conjugate.

If $P$ is a property which a complex number may enjoy or not we write $[P(\lambda)]$ for the set of all complex numbers satisfying that property, e.g.

$$[\operatorname{Re}\lambda \leq 0] = \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \leq 0\}.$$

If $n$ is an integer and $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are elements of $\mathbb{K}^n$ we set

$$(x|y) := \sum_{k=1}^{n} x_k \bar{y}_k$$

and

$$|x| := \sqrt{(x|x)}.$$

Thus, $(\cdot\,|\,\cdot)$ and $|\cdot|$ denote the euclidean inner product and the euclidean norm on $\mathbb{K}^n$, respectively. Finally we shall use the notation

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n;\ x_i \geq 0\ i = 1, 2, \dots, n\}.$$

If above $n = 1$ we shall just write $\mathbb{R}_+$.

**B. Topological spaces:** Suppose $X$ is a topological space and let $A$ be a subset of $X$. We shall often think of $A$ as a topological space by its own right, endowing it with the relative topology. We write

$$\mathring{A}, \quad \bar{A}, \quad \partial A$$

to mean the *interior*, the *closure* and the *boundary* of $A$ in $X$, respectively. If the choice of the reference space $X$ is not obvious from the context we shall also write $\mathrm{int}_X(A)$, $\mathrm{cl}_X(A)$ and $\partial_X(A)$ for the above sets. The subset $A$ is said to be *relatively compact in* $X$ if its closure $\bar{A}$ is a compact subset of $X$.

If $Y$ is a further topological space we denote the set of all continuous functions from $X$ into $Y$ by

$$C(X, Y).$$

If $X$ is a metric space with metric $d$ we set for any $x_0 \in X$ and $\varepsilon > 0$

$$\mathbb{B}_X(x_0, \varepsilon) := \{x \in X;\ d(x_0, x) < \varepsilon\}.$$

The open set $\mathbb{B}_X(x_0, \varepsilon)$ is called the *open ball centered at* $x_0$ *with radius* $\varepsilon$. The *closed ball* is defined similarly substituting '$<$' by '$\leq$'. We usually write $\mathbb{B}_X$ instead of $\mathbb{B}_X(0, 1)$, and just $\mathbb{B}^n$ whenever $X = \mathbb{R}^n$, $n \geq 1$.

**C. Linear operators:** Let now $\mathbb{K}$ be either one of the fields $\mathbb{R}$ or $\mathbb{C}$. Suppose that $X$ and $Y$ are linear metric spaces (mostly they will be Banach spaces). The linear space of all continuous linear operators from $X$ into $Y$ shall be denoted by $\mathcal{L}(X, Y)$. We shall write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$.

A subset $B$ of a linear metric space $Z$ is called *topologically bounded* (or just *bounded*) if it is absorbed by any zero-neighbourhood, i.e. if to each zero-neighbourhood $U$ there exists a scalar $\lambda$ such that $B \subset \lambda U$. Observe that in general this is not equivalent to $B$ being bounded with respect to the metric in $Z$, i.e. $\sup_{x\in B} d(0, x) < \infty$. But if $Z$ is a normed space then the boundedness of $B$ is equivalent to its *norm boundedness*, i.e. $\sup_{x\in B} \|x\| < \infty$.

Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on the vector space $Z$. Recall that $\|\cdot\|_1$ is said to be *weaker* than $\|\cdot\|_2$ if $\|x\|_1 \leq c\|x\|_2$ holds for some constant $c > 0$ and every $x \in Z$. In this case $\|\cdot\|_2$ is said to be *stronger* than $\|\cdot\|_1$. These norms are said to be *equivalent* if $\|\cdot\|_2$ is both stronger and weaker than $\|\cdot\|_1$. In this case they generate the same topology on $Z$.

A linear operator $T: X \to Y$ is called *bounded* if it maps bounded subsets of $X$ into bounded subsets of $Y$. Then $T$ is bounded if and only if $T$ is continuous. Because of this we shall use terms bounded and continuous linear operators synonymously.

A linear operator $T \in \mathcal{L}(X,Y)$ is called *invertible* if it is bijective and its inverse $T^{-1}: Y \to X$ is bounded. If $X$ and $Y$ are *Fréchet spaces*, i.e. complete linear metric locally convex spaces, the open mapping theorem implies that $T \in \mathcal{L}(X,Y)$ is invertible if and only if it is bijective. We denote the set of all invertible bounded operators from $X$ to $Y$ by $\mathrm{Isom}(X,Y)$ and set $\mathcal{GL}(X) := \mathrm{Isom}(X,X)$. Note that $\mathrm{Isom}(X,Y)$ is not a subspace of $\mathcal{L}(X,Y)$ but only an open subset.

An operator $T \in \mathcal{L}(X,Y)$ is said to be *compact* if it maps bounded subsets of $X$ into relatively compact subsets of $Y$. We denote the subspace of $\mathcal{L}(X,Y)$ consisting of all compact operators by $\mathcal{K}(X,Y)$ and put $\mathcal{K}(X) := \mathcal{K}(X,X)$. Consider a third linear metric space $Z$. If $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$ then $ST := S \circ T \in \mathcal{K}(X,Z)$ whenever $T$ or $S$ is compact. This shall be referred to as the *stability property* of compact operators.

We shall often deal with linear operators $A$ taking values in $Y$ which are not defined on the whole space $X$ but only on a subspace $D(A)$ of $X$, called the *domain of definition* of $A$:

$$A: X \supset D(A) \to Y.$$

Such an operator is called *closed* if $\mathrm{graph}(A)$ is a closed subset of $X \times Y$ with respect to the product topology. Furthermore, $A$ is said to be *densely defined* if $D(A)$ is a dense subset of $X$.

Suppose we can write $X$ as the topological direct sum of two subspaces $X_1$ and $X_2$, i.e.

$$X = X_1 \oplus X_2.$$

This decomposition is said to *reduce* the linear operator $A: X \supset D(A) \to X$ if both $X_1$ and $X_2$ are invariant under $A$, i.e. $A(X_i \cap D(A)) \subset X_i$ for $i = 1, 2$. If this is the case we can write in an obvious way

$$A = A_1 \oplus A_2\,,$$

where the operators $A_i: X_i \supset D(A_i) \to X_i$ are defined by $D(A_i) := X_i \cap D(A)$ and $A_i x := Ax$ for any $x \in X_i$, $i = 1, 2$. If $A$ is closed or bounded this property is inherited by both $A_1$ and $A_2$.

If $X$ is a normed space we shall write $\|\cdot\|$ for its norm. This means that if we consider several Banach spaces we shall denote all norms by the same symbol $\|\cdot\|$. If there should be any ambiguity we shall provide the norms with subscripts such as $\|\cdot\|_X$. Suppose now that $X$ and $Y$ are normed spaces. Then we can define a norm on $\mathcal{L}(X,Y)$ by setting

$$\|T\| := \sup_{\|x\|\leq 1} \|Tx\| = \inf\{c \geq 0;\ \|Tx\| \leq c\|x\| \text{ for all } x \in X\}.$$

The topology on $\mathcal{L}(X,Y)$ induced by this norm is called the *uniform operator topology.* Convergence of a sequence of operators or continuity of a function taking values in $\mathcal{L}(X,Y)$ shall always be understood with respect to this topology, unless explicitly stated. If $Y$ is a Banach space then $\mathcal{L}(X,Y)$ equipped with the above norm is also a Banach space.

Observe that if $X$ is a Banach space, $\mathcal{L}(X)$ is actually a Banach algebra and, by the stability property of compact operators, $\mathcal{K}(X)$ a two-sided ideal in this algebra. The unit element of $\mathcal{L}(X)$, i.e. the identity map, shall be denoted by $\mathbb{1}_X$ or just $\mathbb{1}$ if no confusion seems likely.

Suppose now that $A\colon X \supset D(A) \to Y$ is a linear operator. Then we can define a norm on $D(A)$ by setting

$$\|x\|_{D(A)} := \|x\| + \|Ax\|,$$

for all $x \in D(A)$. We shall always think of $D(A)$ as being equipped with this norm which is called the *graph-norm* on $D(A)$. The closedness of $A$ is then equivalent to $D(A)$ being a Banach space. If $A$ is bijective from its domain of definition to $Y$ and its inverse $A^{-1}$ is a bounded operator from $Y$ into $X$ then

$$|\!|\!|x|\!|\!|_{D(A)} := \|Ax\| \qquad \text{for all } x \in D(A)$$

defines a norm which is equivalent to the graph-norm on $D(A)$.

Finally, for any linear operator $A\colon X \supset D(A) \to Y$ we set

$$\ker(A) := \{x \in D(A);\ Ax = 0\} \qquad \text{and} \qquad \operatorname{im}(A) := \{Ax;\ x \in D(A)\}.$$

The subspaces $\ker(A)$ of $X$ and $\operatorname{im}(A)$ of $Y$ are called *kernel* (or *null-space*) and *range* (or *image*) of $A$, respectively.

**D. Imbeddings:** Suppose that $X$ and $Y$ are linear metric spaces. If $X \subset Y$ holds (as sets) and the inclusion map $i\colon X \to Y$ is continuous we say that $X$ is *continuously imbedded* in $Y$ and write

$$X \hookrightarrow Y.$$

This is equivalent to saying that if $X$ is endowed with the relative topology of $Y$ this topology is weaker than the original one.

$X$ is said to be *densely imbedded* in $Y$ if $X \hookrightarrow Y$ and $X$ is a dense subset of $Y$. In this case we write

$$X \overset{d}{\hookrightarrow} Y.$$

If $X \hookrightarrow Y$ holds and the inclusion map $i$ is compact then $X$ is said to be *compactly imbedded* in $Y$. We write for a compact imbedding and a compact dense imbedding

$$Y \hookrightarrow\hookrightarrow X \qquad \text{and} \qquad Y \overset{d}{\hookrightarrow\hookrightarrow} X,$$

respectively.

Whenever $X$ is continuously imbedded in $Y$ and $Z$ is another linear metric space we can view $\mathcal{L}(Y,Z)$ as a linear subspace of $\mathcal{L}(X,Z)$ by identifying $T \in \mathcal{L}(Y,Z)$ with $T \circ i \in \mathcal{L}(X,Z)$. If $X$ is densely imbedded this correspondence is one-to-one. If $X$ is compactly imbedded in $Y$ then $\mathcal{L}(Y,Z) \subset \mathcal{K}(X,Z)$ by the stability property of compact operators.

Suppose that $X$ and $Y$ are normed spaces and that $X \hookrightarrow Y$ holds. Then we have

$$\|x\|_Y \le \|i\|_{\mathcal{L}(X,Y)} \|x\|_X$$

for all $x \in X$. We call $\|i\|_{\mathcal{L}(X,Y)}$ the *imbedding constant* of $X \hookrightarrow Y$.

Finally if two normed spaces $X$ and $Y$ are equal as sets and their norms are equivalent we write

$$X \doteq Y.$$

Let $X$ and $Y$ be Fréchet spaces satisfying $X \hookrightarrow Y$ and let $A\colon Y \supset D(A) \to Y$ be a closed linear operator. The operator $A_X : X \supset D(A_X) \to X$, called the $X$-*realization* of $A$, is defined by

$$D(A_X) = \{x \in D(A) \cap X;\ Ax \in X\} \qquad \text{and} \qquad A_X x = Ax \text{ for all } x \in D(A_X).$$

Observe that $A_X$ is a closed operator. The same notions are used if $Y$ is not a Fréchet space but only a locally convex space (for example the space $\mathcal{D}'$ of Distributions when dealing with differential operators in function spaces).

**E. Duality:** Let $X$ be a Banach space. By a *linear functional* on $X$ we mean a linear operator from $X$ into $\mathbb{K}$. The *topological dual space* of $X$ is the Banach space $X' := \mathcal{L}(X, \mathbb{K})$. If $x' \in X'$ and $x \in X$ we write

$$<x', x> := x'(x).$$

Let $Y$ be a further Banach space and $T \in \mathcal{L}(X,Y)$. The *adjoint* or *dual* operator of $T$ is the uniquely determined operator $T' \in \mathcal{L}(Y', X')$ satisfying

$$<T'y', x> = <y', Tx>$$

for all $y' \in Y'$ and $x \in X$.

If $M$ is a subspace of $X$ and $N$ a subspace of $X'$ we define their *annihilators* to be the closed subspaces

$$M^\perp := \{x' \in X';\ <x', x> = 0 \text{ for all } x \in M\}$$

$$^\perp N := \{x \in X;\ <x', x> = 0 \text{ for all } x' \in N\}$$

of $X'$ and $X$, respectively. The following identities hold for any operator $T \in \mathcal{L}(X, Y)$ (cf. [**106**], Theorem 4.12)

$$\ker(T') = \operatorname{im}(T)^\perp \qquad \text{and} \qquad \ker(T) = {}^\perp \operatorname{im}(T').$$

**F. Spectral theory:** Let $X$ be a Banach space and $A: X \supset D(A) \to X$ a densely defined closed linear operator. If the space is real the notions below are defined in the context of its complexification $X_\mathbb{C}$. This Banach space consists of the formal expressions of the form $z := x + iy$ with $x, y \in X$. The operations on $X_\mathbb{C}$ are defined as

$$z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2)$$

and

$$\lambda z := (\lambda_1 x - \lambda_2 y) + i(\lambda_2 x + \lambda_1 y)$$

whenever $z, z_1, z_2 \in X_\mathbb{C}$ and $\lambda + \lambda_1 + i\lambda_2 \in \mathbb{C}$. The norm on $X_\mathbb{C}$ is given by

$$\|z\| := \max_{0 \le \phi \le 2\pi} \|\cos(\phi)x + \sin(\phi)y\|.$$

As was to be expected $\mathbb{R}_\mathbb{C} = \mathbb{C}$. The complexification of $A$ is then the operator $A_\mathbb{C}: X_\mathbb{C} \supset D(A_\mathbb{C}) \to X_\mathbb{C}$ defined by $D(A_\mathbb{C}) := \{\, z = x + iy;\ x, y \in D(A) \,\}$ and $A_\mathbb{C} z := Ax + iAy$ for $z \in D(A_\mathbb{C})$.

The *resolvent set*, $\varrho(A)$, of $A$ is the open subset of $\mathbb{C}$ defined by

$$\varrho(A) := \{\lambda \in \mathbb{C};\ (\lambda - A)^{-1} \text{ exists and lies in } \mathcal{L}(X)\}.$$

Here we use the notation $\lambda - A := \lambda \mathbb{1} - A$. The *spectrum*, $\sigma(A)$, of $A$ is the complement of the resolvent set in $\mathbb{C}$, i.e. $\sigma(A) := \mathbb{C} \setminus \varrho(A)$. Hence, $\sigma(A)$ is a closed set. The spectrum of $A$ can be writen as the disjoint union of the following sets

$$\sigma_p(A) := \{\lambda \in \mathbb{C};\ (\lambda - A) \text{ is not injective}\},$$

$$\sigma_c(A) := \{\lambda \in \mathbb{C};\ (\lambda - A) \text{ is injective, } \operatorname{im}(\lambda - A) \text{ dense in } X,\ (\lambda - A)^{-1} \text{ not bounded}\},$$

$$\sigma_r(A) := \{\lambda \in \mathbb{C};\ (\lambda - A) \text{ is injective and } \operatorname{im}(\lambda - A) \text{ is not dense in } X\}.$$

These sets are called *point spectrum, continuous spectrum* and *residual spectrum*, respectively. An element $\lambda$ of $\sigma_p(A)$ is an *eigenvalue* of $A$, $\ker(\lambda - A)$ is the *eigenspace* to $\lambda$ and any non-zero element in it is called an *eigenvector* to $\lambda$.

If $A \in \mathcal{L}(X)$ then $\sigma(A)$ can be shown to be compact and we set

$$\mathrm{r}(A) := \sup\{|\lambda|; \lambda \in \sigma(A)\}.$$

This number is called the *spectral radius* of $A$. It can be calculated by the well known formula

$$\mathrm{r}(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

A subset $\sigma_1$ of $\sigma(A)$ is called *spectral set* of $A$ if it is open and closed in $\sigma(A)$ (for instance a set consisting of an isolated point of $\sigma(A)$). If $\sigma_1$ is a compact spectral set we put $\sigma_2 := \sigma(A) \setminus \sigma_1$ and take a bounded domain $U \in \mathbb{C}$, such that

$$\sigma_1 \in U, \quad \sigma_2 \in \mathbb{C} \setminus \bar{U},$$

and

$$\partial U = \bigcup_{j=i}^{m} \Gamma_j\,,$$

where $(\Gamma_j)_{1\leq m}$ is a finite collection of disjoint smooth Jordan-curves which are positively oriented with respect to $U$. Then we can define the following *Dunford-type* integral:

$$P_1 := \frac{1}{2\pi i}\int_{\partial U} (\lambda - A)^{-1}\, d\lambda \qquad \text{in } \mathcal{L}(X).$$

The operator $P_1$ is a projection which is independent of the particular choice of a domain $U$ with the above properties. We call $P_1$ the *spectral projection of $A$ with respect to $\sigma_1$*. Setting $X_1 := P_1(X)$ and $X_2 := (\mathbb{1} - P_1)(X)$ we obtain a decomposition

$$X = X_1 \oplus X_2\,,$$

which reduces the operator $A$

$$A = A_1 \oplus A_2\,.$$

Furthermore, we have

$$\sigma(A_i) = \sigma_i,\ i = 1, 2.$$

**G. Derivatives:** Let $X$ and $Y$ be Banach spaces. If the function $f\colon U \to Y$, defined on the open subset $U$ of $X$, is (Fréchet)-differentiable at a point $x_0 \in U$ then its derivative at $x_0$ is a linear operator in $\mathcal{L}(X, Y)$ and we write either $Df(x_0)$ or $f'(x_0)$ to denote it. Suppose $X := X_1 \times X_2$ and $U := U_1 \times U_2$, where $U_i$ is an open subset of the Banach space $X_i$, $i = 1, 2$. If $f\colon U \to Y$ is differentiable at $(x_0^1, x_0^2) \in U$ we denote its $i$-th partial derivative, which is an operator in $\mathcal{L}(X_i, Y)$ by $D_i f(x_0^1, x_0^2)$, $i = 1, 2$.

If $f\colon \Omega \to \mathbb{R}$ is a differentiable function defined on an open subset $\Omega$ of $\mathbb{R}^n$, $n \geq 1$, we write for any $x_0 \in \Omega$ and $i = 1, \ldots, n$

$$\partial_i f(x_0) := D_i f(x_0).$$

The *gradient* of $f$ at $x_0 \in \Omega$ shall be denoted by

$$\operatorname{grad} f(x_0) := \nabla f(x_0) := \big(\partial_1 f(x_0), \ldots, \partial_n f(x_0)\big) \in \mathbb{R}^n.$$

The *Laplace operator* or *Laplacian* of $f$ at $x_0$ is defined as

$$\Delta f(x_0) := \partial_1^2 f(x_0) + \cdots + \partial_n^2 f(x_0) \in \mathbb{R}.$$

We call an element $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathbb{N}^n$ a *multiindex* (*of rank* $n$). The *order* of $\alpha$ is given by $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Let $\alpha \in \mathbb{N}^n$. If $f\colon \Omega \to \mathbb{R}$ is sufficiently smooth in $x_0 \in \Omega$, we write

$$\partial^\alpha f(x_0) = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} f(x_0).$$

If $\alpha$ and $\beta$ are two multiindices we write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, \ldots, n$. We define

$$\alpha! := \prod_{i=1}^n \alpha_i! \qquad \text{and} \qquad \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}\,,$$

where $\beta \leq \alpha$ and $(\alpha - \beta)_i = \alpha_i - \beta_i$ for all $i = 1, \ldots, n$. Let now $f, g\colon \Omega \to \mathbb{R}$ be two functions. Then, *Leibniz's rule*

$$\partial^\alpha \big(fg\big)(x_0) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x_0) \partial^{\alpha-\beta} g(x_0)$$

holds, whenever it makes sense.

If $f\colon \Omega \times I \to \mathbb{R}$ is defined on the product of an open subset $\Omega$ of $\mathbb{R}^n$ and an open interval $I$ in $\mathbb{R}$, we think of the point $(x, t) \in \Omega \times I$ as consisting of the *space-variable* $x$ and the *time-variable* $t$. Furthermore we write

$$\partial_t f(x_0, t_0) := D_{n+1} f(x_0, t_0),$$

$$\partial_i f(x_0, t_0) := D_i f(x_0, t_0),$$

$$\operatorname{grad} f(x_0, t_0) := \nabla f(x_0, t_0) := \big(\partial_1 f(x_0, t_0), \ldots, \partial_n f(x_0, t_0)\big)$$

and

$$\Delta f(x_0, t_0) := \partial_1^2 f(x_0, t_0) + \cdots + \partial_n^2 f(x_0, t_0)$$

for any $(x_0, t_0) \in \Omega \times I$ and $i = 1, \dots, n$. Note that in this situation we take the gradient and the Laplacian only with respect to the space-variable. With $D_x f(x_0, t_0)$ we shall denote the full derivative with respect to the space-variable.

Suppose now that $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a $C^1$ boundary. Let $f \in C^1(\bar{\Omega}, \mathbb{R})$ and assume there is given a vectorfield $b: \partial\Omega \to \mathbb{R}^n$. Then we put for $x_0 \in \partial\Omega$

$$\partial_b f(x_0) := Df(x_0)b(x_0).$$

Thus, $\partial_b f(x_0)$ denotes the derivative of $f$ at $x_0$ in the direction of the vectorfield $b$ at that point. Of course if we have a function $f: \Omega \times I \to \mathbb{R}$ as above we define

$$\partial_b f(x_0, t_0) := D_2 f(x_0, t_0) b(x_0)$$

for any $(x_0, t_0) \in \Omega \times I$.

**H. Strong continuity and strong differentiability:** Let $X$ and $Y$ be Banach spaces over $\mathbb{K}$ and $\Lambda$ a metric space. Consider a function $f: \Lambda \to \mathcal{L}(X, Y)$. We shall frequently encounter the situation where $f$ fails to be continuous at $\lambda_0 \in \Lambda$, with respect to the uniform operator topology of $\mathcal{L}(X, Y)$, but turns out to be continuous in the weaker sense, that for every $x \in X$ the function $f(\cdot)x: \Lambda \to Y$ is continuous at $\lambda_0$. Such a mapping is said to be *strongly continuous* at $\lambda_0$.

The space $Y^X$ carries the product topology of $\prod_{x \in X} Y$– which is also the topology of pointwise convergence – and we can view $\mathcal{L}(X, Y)$ as a closed subspace of $Y^X$. We write $\mathcal{L}_s(X, Y)$ for $\mathcal{L}(X, Y)$ endowed with the topology induced by $Y^X$ and call this topology the *strong operator topology* on $\mathcal{L}(X, Y)$. Strong continuity is then nothing else but continuity with respect to this topology. Thus

$$C\big(\Lambda, \mathcal{L}_s(X, Y)\big)$$

is the space of all functions $\Lambda \to \mathcal{L}(X, Y)$ which are strongly continuous at all points.

The uniform boundedness principle immediately implies that if $f: \Lambda \to \mathcal{L}(X, Y)$ is strongly continuous at $\lambda_0$, there exists an $\varepsilon > 0$, such that

$$\sup_{\lambda \in \bar{\mathbb{B}}_\Lambda(\lambda_0, \varepsilon)} \|f(\lambda)\| < \infty.$$

This means that $f$ is *locally bounded* at $\lambda_0$. In particular if $f \in C\big(\Lambda, \mathcal{L}_s(X, Y)\big)$ and $\Lambda$ is compact, then

$$\sup_{\lambda \in \Lambda} \|f(\lambda)\| < \infty,$$

that is , $f$ is *bounded.* It is an easy exercise to prove the following two results:

*(a)* $f \in C(\Lambda, \mathcal{L}_s(X,Y))$ *and* $u \in C(\Lambda, X)$ *imply that* $[\lambda \mapsto f(\lambda)u(\lambda)] \in C(\Lambda, Y)$ *holds.*

*(b) Let* $f\colon \Lambda \to \mathcal{L}(X,Y)$ *be a mapping and* $\lambda_0 \in \Lambda$*. Suppose that* $f$ *is locally bounded at* $\lambda_0$ *and that there is a dense subspace* $X_1$ *of* $X$*, such that* $\lim_{\lambda \to \lambda_0} f(\lambda)x = f(\lambda_0)x$ *for all* $x \in X_1$*. Then* $f$ *is strongly continuous at* $\lambda_0$*.*

Let now $\Lambda$ be an open subset of a Banach space $Z$ and $f\colon \Lambda \to \mathcal{L}(X,Y)$ a mapping. $f$ is said to be *strongly differentiable* at $\lambda_0 \in \Lambda$, if $f(\cdot)x\colon \Lambda \to Y$ is differentiable at $\lambda_0$. We write

$$f \in C^1\big(\Lambda, \mathcal{L}_s(X,Y)\big),$$

whenever $f(\cdot)x \in C^1(\Lambda, Y)$ for every $x \in X$, and call $f$ *strongly continuously differentiable.* It is clear how to define the spaces $C^r(\Lambda, \mathcal{L}_s(X,Y))$ of *$r$-times strongly continuously differentiable functions* for any $r \in \mathbb{N} \cup \{\infty\}$. If $f \in C^r(\Lambda, \mathcal{L}_s(X,Y))$, $\lambda \in \Lambda$ and $0 \le k \le r$, we write $D^k f(\lambda_0)$ for the $k$-th derivative of $f(\cdot)x$ at $\lambda_0 \in \Lambda$. Obviously, $D^k f \in C(\Lambda, \mathcal{L}_s(X,Y))$ for all $0 \le k \le r$ in this case.

Of course one could also define the class $C^\omega(\Lambda, \mathcal{L}_s(X,Y))$ of strongly analytic functions in the obvious way. In case that $\mathbb{K} = \mathbb{C}$ this class coincides with the class of norm–analytic mappings (cf. [**75**], Theorem III.3.12).

Analogously to the case of strong continuity we have the following result:

*Let* $r \in \mathbb{N} \cup \{\infty\}$ *and suppose that* $f \in C^r(\Lambda, \mathcal{L}_s(X,Y))$ *and* $u \in C^r(\Lambda, X)$*. Then,* $[\lambda \mapsto f(\lambda)u(\lambda)] \in C^r(\Lambda, Y)$*. Furthermore, if* $r \ge 1$*, we have for each* $\lambda_0 \in \Lambda$ *that the product rule*

$$D[f(\cdot)u(\cdot)](\lambda_0) = D[f(\lambda_0)]u(\lambda_0)] + f(\lambda_0)Du(\lambda_0)$$

*holds.*

# I. Linear evolution equations of parabolic type

In this first chapter we study abstract linear evolution equations of parabolic type and derive some estimates for the evolution operator in interpolation spaces which turn out to be useful in the treatment of the semilinear problem to be considered in the third and fifth chapter.

## 1. Analytic $C_0$-semigroups

In this section we collect – for the readers convenience – some well-known facts from the theory of strongly continuous analytic semigroups on Banach spaces. The style will be informal, the purpose being to freshen up the readers memory rather than to give a logically self contained account of the theory.

**A. $C_0$-semigroups:** Let $X$ be a Banach space. A family $\big(T(t)\big)_{t\geq 0}$ of bounded linear operators on $X$, satisfying:

$(S1)$ $T(0) = \mathbb{1}$, $T(s+t) = T(s)T(t)$ for $s, t \geq 0$,

$(S2)$ for all $x \in X$: $\lim\limits_{t \searrow 0} T(t)x = x$,

is called a *strongly continuous semigroup* on $X$, or, $C_0$-*semigroup*, for short. Property $(S2)$ means that $T$ is strongly continuous at 0. It is not difficult to show that by $(S2)$ $T$ is strongly continuous at every point of $\mathbb{R}_+$.

The *infinitesimal generator* of the $C_0$-semigroup $\big(T(t)\big)_{t\geq 0}$ is the linear operator

$$B : X \supset D(B) \to X,$$

defined by

$$Bx := \lim_{t \searrow 0} \frac{1}{t}\big(T(t)x - x\big), \quad \text{for } x \in D(B),$$

where

$$D(B) := \big\{ x \in X;\ \lim_{t \searrow 0} \frac{1}{t}\big(T(t)x - x\big) \text{ exists } \big\}.$$

$B$ is then a densely defined, closed linear operator in $X$ and uniquely determines the semigroup. We will usually consider the operator $A := -B$ rather than B itself. This is the standard usage in abstract parabolic equations.

The connection with differential equations is given by the fact that for each $x \in D(A)$, the function $u := T(\cdot)x$ is continuously differentiable from $\mathbb{R}_+$ to $X$, and solves the *abstract Cauchy-problem*:

$$(1.1) \qquad \begin{cases} \partial_t u(t) + Au(t) = 0 & \text{for } t \geq 0 \\ u(0) = x. \end{cases}$$

In case that $A$ is a bounded operator on $X$, we can define

$$e^{-tA} := \sum_{k \geq 0} \frac{1}{k!}(-t)^k A^k,$$

for all $t \in \mathbb{R}$. Then $(e^{-tA})_{t \geq 0}$ is a $C_0$-semigroup on $X$ with infinitesimal generator $-A$. More is actually true: $(e^{-tA})_{t \in \mathbb{R}}$ is a uniformly continuous group of operators on $X$, i.e. $(S1)$ holds for all $s, t \in \mathbb{R}$, and $(S2)$ can be replaced by $\lim_{t \to 0} T(t) = \mathbb{1}_X$ in $\mathcal{L}(X)$.

The above example justifies the notation:

$$e^{-tA} := T(t), \qquad \text{for all } t \geq 0,$$

whenever $\big(T(t)\big)_{t \geq 0}$ is a $C_0$-semigroup with infinitesimal generator $A$.

A $C_0$-semigroup is always *exponentially bounded*, i.e. there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$, such that:

$$\|e^{-tA}\| \leq Me^{t\omega} \qquad \text{for all } t \geq 0.$$

We can therefore define the *growth-bound*, or *exponential-type*, of $(e^{-tA})_{t \geq 0}$ by:

$$(1.2) \qquad \begin{aligned} \omega(-A) &:= \omega(e^{-tA}) \\ &:= \inf\{\omega \in \mathbb{R};\ \text{there is } M \geq 1 \text{ with } \|e^{-tA}\| \leq Me^{t\omega} \text{ for all } t \geq 0\}. \end{aligned}$$

Alternatively, we could have defined $\omega(-A)$ by the formula:

$$(1.3) \qquad \omega(-A) = \lim_{t \to \infty} \frac{1}{t} \log \|e^{-tA}\|.$$

The *spectral bound* of $-A$, is given by:

$$(1.4) \qquad s(-A) := \sup\{\mathrm{Re}(\lambda);\ \lambda \in \sigma(-A)\}.$$

We always have $s(-A) \leq \omega(-A)$. Furthermore, the identity

$$(1.5) \qquad (\lambda + A)^{-1} y = \int_0^\infty e^{-t\lambda} e^{-tA} y \, dt$$

holds for all $y \in X$, and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega(-A)$. By means of (1.5) we can construct the infinitesimal generator, or rather its resolvent, from the semigroup. But most of the time the situation is just the opposite: we know the generator and would like to construct the corresponding semigroup. One way of achieving this is by means of the formula:

$$(1.6) \qquad e^{-tA}x = \lim_{n\to\infty} \big(\mathbb{1} + \frac{t}{n}A\big)^{-n} x,$$

which holds for all $x \in X$. There is still another way, based on the following strong approximation of the identity:

$$(1.7) \qquad \lim_{n\to\infty} \big(\mathbb{1} + \frac{1}{n}A\big)^{-1} x = x \quad \text{for all } x \in X.$$

Setting $A_n := A(\mathbb{1} + \frac{1}{n}A)^{-1} = -n\big((\mathbb{1} + \frac{1}{n}A)^{-1} - \mathbb{1}\big) \in \mathcal{L}(X)$, for $n > \omega(-A)$, we immediately obtain a strong-approximation of the generator:

$$(1.8) \qquad \lim_{n\to\infty} A_n x = Ax \quad \text{for all } x \in D(A).$$

The sequence $(A_n)_{n>\omega(-A)}$ of bounded operators on $X$, is called the *Yosida-approximation* of $A$. We are now ready to give a second formula for obtaining the semigroup out of its infinitesimal generator:

$$(1.9) \qquad \lim_{n\to\infty} e^{-tA_n} x = e^{-tA} x \quad \text{for all } x \in X.$$

Note that because $A_n$ is bounded, the corresponding semigroup is known.

**B. Analytic semigroups:** Before giving the definitions of holomorphic and analytic semigroups we introduce the following notation:

$$(1.10) \qquad S_\alpha := \{z \in \mathbb{C}\backslash\{0\};\ |\arg(z)| < \alpha\} \cup \{0\},$$

for $\alpha \in [0, \pi]$. $S_\alpha$ is thus an open sector in the complex-plane.

If $X$ is a complex Banach space, $\alpha \in (0, \frac{\pi}{2}] \cup \{\pi\}$, and $\big(P(z)\big)_{z\in S_\alpha}$ a family of bounded linear operators on $X$, such that:

- $(H1)$ $P(0) = \mathbb{1}_X$, $P(z_1 + z_2) = P(z_1)P(z_2)$ for all $z_1, z_2 \in S_\alpha$ ,
- $(H2)$ $P\colon \mathrm{int}_{\mathbb{C}}(S_\alpha) \to \mathcal{L}(X)$ is a holomorphic function,
- $(H3)$ For all $x \in X$ and $\varepsilon \in (0, \alpha)$: $P(z)x \to x$ as $z \in S_{\alpha-\varepsilon}$ approaches 0,

then $\big(P(z)\big)_{z\in S_\alpha}$ is called a *holomorphic semigroup* on $X$ (*with angle* $\alpha$).

A $C_0$-semigroup $\big(T(t)\big)_{t\geq 0}$ on a (real or complex) Banach space $X$ is called an *analytic semigroup* (*of angle* $\alpha \in (0, \frac{\pi}{2}] \cup \{\pi\}$) on $X$ if we can extend it to a holomorphic semigroup on the sector $S_\alpha$. If the space is real we have to apply this definition to the complexification. In this case it is not difficult to prove that

$$\left.\begin{array}{l}\big(T(t)\big)_{t\geq 0} \text{ is a } C_0\text{-semigroup on } X \\ \text{with infinitesimal generator } -A\end{array}\right\} \iff \left\{\begin{array}{l}\big(T(t)_{\mathbb{C}}\big)_{t\geq 0} \text{ is a } C_0\text{-semigroup on } X_{\mathbb{C}} \\ \text{with infinitesimal generator } -A_{\mathbb{C}}\end{array}\right.$$

For the notation on complexifications consult Section 0.F.

Observe that if $A \in \mathcal{L}(X)$ then $(e^{-tA})_{t\geq 0}$ is an analytic semigroup on $X$.

Among the many important properties of analytic semigroups, we single out the very typical *smoothing property* ([**55**], Lemma 4.1.1):

$$e^{-tA}(X) \subset D(A) \qquad \text{for all } t > 0. \tag{1.11}$$

Together with $(H2)$, this implies that for every $x \in X$ the function

$$[t \mapsto e^{-tA}x] \colon (0, \infty) \to X$$

is analytic and solves the abstract Cauchy-problem:

$$\begin{cases} \dot{u}(t) + Au(t) = 0 & \text{for } t > 0 \\ \qquad\quad u(0) = x. \end{cases} \tag{1.13}$$

Another nice property of analytic semigroups is the following *spectral mapping theorem* ([**98**]):

$$\sigma(e^{-tA})\backslash\{0\} = \{e^{-t\lambda};\ \lambda \in \sigma(-A)\}. \tag{1.13}$$

Furthermore, one can show that for an analytic semigroup $(e^{-tA})_{t\geq 0}$, the spectral bound is equal to the exponential-type (cf. [**98**]):

$$\omega(-A) = s(-A). \tag{1.14}$$

If $-A$ is the generator of a strongly continuous analytic semigroup of angle $\alpha \in (0, \frac{\pi}{2}]$ one can show that

$$\omega(-A) + \mathrm{int}_{\mathbb{C}} S_{\frac{\pi}{2}+\alpha} \subset \varrho(-A). \tag{1.15}$$

and that for any $\lambda > \omega(-A)$ and $\varepsilon > 0$ there exists a constant $c > 0$ such that

$$\|(\lambda + A)^{-1}\| \leq \frac{c}{|\lambda - \lambda_0|} \tag{1.16}$$

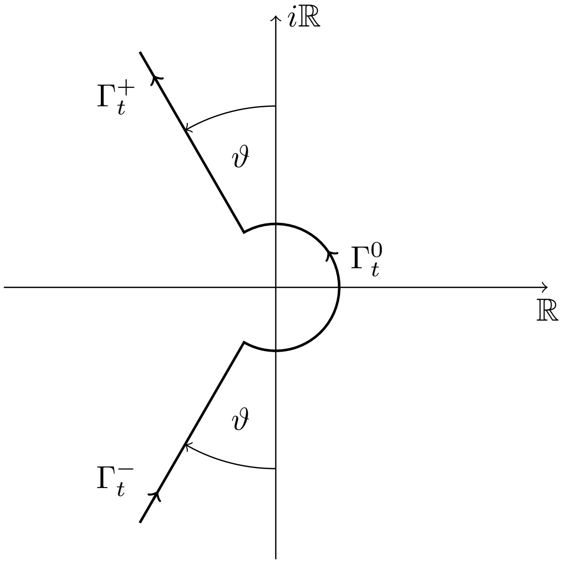
for all $\lambda \in \lambda_0 + S_{\frac{\pi}{2}+\alpha-\varepsilon}\backslash\{0\}$. For simplicity, let us assume now that $\omega(-A) < 0$. In this case we may choose $\lambda_0 = 0$. Then the following representation formula holds:

$$e^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A)^{-1}d\lambda, \tag{1.17}$$

where $\Gamma$ is a piecewise smooth Jordan curve in $\mathrm{int}_{\mathbb{C}}\, S_{\frac{\pi}{2}+\alpha}$ running from $\infty e^{-i\vartheta}$ to $\infty e^{i\vartheta}$ with $\vartheta \in (0,\alpha)$ arbitrary. For example for fixed $t > 0$ we write $\Gamma_t = \Gamma_t^+ + \Gamma_t^0 + \Gamma_t^-$, where

$$\Gamma_t^\pm := \{r\, e^{\pm i(\vartheta+\frac{\pi}{2})};\ t^{-1} \le r < \infty\} \qquad \text{and} \qquad \Gamma_t^0 := \{t^{-1}\, e^{i\varphi};\ |\varphi| \le \vartheta + \frac{\pi}{2}\}.$$

In the complex plane these paths look as follows:




Using this special path and the resolvent estimate (1.16) we obtain

$$\|A^k e^{-tA}\| = \|\left(\frac{d}{dt}\right)^k e^{-tA}\| \le c_k\, t^{-k} \tag{1.18}$$

for all $t > 0$ and some constant $c_k > 0$ depending only on $k \in \mathbb{N}$ and $c$ of (1.16).

Since, usually, the given object is a differential equation – which induces a linear operator on some Banach space of functions – it is useful to have criteria for deciding whether a given operator is the infinitesimal generator of an analytic semigroup or not. It turns out that the (1.16) and (1.17) or (1.11) and (1.18) are in fact characterizations of generators of analytic $C_0$-semigroups. The precise assertions we collect in the following generation theorem (see e.g. [**55**], [**100**], [**119**]):

**1.1 Theorem**
*The linear operator* $-A\colon X \supset D(A) \to X$ *is the generator of an analytic semigroup on* $X$ *if and only if the following two sets of conditions are satisfied:*

(1) $A$ *is densely defined and closed.*

(2) *There exists a* $\lambda_0 \in \mathbb{R}$ *such that:*

$$[\operatorname{Re}\lambda \le \lambda_0] \subset \varrho(A),$$

*and an* $M \ge 1$ *with:*

$$\|(\lambda + A)^{-1}\| \le \frac{M}{1+|\lambda|}$$

*for all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \ge \lambda_0$.

*or*

(1′) $-A$ *is the infinitesimal generator of a* $C_0$*-semigroup with the smoothing property* (1.11).

(2′) $\limsup\limits_{t\to 0} t\|Ae^{-tA}\| < \infty$.

**1.2 Remark**
The estimate in (2) of Theroem 1.1 implies always an estimate of the form (1.16) in some sector $\lambda_0 + S_{\frac{\pi}{2}+\alpha}$ with $\alpha \in (0, \frac{\pi}{2}]$. The constant $c$ and the angle $\alpha$ appearing in 1.16 depend only on the constant $M$ in (2). To see this observe that there is a constant $c_0 > 0$ such that

$$\frac{|\lambda - \lambda_0|}{1+|\lambda|} \le c_0 \tag{1.19}$$

holds for all $\lambda \in [\operatorname{Re}\mu \ge \lambda_0] \setminus \{\lambda_0\}$. Thus one finds that

$$\|(\lambda + A)^{-1}\| \le \frac{Mc_0}{|\lambda - \lambda_0|} \tag{1.20}$$

for all such $\lambda$. Then one can find a constant $C \ge 1$ dependent only on $Mc_0$, such that (1.16) holds with $\alpha = \arcsin\bigl(1/Mc_0\bigr)$ (see e.g. [**55**], Lemma 4.2.3). □

A useful method of producing a new semigroup from a known one is by perturbation. We give here two of the well-known perturbation results for analytic semigroups:

**1.3 Theorem**
*Let $-A$ be the infinitesimal generator of an analytic semigroup and suppose that*

$$[\operatorname{Re}\mu \geq \lambda_0] \subset \varrho(-A) \qquad \text{and} \qquad \|(\lambda + A)^{-1}\| \leq \frac{M}{1+|\lambda|}$$

*holds for all $\lambda \in [\operatorname{Re}\mu \geq \lambda_0]$, where $\lambda_0 \in \mathbb{R}$ and $M \geq 1$ are suitably chosen. Moreover, let $B$ satisfy one of the following conditions:*

(*a*) *$B\colon X \supset D(B) \to X$ is a linear operator on $X$ with $D(B) \supset D(A)$, such that for any $a > 0$, there is a $b > 0$ with*

$$\|Bx\| \leq a\|Ax\| + b\|x\| \qquad \text{for all } x \in D(A).$$

(*b*) *Suppose that $D(A)$ is equipped with the graph norm and let*

$$B \in \mathcal{K}\big(D(A), X\big)$$

*Then $-(A+B)\colon X \supset D(A+B) = D(A) \to X$ is also the infinitesimal generator of an analytic semigroup and there exist constants $\tilde{M} \geq 1$ and $\lambda_1 \in \mathbb{R}$ such that*

$$[\operatorname{Re}\mu \geq \lambda_1] \subset \varrho\big(-(A+B)\big) \qquad \text{and} \qquad \|(\lambda + A + B)^{-1}\| \leq \frac{\tilde{M}}{1+|\lambda|}$$

*holds for all $\lambda \in [\operatorname{Re}\mu \geq \lambda_1]$. Moreover, $\lambda_1$ and $\tilde{M}$ depend only on $\lambda_0$, $M$, $a$ and $b$ in the case $(a)$ and on $\lambda_0$, $M$ and a compact set in $X$ containing $\{Bx;\ \|x\|_{D(A)} \leq 1\}$ in case $(b)$.*

**Proof**
A proof may be found for example in [**55**], Theorem 5.3.6 in case $(a)$ and [**31**], Theorem 5.6 in case $(b)$. The statements about $\lambda_1$ and $\tilde{M}$ can be easily verified examining the proofs of these perturbation theorems. □

The following theorem shows that the set of generators of analytic semigroups is open in $\mathcal{L}\big(D(A), X\big)$.

**1.4 Theorem**
*Let $-A$ be the generator of an analytic $C_0$-semigroup satisfying*

$$\|(\lambda + A)^{-1}\| \leq \frac{M}{1+|\lambda|}$$

*for all $\lambda \in [\operatorname{Re}\mu \geq \mu_0]$, where $\mu_0$ is chosen suitably. Then, there exists a neighbourhood of $A$ in $\mathcal{L}\big(D(A), X\big)$, such that*

$$\|(\lambda + A + B)^{-1}\| \leq 2\|(\lambda + A)^{-1}\| \leq \frac{2M}{1+|\lambda|}$$

*holds for all $\lambda \in [\operatorname{Re}\mu \geq \mu_0]$ and $B \in \mathcal{L}\big(D(A), X\big)$ small.*

**Proof**

The assertion follows from Lemma 5.2 and the proof of Theorem 5.3 in [**31**]. □

**C. Elliptic boundary value problems I:** For the function spaces appearing in this subsection consult the corresponding appendix. Let $n \geq 1$ and $\eta \in [0,1)$. Consider a triple

$$\big(\Omega, \mathcal{A}(x,D), \mathcal{B}(x,D)\big)$$

such that

(a) $\Omega$ is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of class $C^\infty$.

(b) $\mathcal{A} := \mathcal{A}(x,D)$ is a *linear uniformly elliptic differential expression of second order*, i.e.

$$\mathcal{A}(x,D) := -\sum_{j,k=1}^{n} a_{jk}(x)\partial_j\partial_k + \sum_{j=1}^{n} a_j(x)\partial_j + a_0(x),$$

where the coefficient functions $a_{jk} = a_{kj}$, $a_j$, and $a_0$, for $j,k = 1,\dots,n$, belong to $C^\eta(\overline{\Omega})$ and satisfy

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq \underline{\alpha}|\xi|^2$$

for $x \in \overline{\Omega}$ and $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$, for some positive constant $\underline{\alpha}$.

(c) $\mathcal{B} := \mathcal{B}(x,D)$ is a *boundary differential operator of first order*, i.e

$$\mathcal{B}(\cdot\,,D)u := \begin{cases} u \restriction_{\partial\Omega} & \text{(Dirichlet boundary conditions)} \\ \partial_b u & \text{(Neumann boundary conditions)} \\ \partial_b u + b_0(\cdot)u \restriction_{\partial\Omega} & \text{(Robin boundary conditions),} \end{cases}$$

where $b\colon \partial\Omega \to \mathbb{R}^n$ is a vectorfield on $\partial\Omega$ satisfying $(b(x)|\nu(x)) > 0$ for all $x \in \partial\Omega$ and $b_0\colon \partial\Omega \to \mathbb{R}$ a given non-zero function, both of class $C^{1+\eta}$. Here, $\nu$ denotes the outer unit normal on $\partial\Omega$.

We call such a triple a (*second order*) *elliptic boundary value problem of class* $C^\eta$.

We have chosen to deal only with domains of class $C^\infty$, although weaker regularity assumptions would suffice. The main reason for doing this is that at a later stage we shall need to consider interpolation spaces between the domain of definition of the $L_p$-realization of an elliptic boundary value problem and $L_p$ (for a definition of this $L_p$-realization see below). For these results the only precise reference we were able to find is for the $C^\infty$-case. This is by no means a great restriction. In our point of view, weaker regularity of the boundary is interesting only from Lipschitz-continuity downwards since then the results become interesting also for numerical analysts.

We set:

$$D(A_0) := C^2_{\mathcal{B}}(\overline{\Omega}) := \{u \in C^2(\overline{\Omega});\ \mathcal{B}(x, D)u(x) = 0 \text{ for all } x \in \partial\Omega\}$$

and

$$A_0 u := \mathcal{A}(\,\cdot\,, D)u(\cdot) \qquad \text{for } u \in D(A_0).$$

Let now $p \in (1, \infty)$ and set

$$X := L_p(\Omega),$$

The operator $A_0 \colon X \supset D(A_0) \to X$ is closable. We denote its closure by $A_p$, i.e.

$$A_p = \overline{A_0}^X.$$

$A_p$ is called the *$X$-realization* of $(\Omega, \mathcal{A}, \mathcal{B})$. The following essential result holds (see e.g. [**55**], Theorem 4.9.1, [**59**], Section 1.19, [**100**], Chapter 7):

> *The operator $-A_p$ is the infinitesimal generator of an analytic semigroup of compact operators on $X$.*

This assertion is shown by means of a priori estimates for elliptic boundary value problems going back to Agmon, Douglis and Nirenbeg [**3**] and [**4**] (compare also Lemma 26.5 in the last Section).

For a precise description of $D(A_p)$ we need some facts on traces. For $u \in C(\overline{\Omega})$ we set:

$$\gamma(u) := u \restriction \partial\Omega.$$

Then

$$\gamma \in \mathcal{L}\big(C^k(\overline{\Omega}), C^k(\partial\Omega)\big) \quad \text{for } k = 0, 1, 2.$$

The operator $\gamma$ can then be extended continuously to the *trace operator* (see Appendix 4):

$$\gamma \in \mathcal{L}\big(W^k_p(\Omega), W^{k-\frac{1}{p}}_p(\partial\Omega)\big) \quad \text{for } k = 1, 2,$$

where we denote the extensions of $\gamma$ by the same symbol.

The trace operator allows us to speak of the boundary values of a function in $W^1_p(\Omega)$ though they certainly do not exist in a classical sense since the boundary $\partial\Omega$ is a set of Lebesgue-measure zero. We now define

$$B_p := \begin{cases} \gamma \quad \in \mathcal{L}\big(W^2_p(\Omega), W^{2-\frac{1}{p}}_p(\partial\Omega)\big) & \text{(Dirichlet b.c.)} \\ D_b \quad \in \mathcal{L}\big(W^2_p(\Omega), W^{1-\frac{1}{p}}_p(\partial\Omega)\big) & \text{(Neumann b.c.)} \\ D_b + M_{b_0} \circ \gamma \quad \in \mathcal{L}\big(W^2_p(\Omega), W^{1-\frac{1}{p}}_p(\partial\Omega)\big) & \text{(Robin b.c.),} \end{cases}$$

where by $D_b$ we denote the bounded operator $W_p^2(\Omega) \to W_p^{1-\frac{1}{p}}(\partial\Omega)$ defined by

$$D_b u(\cdot) := \big(b(\cdot)|\gamma(\nabla u)(\cdot)\big)_{\mathbb{R}^n},$$

and by $M_{b_0}$ the multiplication operator on $W_p^{2-\frac{1}{p}}(\partial\Omega)$ induced by $b_0$. Here we took the liberty of writing $\gamma$ for the trace operator acting on $\mathbb{R}^n$-valued functions. With this notation we have (see e.g. [**55**], [**59**])

$$D(A_p) = W^2_{p,\mathcal{B}}(\Omega) := W_p^2(\Omega) \cap \ker(B_p).$$

**D. Elliptic boundary value problems II:** We already mentioned in the introduction that there is some degree of freedom in the choice of the right function space for the abstract formulation of a given equation. We will now describe another possible setting for the elliptic boundary value problems of the previous subsection. These results were obtained by H.B. Stewart in [**115**] and [**116**]. We essentially follow [**9**].

Let $\big(\Omega, \mathcal{A}(x,D), \mathcal{B}(x,D)\big)$ be a second order elliptic boundary value problem of class $C^\eta$ for some $\eta \in (0,1)$. Define now:

$$X := \begin{cases} C_0(\bar{\Omega}) & \text{(Dirichlet boundary conditions)} \\ \\ C(\bar{\Omega}) & \text{(Neumann or Robin boundary conditions)} \end{cases}$$

where $C_0(\bar{\Omega})$ denotes the space of continuous functions on $\bar{\Omega}$ which vanish at the boundary. We now set

$$D(A) := \{u \in X \cap W^2_{p,\mathcal{B}}(\Omega);\ \mathcal{A}(\cdot\,, D)u \in X\}$$

for some $p \in (1,\infty)$, and

$$Au := \mathcal{A}(\cdot\,, D)u$$

for all $u \in D(A)$. One can show that $D(A)$ is independent of the special choice of $p \in (1,\infty)$ and that the following holds:

> *The operator $-A$ is the infinitesimal generator of an analytic semigroup of compact operators on $X$.*

**E. The Laplace operator on $\mathbb{R}^n$:** It is a classical result in partial differential equations that the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(\cdot\,,0) = u_0 & \text{in } \mathbb{R}^n \end{cases} \tag{1.21}$$

has for each $u \in \mathcal{S}'$(=space of tempered distributions) a unique solution $u$, which can be obtained by convolution of the initial value $u_0$ with the *Gauss-Weierstrass* or *heat kernel*

$$w_t(x) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} .$$

Recall that if $u$ and $v$ are measurable functions, the convolution $u * v$ is the function defined by

$$u * v(x) = \int_{\mathbb{R}^n} u(x-y)v(y)dy$$

for all $x \in \mathbb{R}^n$ such that the integral exists. For $\varphi \in \mathcal{S}$(=Schwartz space) and $u \in \mathcal{S}'$, the convolution is given by

$$u * \varphi(x) = <u, \varphi(x - \cdot)>$$

for all $x \in \mathbb{R}^n$. With these definitions, the solution of (1.21) is representated by

$$U(t)u_0 := u_0 * w_t \tag{1.22}$$

for $t > 0$ and $U(0) := \mathbb{1}$. Observe that $w_t \in \mathcal{S}$ for all $t > 0$. It is now possible to prove the following generation theorem. For the function spaces appearing there consult the corresponding appendix.

**1.5 Theorem**
*Let $X$ be one of the Banach spaces $BUC(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ or $L_p(\mathbb{R}^n)$ $(1 \le p < \infty)$. Then $\big(U(t) \restriction X\big)_{t\ge 0}$ is a strongly continuous analytic semigroup of contractions on $X$. Moreover, the $X$-realisation of $\Delta \in \mathcal{L}\big(\mathcal{D}'(\mathbb{R}^n)\big)$, which we denote by $\Delta_X$, is its infinitesimal generator and $BUC^2(\mathbb{R}^n) \subset D(\Delta_{BUC})$, $C_0^2(\mathbb{R}^n) \subset D(\Delta_{C_0})$, $D(\Delta_{L_p}) \doteq W_p^2(\mathbb{R}^n)$ $(1 < p < \infty)$, and $W_1^2(\mathbb{R}^n) \subset D(\Delta_{L_1})$.*

The semigroup $\big(U(t)\big)_{t>0}$ is called *Gauss-Weierstrass semigroup.* If $X = L_p(\Omega)$, one finds a proof in [**122**], Lemma 2.5.2. The assertion in the other two cases is mathematical folklore, but we were not able to find a proof in the literature. For completeness we give a proof we learned from H. Amann.

But let us first give some facts on convolution and the Fourier transform, on which the proof of the theorem is based. It can be shown that $(u, v) \mapsto u * v$ is a continuous bilinear mapping from $\mathcal{S} \times \mathcal{S}$ into $\mathcal{S}$ and from $\mathcal{S}' \times \mathcal{S}$ into $\mathcal{S}' \cap C^\infty(\mathbb{R}^n)$ (cf. [**106**], Theorem 7.19). Moreover, the convolution is a continuous bilinear mapping from $L_1 \times X$ into $X$ with norm one, where $X$ is any of the Banach spaces from the above theorem. In particular – using definition (1.22) – it is clear that

$$U(t)X \subset X \tag{1.23}$$

for all $t \geq 0$.

Let now $\mathcal{F} \in \mathcal{GL}(\mathcal{S}) \cap \mathcal{GL}(\mathcal{S}')$ be the *Fourier transformation*, which is for each $u \in L_1(\mathbb{R}^n)$ defined by

$$\hat{u}(\xi) := \mathcal{F}u(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i(x|\xi)_{\mathbb{R}^n}} u(x)\, dx \tag{1.24}$$

for all $\xi \in \mathbb{R}^n$. For $u \in \mathcal{S}'$, the Fourier transform is given by

$$< \mathcal{F}u, \varphi > := < u, \mathcal{F}\varphi > \tag{1.25}$$

for all $\varphi \in \mathcal{S}$. It is not hard to check that

$$u * w_t = \mathcal{F}^{-1} e^{-t|x|^2} \mathcal{F}u \tag{1.26}$$

holds for all $u \in \mathcal{S}'$ (cf. [**70**], Sect. 7.6), which immediately implies the semigroup property

$$U(t+s)u_0 = U(t)U(s)u_0 \tag{1.27}$$

for all $u_0 \in \mathcal{S}'$. Now we are ready to give the proof of Theorem 1.5.

**Proof of Theorem 1.5**

*Step 1:* First we want to show that $U \restriction X$ defines a strongly continuous semigroup on $X$. By (1.23), X is invariant under $U(t)$ for all $t \geq 0$. Put

$$w(x) := (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}.$$

Then $w_t(x) = t^{-\frac{n}{2}} w\big(\frac{x}{\sqrt{t}}\big)$ and $\|w\|_{L_1} = \|w_t\|_{L_1} = 1$. This implies that

$$\lim_{t \to 0} u * w_t = u$$

in $X$ (see e.g. [**57**], Prop. 8.14]), i.e. strong continuity. Taking into account that the norm of the convolution $* : L_1 \times X \to X$ is one, the assertion follows.

*Step 2:* Next we want to show that $\Delta_X$ is the generator of $\big(U(t)\big)_{t \geq 0}$ in X. To do this we need that

$$\lim_{t \to 0} \frac{U(t)u - u}{t} = \Delta u \tag{1.28}$$

in $\mathcal{S}$ and $\mathcal{S}'$ whenever $u \in \mathcal{S}$ or $\mathcal{S}'$ respectively. By the semigroup property (1.27) it is now clear that for any $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$ the right derivative $\partial_t^+ < u, U(t)\varphi > = < u, \Delta U(t)\varphi >$ exists for $t \geq 0$ and is continuous. Therefore,

$$< u, U(t)\varphi > \in C^1(\mathbb{R}_+) \quad \text{and} \quad \partial_t < u, U(t)\varphi > = < u, \Delta U(t)\varphi > \tag{1.29}$$

for all $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. To prove (1.28) observe that by (1.26)

$$\frac{U(t)u - u}{t} - \Delta u = \mathcal{F}^{-1}\frac{e^{-t|\xi|^2} - 1 + t|\xi|^2}{t}\mathcal{F}u \tag{1.30}$$

for all $u \in \mathcal{S}'$. Taylor's Formula applied to $t \mapsto e^{-t|\xi|^2}$ gives then

$$\frac{e^{-t|\xi|^2} - 1 + t|\xi|^2}{t} = \frac{|\xi|^4}{2t}\int_0^t (t-\tau)e^{-\tau|\xi|^2}\,d\tau =: f_t(\xi)\,.$$

Leibniz's rule and the definition of the seminorms in $\mathcal{S}$ (see the appendix) imply that for any $\varphi \in \mathcal{S}$ $f_t\varphi$ tends to zero when $t$ approaches zero. Since $\mathcal{F} \in \mathcal{GL}(\mathcal{S})$, the limit (1.28) exists in $\mathcal{S}$ for all $u \in \mathcal{S}$. On the other hand, we have

$$<\frac{U(t)u - u}{t}, \varphi> = <u, \frac{U(t)\varphi - \varphi}{t}>$$

and thus

$$\lim_{t\to 0}<\frac{U(t)u - u}{t}, \varphi> = <u, \Delta\varphi> = <\Delta u, \varphi>$$

holds for all $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. Since pointwise convergence implies convergence in $\mathcal{S}'$, (1.28) is proved.

Let now $u \in D(A_X)$, where $A_X$ is the infinitesimal generator of the semigroup $\big(U(t)\big)_{t\geq 0}$ in $X$. Then – by definition of the generator and (1.28), it follows that

$$\lim_{t\to 0}\frac{U(t)u - u}{t} = \begin{cases} A_X u & \text{in } X \\ \Delta u & \text{in } \mathcal{S}'\,. \end{cases} \tag{1.31}$$

Since $X \hookrightarrow \mathcal{S}'$, it follows that $A_X u = \Delta_X u$. Conversely, if $u \in D(\Delta_X)$, we have $\Delta_X \in X$ and because $\big(U(t)\big)_{t\geq 0}$ is strongly continuous we get

$$\lim_{t\to 0}\frac{1}{t}\int_0^t U(t)\Delta u\,d\tau = \Delta u\,.$$

If we can show that

$$\frac{1}{t}\int_0^t U(t)\Delta u\,d\tau = \frac{1}{t}\big(U(t)u - u\big) \tag{1.32}$$

it follows immediately that $u \in D(A_X)$ and $A_X u = \Delta u$. To prove this observe first, that $\mathcal{S} \hookrightarrow X \hookrightarrow \mathcal{S}'$ and $\mathcal{S} \overset{d}{\hookrightarrow} \mathcal{S}'$ yields $X \overset{d}{\hookrightarrow} \mathcal{S}'$, and, by reflexivity of $\mathcal{S}$, $\mathcal{S} = \mathcal{S}'' \hookrightarrow X'$. We can therefore identify the elements of $\mathcal{S}$ with continuous functionals on $X$. In this sense, using (1.29), we get for all $\varphi \in \mathcal{S}$

$$
\begin{aligned}
<\int_0^t U(\tau)\Delta u \, d\tau, \varphi>_{\mathcal{S}} &= <\varphi, \int_0^t U(\tau)\Delta u \, d\tau>_X = \int_0^t <\varphi, U(\tau)\Delta u>_X \, d\tau \\
&= \int_0^t <U(\tau)\Delta u, \varphi>_{\mathcal{S}} \, d\tau = \int_0^t <u, \Delta U(\tau)\varphi>_{\mathcal{S}} \, d\tau = \int_0^t \partial_\tau <u, U(\tau)\varphi>_{\mathcal{S}} \, d\tau \\
&= <U(t)u - u, \varphi>_{\mathcal{S}}
\end{aligned}
$$

and (1.32) follows.

*Step 3:* To show that the semigroup $\big(U(t)\big)_{t\geq 0}$ is analytic as a semigroup on $X$, we are going to verify conditions $(1')$ and $(2')$ of Theorem 1.1. In order to verify $(1')$, we have only to show that $\big(U(t)\big)_{t\geq 0}$ has the smoothing property (1.11). Strong continuity was proved in Step 1. It is not hard to see that

$$
(1.33) \qquad [t \mapsto w_t] \in C^1\big(0,\infty), L_1(\mathbb{R}^n)\big) \qquad \text{and} \qquad \dot{w}_t(x) = \frac{1}{t}\Big(-\frac{n}{2} + \frac{|x|^2}{4t}\Big) w_t(x).
$$

Since $* : L_1 \times X \to X$ is a continuous bilinear mapping, it follows from (1.33), that $[t \mapsto w_t * u] \in C^1\big((0,\infty), X\big)$ and that $\frac{d}{dt}(w_t * u) = \dot{w}_t * u \in X$, that is $U(t)u \in D(A_X)$ for all $u \in X$. It remains to prove $(2')$. We can write

$$
t\,\dot{w}_t = -\frac{n}{2}\, w_t + g_t \qquad \text{with} \qquad g_t(x) := \frac{|x|^2}{4t}\, w_t\, .
$$

Using polar coordinates and the $\Gamma$-function, it follows that $\|g_t\|_{L_1} = \frac{n}{2}$ and thus

$$
t\|\dot{w}_t\|_{L_1} \leq n
$$

holds for all $t > 0$. Since $tA_X U(t)u = t\,\dot{w}_t * u$, the assertion follows. □

**F. Diagonal operators:** Suppose that $X^1, \dots, X^n$ are Banach spaces and that for each $i = 1, \dots, N$, there is given a semigroup $(e^{-tA_i})_{t\geq 0}$. Set

$$
X := \prod_{i=1}^N X^i
$$

$$
T(t)(x_1, \dots, x_N) := (e^{-tA_1}x_1, \dots, e^{-tA_N}x_N) \qquad \text{for } (x_1, \dots, x_N) \in X,
$$

$$
D(A) := \prod_{i=1}^N D(A_i),
$$

and

$$A(x_1,\dots,x_N) := (A_1x_1,\dots,A_Nx_N) \qquad \text{for } (x_1,\dots,x_N)\in D(A).$$

Then:

> $\big(T(t)\big)_{t\ge 0}$ *is a $C_0$-semigroup on $X$ with infinitesimal generator $-A$. If each of the semigroups $(e^{-tA_i})_{t\ge0}$, $i=1,\dots,N$, is analytic, then so is $(T(t))_{t\ge0}$.*

**Notes and references:** A few books on semigroups are Clément et al. [**31**], Butzer and Berens [**26**], Davies [**43**], Fattorini [**55**], Goldstein [**63**], Hille and Philips [**69**], Pazy [**100**]. We are particularly fond of Clément et al. [**31**], Pazy [**100**] and Fattorini [**55**].

The perturbation result in Theorem 1.3($b$) goes back to Desch and Schappacher [**46**], see also [**31**].

In Robinson [**105**], Theorem V.2.7 a proof of Theorem 1.5 in much greater generality can be found.

## 2. The evolution operator

Let $X_0$ and $X_1$ be Banach spaces satisfying

$$X_1 \overset{d}{\hookrightarrow} X_0.$$

We denote their norms by $\|\cdot\|$ and $\|\cdot\|_1$ respectively.

We fix a number T>0 and consider a family

$$\big(A(t)\big)_{0\le t\le T}$$

of closed linear operators in $X_0$ having the following properties:

($A1$) $D(A(t)) = X_1$ for all $t\in[0,T]$.

($A2$) For all $t\in[0,T]$ we have

$$[\operatorname{Re}\mu\ge 0]\subset \varrho(-A(t)).$$

Furthermore, there exists a constant $M\ge 1$, independent of $t\in[0,T]$, such that

$$\|\big(\lambda + A(t)\big)^{-1}\| \le \frac{M}{1+|\lambda|}$$

for all $(\lambda,t)\in[\operatorname{Re}\mu\ge0]\times[0,T]$.

($A3$) There exists a constant $\rho\in(0,1)$, such that

$$A(\cdot)\in C^{\rho}\big([0,T],\mathcal{L}(X_1,X_0)\big).$$

**2.1 Remarks**

(a) Assumptions $(A1)$ and $(A2)$ imply by Theorem 1.1 that for each $t \in [0,T]$ the operator $-A(t)$ is the infinitesimal generator of a strongly continuous analytic semigroup on $X_0$ .

(b) Recall that the set $\mathrm{Isom}(X_1, X_0)$ is open in $\mathcal{L}(X_1, X_0)$ and that the mapping

$$[B \mapsto B^{-1}]\colon \mathrm{Isom}(X_1, X_0) \to \mathcal{L}(X_0, X_1)$$

is analytic. This, together with $(A3)$, implies that

$$A^{-1}(\cdot) \in C^{\rho}\big([0,T], \mathrm{Isom}(X_0, X_1)\big)$$

holds.

(c) Assume $t \in [0,T]$. Since $0 \in \varrho(A(t))$, the norm on $X_1$ defined by $|||x|||_t := \|A(t)x\|$, is equivalent to the graph norm with respect to $A(t)$. Furthermore, for each $x \in X_1$ we have by $(A3)$ that

$$|||x|||_t = \|A(t)x\| \le \|A(t)\|_{\mathcal{L}(X_1,X_0)} \|x\|_1 \le c_1 \|x\|_1,$$

with a positive constant $c_1$ not depending on $t \in [0,T]$. Furthermore, the preceding remark implies that for each $x \in X_1$

$$\|x\|_1 \le \|A(t)^{-1}\|\, \|A(t)x\| \le c_2 |||x|||_t$$

holds with a positive constant $c_2$ which is also independent of $t \in [0,T]$. These two inequalities imply then that the norms $|||\cdot|||_t$ and $\|\cdot\|_1$ are equivalent, uniformly in $t \in [0,T]$. We will often use this fact without further mention.

(d) Condition $(A2)$ implies that there exists an angle $\alpha \in (0, \frac{\pi}{2}]$ and a constant $C \ge 1$ – both independent of $t \in [0,T]$ – such that (1.19) and (1.20) hold for all $t \in [0,T]$. □

We are interested in solving the following abstract *Cauchy-problem*

$$\text{(2.1)} \qquad \begin{cases} \partial_t u + A(t)u = f(t) & \text{for } t \in (s,T] \\ u(s) = x, & \end{cases}$$

where $s \in [0,T)$, $x \in X_0$, and $f \in C([s,T], X_0)$ are the *initial time, initial value* and *inhomogenity* respectively.

**2.2 Definition**

By a *solution* of (2.1) we mean a function

$$u \in C\big([s,T], X_0\big) \cap C^1\big((s,T], X_0\big)$$

such that $u(t) \in X_1$ for $t \in (s,T]$ and $u(s) = x$, satisfying $\partial_t u(t) + A(t)u(t) = f(t)$ for $t \in (s,T]$. □

At this point it is convenient to introduce the notion of an evolution operator for $\big(A(t)\big)_{0\le t\le T}$. Our definition is a specialization of the one used by Amann in [**13**]. We will need the following notation:

$$\Delta_T := \{(t,s);\ 0 \le s \le t \le T\} \qquad \text{and} \qquad \dot{\Delta}_T := \{(t,s);\ 0 \le s < t \le T\}.$$

**2.3 Definition**
An *evolution operator* for the family $\big(A(t)\big)_{0\le t\le T}$ is a mapping

$$U\colon \Delta_T \to \mathcal{L}(X_0)$$

satisfying the following properties:

$(U1)$ $U \in C\big(\Delta_T, \mathcal{L}_s(X_0)\big) \cap C\big(\Delta_T, \mathcal{L}_s(X_1)\big) \cap C\big(\dot{\Delta}_T, \mathcal{L}(X_0, X_1)\big)$.

$(U2)$ $U(t,t) = \mathbb{1}_{X_0}$, $U(t,s) = U(t,\tau)U(\tau,s)$ for all $0 \le s \le \tau \le t \le T$.

$(U3)$ $[(t,s) \mapsto A(t)U(t,s)] \in C\big(\dot{\Delta}_T, \mathcal{L}(X_0)\big)$ and

$$\sup_{(t,s)\in\dot{\Delta}_T} (t-s)\|A(t)U(t,s)\| < \infty.$$

$(U4)$ $U(\cdot\,,s) \in C^1\big((s,T], \mathcal{L}(X_0)\big)$ for each $s \in [0,T)$, and for all $t \in (s,T]$:

$$\partial_1 U(t,s) = -A(t)U(t,s)$$

$U(t,\cdot) \in C^1\big([0,t), \mathcal{L}_s(X_1,X_0)\big)$ for each $t \in (0,T]$, and for all $s \in [0,t)$:

$$\partial_2 U(t,s)x = U(t,s)A(s)x$$

for all $x \in X_1$. □

**2.4 Remark**
By Remark 2.1(c) and $(U3)$ one immediately obtains

$$\|U(t,s)\|_{\mathcal{L}(X_0,X_1)} \le c(t-s)^{-1}$$

for all $(t,s) \in \dot{\Delta}_T$ and a suitable positive constant $c$. Here we recover the estimate (1.18) of analytic semigroups. □

The usefulness of an evolution operator is illustrated by the following

**2.5 Theorem**
*There exists at most one evolution operator $U$ for the family $\big(A(t)\big)_{0\le t\le T}$. Furthermore, if $u$ is any solution of* (2.1) *it follows that*

$$u(t) = U(t,s)x + \int_s^t U(t,\tau)f(\tau)\,d\tau \tag{2.2}$$

*holds for every $t\in[0,T]$.*
**Proof**
Let u be a solution of (2.1). By $(U4)$ we have for any $s<r<t\le T$

$$\partial_2 U(t,r)u(r) = U(t,r)u'(r) + U(t,r)A(r)u(r) = U(t,r)f(r)$$

Integrating both sides from $\varepsilon\in(s,t)$ to $t$ and letting $\varepsilon\to s$ we obtain the representation formula.

It is clear by $(U4)$ that the uniqueness of an evolution operator follows from this formula. □

The above representation formula – which will be instrumental in the treatment of semilinear equations in Chapter IV – is usually referred to as the *variation-of-constants formula*. We will adhere to this terminology and use it without further reference.

The preceding theorem shows that the following result, which was proved independently by Sobolevskii [**114**] and Tanabe [**118**], is crucial for the study of the solvability of (2.1):

**2.6 Theorem**
*There exists a unique evolution operator for the family $\big(A(t)\big)_{0\le t\le T}$. Moreover,*

$$\|U(t,s)\|_{\mathcal{L}(X_i)}\quad (i=0,1)\qquad\text{and}\qquad (t-s)\|A(t)U(t,s)\|$$

*are bounded uniformly in $(t,s)\in\Delta_T$ with a constant only depending on $M$, $\rho$, the Hölder norm of $A(\cdot)$ and a bound for $\|A(t)A^{-1}(s)\|$.*

We do not want to give the complete proof of this theorem, since it is rather long and technical. A complete proof may be found in [**100**], Section 5.6, [**119**], Section 5.2 or [**74**].

**Sketch of the proof**

Assume that there exists an evolution operator $U$ for the family of operators $\big(A(t)\big)_{0\le t\le T}$ of closed operators. We can write $U(t,s)$ as a perturbation of the semigroup generated by $-A(s)$:

$$U(t,s) = e^{-(t-s)A(s)} + W(t,s) \qquad \big((t,s)\in\Delta_T\big). \tag{2.3}$$

It is obvious, that $W(t,t)=0$ for all $t\in[0,T]$. Then, by $(U4)$ and (2.3), we get that

$$\begin{aligned}\partial_1 U(t,s) &= -A(t)U(t,s) = -A(t)\big(e^{-(t-s)A(s)} + W(t,s)\big)\\ &= -A(s)e^{-(t-s)A(s)} + \partial_1 W(t,s).\end{aligned}$$

But this implies that $W$ satisfies the equation

$$\begin{cases}\partial_1 W(t,s) + A(t)W(t,s) = -\big(A(t)-A(s)\big)e^{-(t-s)A(s)} =: R_1(t,s) & \text{for } (t,s)\in\dot{\Delta}_T\\ W(s,s)=0\end{cases}$$

By the variation-of-constants formula (2.2), $W(t,s)$ can be written as

$$W(t,s) = \int_s^t U(t,\tau)R_1(\tau,s)\,d\tau \tag{2.4}$$

and thus $U$ solves the integral equation

$$U(t,s) = e^{-(t-s)A(s)} + \int_s^t U(t,\tau)R_1(\tau,s)\,d\tau\,. \tag{2.5}$$

This is a Volterra integral equation of the second kind. Since, in general, $X_1\neq X_0$, it is clear that the kernel

$$R_1(t,s) = -\big(A(t)-A(s)\big)e^{-(t-s)A(s)}$$

has a singularity at $t=s$ as an operator from $\dot{\Delta}_T$ to $\mathcal{L}(X_0)$. Using (1.18), Remark 2.1$(c)$ and assumption $(A3)$, we find a constant $c>0$ independent of $(t,s)\in\dot{\Delta}_T$ such that

$$\|R_1(t,s)\| \le \|A(t)-A(s)\|_{\mathcal{L}(X_1,X_0)}\|e^{-(t-s)A(s)}\|_{\mathcal{L}(X_0,X_1)} \le c(t-s)^{\rho-1}.$$

Thus the singularity is integrable and the integral (2.4) really exists in $\mathcal{L}_s(X_0)$. Moreover, this makes clear why Hölder continuity and not only continuity is required for $A(\cdot)$. The general procedure to solve an integral equation like (2.5) is will be described in Section 9.A. □

**2.7 Remarks**

(a) From the variation-of-constants formula it follows that the Cauchy-problem has at most one solution. Moreover, to prove existence it suffices to prove that the function $v\colon[0,T]\to X_0$ defined by

$$v(t) := \int_s^t U(t,\tau)f(\tau)\,d\tau \quad \text{for } t\in[s,T]$$

lies in $C\big([s,T],X_0\big)\cap C^1\big((s,T],X_0\big)$ and solves the Cauchy-problem (2.1) if $x=0$. This is clear from the fact that $u(\cdot):=U(\cdot\,,s)x$ solves (2.1) with $f\equiv 0$ and thus, by linearity, $u+v$ solves (2.1).

(b) The above results remain valid if we replace assumption $(A2)$ by

$(A2')$ For all $t\in[0,T]$ we have for some $\lambda_0\geq 0$,

$$[\operatorname{Re}\mu\geq\lambda_0]\subset\varrho\big(-A(t)\big).$$

Furthermore, there exists a constant $M>0$, independent of $t\in[0,T]$ such that

$$\|(A(t)+\lambda)^{-1}\|\leq\frac{M}{1+|\lambda|}$$

for all $(\lambda,t)\in[\operatorname{Re}\mu\geq\lambda_0]\times[0,T]$.

Indeed, setting $A_1(t):=\lambda_0+A(t)$ we see that the family $\big(A_1(t)\big)_{t\in[0,T]}$ satisfies $(A1)$–$(A3)$. Let $U_1$ be the corresponding evolution operator. Then

$$U(t,s):=e^{(t-s)\lambda_0}U_1(t,s)$$

defines the evolution operator for $\big(A(t)\big)_{t\in[0,T]}$. □

A partial answer to the question of solvability of (2.1) is given by the following well known result which also goes back to Sobolevskiĭ [**114**] and Tanabe [**118**]:

**2.8 Theorem**

*For any $s\in[0,T)$, $x\in X_0$ and $f\in C^\nu\big([s,T],X_0\big)$, $\nu\in(0,1)$, there exists a unique solution of* (2.1)*. The solution is given by the variation-of-constants formula.*

The proof of this theorem uses estimates and differentiability properties of $W$ defined in (2.14). The Hölder continuity is needed for the existence of certain integrals. For a proof we refer to [**119**], Theorem 5.2.3 or [**100**] Theorem 5.7.1.

**2.9 Examples**

(a) Assume that we have $A:=A(0)=A(t)$ for every $t\in[0,T]$. Then the evolution operator is given by $U(t,s)=e^{-(t-s)A}$ for $0\leq s\leq t<\infty$.

(b) Assume that $-A\colon X_1\to X_0$ is the infinitesimal generator of a strongly continuous analytic semigroup on $X_0$, and that $k\in C\big([0,T],\mathbb{R}\big)$ is a strictly positive function. Define $A(t)$ by $k(t)A$ for each $t\in[0,T]$. Then

(2.6) $$U(t,s):=e^{-\int_s^t k(\tau)\,d\tau A}$$

for $(t,s) \in \Delta_T$ is the evolution operator to the family $\big(A(t)\big)_{0\le t\le T}$.

(c) Let $A(\cdot) \in C\big([0,T], \mathcal{L}(X_0)\big)$. Then there exists a unique evolution operator $U(t,s)$ for the family $\big(A(t)\big)_{0\le t\le T}$. Obviously this evolution operator satisfies the integral equation

$$U(t,s)x = x - \int_s^t A(\tau)U(\tau,s)x\,d\tau \tag{2.7}$$

for all $x \in X_0$ and all assertions in $(U1)$–$(U4)$ hold in the uniform operator topology. Of course, in this case $X_0 = X_1$. For details we refer to Theorem 9.3.

(d) Linear parabolic equations: let $\Omega$ be a bounded domain of class $C^\infty$ in $\mathbb{R}^n$ for some $n \ge 1$ and consider for each $t \in [0,T]$ a *linear elliptic differential expression of second order*

$$\mathcal{A}(x,t,D)u := -\sum_{j,k=1}^{n} a_{jk}(x,t)\partial_j\partial_k u + \sum_{j=1}^{n} a_j(x,t)u + a_0(x,t)u, \tag{2.8}$$

where $a_{jk} = a_{kj}$, $a_j$ and $a_0$ are $C^{\eta,\frac{\eta}{2}}$-functions on $\Omega \times [0,T]$ for some $\eta \in (0,1)$, such that for some constant $\underline{\alpha} > 0$

$$\sum_{j,k=1}^{n} a_{jk}(x,t)\xi_j\xi_k \ge \underline{\alpha}|\xi|^2, \tag{2.9}$$

whenever $(x,t) \in \Omega \times [0,T]$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

Thus

$$\mathcal{L}(x,t,D) := \partial_t + \mathcal{A}(x,t,D)$$

is a *linear uniformly parabolic differential expression of second order*. We now let $\mathcal{B}(x,D)$ be a boundary operator exactly as described in Section 1.C. Thus, for each $t \in [0,T]$ the triple $\big(\Omega, \mathcal{A}(x,t,D), \mathcal{B}(x,D)\big)$ is an elliptic boundary value problem of class $C^\eta$.

Consider the following *inhomogeneous parabolic initial-boundary value problem*:

$$\begin{cases} \partial_t u(x,t) + \mathcal{A}(x,t,D)u(x,t) = h(x,t) & (x,t) \in \Omega \times (0,T] \\ \mathcal{B}(x,D)u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{2.10}$$

where the inhomogenity $h$ lies in $C^{\eta,\frac{\eta}{2}}\big(\bar{\Omega} \times [0,T]\big)$ – consult the Appendix for notation – and the initial value $u_0$ is a given function from $\Omega$ into $\mathbb{R}$.

Let $p \in (1,\infty)$ and set $X_0 = L_p(\Omega)$. Define now $A(t)$, for each $t \in [0,T]$, as the $X_0$-realization of $\big(\Omega, \mathcal{A}(x,t,D), \mathcal{B}(x,D)\big)$ as described in Section 1.C. Then the family $\big(A(t)\big)_{0\le t\le T}$ satisfies assumptions $(A1)$, $(A2')$ and $(A3)$ (For references see Section 1.C).

Furthermore, we set $f(t) := h(\cdot\,,t)$. Then $f \in C^{\frac{\eta}{2}}\big([0,T],X_0\big)$. The $L_p(\Omega)$-*formulation* of (2.10) is then the abstract Cauchy-problem (2.1).

Weaker regularity assumptions on $\Omega$, $a_{jk}$, $a_j$, $a_0$, $b_j$, $b_0$ and $h$ would suffice to put (2.10) in an $L_p$-setting. We shall not take pains to specify these assumptions since we shall need the present regularity in order to prove that $L_p$-solutions of nonlinear problems are in fact classical solutions (see Section 24).

Alternatively, we could have used the 'continuous-setting' of Section 1.D to get the $C(\bar{\Omega})$-*formulation* of (2.10).

(e) As a final example we would like to consider diagonal systems. Suppose that $X_0^i$ and $X_1^i$ $(i = 1,\dots,N)$ are Banach spaces such that $X_1^i \overset{d}{\hookrightarrow} X_0^i$ for all $i = 1,\dots,N$. Moreover, let $\big(A_i(t)\big)_{0\le t\le T}$ be for each $i = 1,\dots,N$ be a family of closed operators on $X_0^i$ with domain $X_1^i$ satisfying conditions $(A1)$–$(A3)$. Set

$$X_0 := \prod_{i=1}^{N} X_0^i \qquad \text{and} \qquad X_1 := \prod_{i=1}^{N} X_1^i$$

and define a linear operator $A(t)\colon X_0 \supset D(A(t)) \to X_0$ by

$$A(x_1,\dots,x_N) := (A_1x_1,\dots,A_Nx_N) \qquad \text{for } (x_1,\dots,x_N) \in D(A)\,,$$

where $D(A(t)) := \prod_{i=1}^{N} D(A_i(t))$. Then, the family $\big(A(t)\big)_{0\le t\le T}$ satisfies conditions $(A1)$–$(A3)$. The evolution operator associated to this family is given by

$$U(t,s)(x_1,\dots,x_N) := \big(U_1(t,s)x_1,\dots,U_N(t,s)x_N\big) \qquad \text{for } (x_1,\dots,x_N) \in X,$$

where $U_i(\cdot\,,\cdot)$ are the evolution operators to the family $\big(A_i(t)\big)_{0\le t\le T}$ $(i = 1,\dots,N)$. □

**Notes and references:** The construction of the evolution operator under the present conditions for abstract parabolic equations goes back to Sobolevskiĭ [**114**] and Tanabe [**118**]. A wholy different method of proof, namely by Yosida approximations was developped by Kato [**74**]. The case where $D(A(t))$ depends on time has also been considered by a variety of authors such as Amann [**13**], Kato[**74**], Lunardi [**92**], Sobolevskiĭ [**114**], Tanabe [**119**] and Yagi [**126**] to name just a few. This kind of situation arises for instance when dealing with initial boundary value problems where the boundary operator depends on time. For a brief description of such a generalization see the Notes and References at the end of Section 5.

## 3. Interpolation spaces

In this section we review those concepts of interpolation theory which are necessary for our purposes. Our intention is to apply interpolation theory rather than to develop it.

This means that we will not dwell at length in the abstract theory, developing it as far it is convenient for our purposes. Actually we will need little more than a few basic definitions in order to deal with abstract evolution equations.

**A. Why interpolation theory?** We start by trying to motivate, quite formally, the use of interpolation theory in semilinear evolution equations. In the introduction we alluded to the problem that in the abstract formulation of a concrete semilinear parabolic equation, the nonlinear term might fail to be well-defined on the underlying space $X_0$, but may well turn out to be a 'nice' function on a smaller space $Z \hookrightarrow X_0$. It is to be expected that in order to develop a systematic theory for such an equation we will have to restrict the choice of $Z$ according to certain criteria. But: which are these criteria?

Let us take a closer look at the problem. The equation under consideration is:

$$(*) \qquad \begin{cases} \partial_t u(t) + A(t)u(t) = g\big(t, u(t)\big) & \text{for } t \in (s, T] \\ u(s) = x \end{cases}$$

where $g\colon [0,T] \times Z \to X_0$ is a given function, $\big(A(t)\big)_{0 \le t \le T}$ is a family of closed linear operators on $X_0$ satisfying $(A1)$–$(A3)$ from the preceding section and $(s,x) \in [0,T) \times Z$. A *solution* of $(*)$ is a function $u \in C\big([s,T], Z\big) \cap C^1\big((s,T], X_0\big)$, such that $u(t) \in X_1$ for all $t > s$, satisfying $(*)$. As usual, we have set $D(A(t)) \doteq X_1$. Because $u(t) \in X_1$ for $t > s$ it is clear that $Z$ should contain $X_1$, i.e. we require

$$X_1 \hookrightarrow Z \hookrightarrow X_0.$$

To prove the existence of solutions of $(*)$, the idea is to mimic the proof in the case of ordinary differential equations. Assuming we have a solution $u$ of $(*)$, we conclude from the variation-of-constants formula that

$$(**) \qquad u(t) = U(t,s)x + \int_s^t U(t,\tau)g\big(\tau, u(\tau)\big)d\tau$$

for all $t \in [0,T]$. Define now for each $T_1 \in (s,T]$, $t \in [s, T_1]$, and $u \in C\big([s,T_1], Z\big)$

$$G_{T_1}(u)(t)$$

by the right-hand side of $(**)$. We would like to show that $G_{T_1}$ is a contraction on $\overline{\mathbb{B}}_{C([s,T_1],Z)}(x,\varepsilon)$ for a suitable $T_1 \in (s,T]$ and $\varepsilon > 0$. From this we get the existence of a unique fixed point which is – ignoring some technicalities – a local solution of $(*)$. Looking at the straight-forward estimates which are needed to show this (see proof of Lemma 16.1) we see that one has to require that $U(t,s)$ can be viewed as a bounded operator on $Z$, and that we have some estimates for $\|U(t,s)\|_{\mathcal{L}(Z)}$.

Let us summarize. We have started with a pair of spaces $X_1 \overset{d}{\hookrightarrow} X_0$ and operators $U(t,s) \in \mathcal{L}(X_1) \cap \mathcal{L}(X_0)$ and would like to find a space $Z$, such that:

(1) $X_1 \hookrightarrow Z \hookrightarrow X_0$,

(2) $U(t,s) \in \mathcal{L}(Z)$,

(3) Estimates for $\|U(t,s)\|_{\mathcal{L}(Z)}$ should be available.

This is just what interpolation theory delivers.

**B. Interpolation spaces:** As always we consider Banach spaces over the fixed field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

A pair $\overline{E} = (E_0, E_1)$ of Banach spaces satisfying $E_1 \overset{d}{\hookrightarrow} E_0$ is called a *Banach couple*. The class of all Banach couples (over $\mathbb{K}$) will be denoted by $\mathfrak{B}_2$ and the class of all Banach spaces (over $\mathbb{K}$) by $\mathfrak{B}_1$. In the sequel $\overline{E} = (E_0, E_1)$, $\overline{F} = (F_0, F_1)$ and $\overline{G} = (G_0, G_1)$ will denote arbitrary Banach couples. A *map*, or *morphism, of Banach couples* $T: \overline{E} \to \overline{F}$ is a linear mapping $T \in \mathcal{L}(E_0, F_0) \cap \mathcal{L}(E_1, F_1)$. A morphism $T: \overline{E} \to \overline{F}$ is called an *isomorphism*, if $T \in \mathrm{Isom}(E_0, F_0)$ and $T \in \mathcal{L}(E_1, F_1)$ is surjective. Note that if this is the case we also have $T \in \mathrm{Isom}(E_1, F_1)$ by the open mapping theorem. We denote by $\mathcal{L}(\overline{E}, \overline{F})$ and $\mathrm{Isom}(\overline{E}, \overline{F})$ the sets of all morphisms and isomorphisms $T: \overline{E} \to \overline{F}$, respectively. Furthermore, we set $\mathcal{L}(\overline{E}) := \mathcal{L}(\overline{E}, \overline{E})$ and $\mathcal{GL}(\overline{E}) := \mathrm{Isom}(\overline{E}, \overline{E})$.

Any Banach space $X$ satisfying $E_1 \hookrightarrow X \hookrightarrow E_0$ is called an *intermediate space with respect to* $\overline{E}$. Note that while the second imbedding is automatically dense, this needs not be the case for the first one.

A pair $(X, Y)$ of Banach spaces (not necessarily a Banach couple) is called *pair of interpolation spaces with respect to the pair* $(\overline{E}, \overline{F})$ *of Banach couples* if the following two conditions are met:

$(I1)$ $X$ and $Y$ are intermediate spaces with respect to $\overline{E}$ and $\overline{F}$, respectively

and

$(I2)$ For any $T \in \mathcal{L}(\overline{E}, \overline{F})$ we have: $T \in \mathcal{L}(X, Y)$.

Observe that condition $(I2)$ is a very strong one indeed. We are actually requiring that $T$ maps $X$ into $Y$ , and that it is continuous with respect to their topologies, and this for any map $T \in \mathcal{L}(\overline{E}, \overline{F})$.

### 3.1 Definition

An *interpolation method* is a mapping

$$\mathfrak{F}: \mathfrak{B}_2 \to \mathfrak{B}_1$$

such that

$(F1)$ For any pair $(\overline{E}, \overline{F})$ of Banach couples
$\big(\mathfrak{F}(\overline{E}), \mathfrak{F}(\overline{F})\big)$ is a pair of interpolation spaces for $(\overline{E}, \overline{F})$. □

The situation described by $(F1)$ is best remembered in the form of the following diagram:

$$\begin{array}{ccccc} F_1 & \hookrightarrow & \mathfrak{F}(\overline{F}) & \hookrightarrow & F_0 \\ \uparrow{\scriptstyle T} & & \uparrow{\scriptstyle T} & & \uparrow{\scriptstyle T} \\ E_1 & \hookrightarrow & \mathfrak{F}(\overline{E}) & \hookrightarrow & E_0 \end{array}$$

The following result shows that an interpolation method is, in a certain way, well behaved. Recall that as usual '$\doteq$' means 'equal up to equivalent norms'.

**3.2 Proposition**
*Let $\mathfrak{F}$ be an interpolation method and $\overline{E} = (E_0, E_1)$ as well as $\overline{F} = (F_0, F_1)$ be Banach couples. Then the following statements hold:*
*$(a)$ If $E_0 = E_1$ then*

$$\mathfrak{F}(\overline{E}) \doteq E_0.$$

*$(b)$ If $E_0 \hookrightarrow F_0$ and $E_1 \hookrightarrow F_1$, then*

$$\mathfrak{F}(\overline{E}) \hookrightarrow \mathfrak{F}(\overline{F}).$$

*In particular, if $E_0 \doteq F_0$ and $E_1 \doteq F_1$ then*

$$\mathfrak{F}(\overline{E}) \doteq \mathfrak{F}(\overline{F}).$$

*$(c)$ If $T : \overline{E} \to \overline{F}$ is an isomorphism the same is true for $T : \mathfrak{F}(\overline{E}) \to \mathfrak{F}(\overline{F})$.*

**Proof**
$(a)$ Since $\mathfrak{F}(\overline{E})$ is an intermediate space we have $E_0 \hookrightarrow \mathfrak{F}(\overline{E}) \hookrightarrow E_0$, which implies the assertion.
$(b)$ Denote by $i$ the injection map $E_0 \to F_0$. Then, by assumption $i \in \mathcal{L}(\overline{E}, \overline{F})$, so that by $(I2)$ it holds that $i \in \mathcal{L}\big(\mathfrak{F}(\overline{E}), \mathfrak{F}(\overline{F})\big)$. This proves the first part of $(b)$. The second part is an easy consequence of the first assertion.
$(c)$ If $T \in \mathrm{Isom}(\overline{E}, \overline{F})$ we obviously have that $T \in \mathcal{L}\big(\mathfrak{F}(\overline{E}), \mathfrak{F}(\overline{F})\big)$ is bijective, which – by the open mapping theorem – implies that $T \in \mathrm{Isom}\big(\mathfrak{F}(\overline{E}), \mathfrak{F}(\overline{F})\big)$. □

Observe that in 3.2 we only assert that $\mathfrak{F}(E_0, E_0)$ and $E_0$ have equivalent – and not equal – norms.

When dealing with systems of differential equations it is useful to have a nice formula like $\mathfrak{F}(\overline{E \times F}) = \mathfrak{F}(\overline{E}) \times \mathfrak{F}(\overline{F})$ at one's disposal. To prove such a formula we need a simple lemma which is inspired by a similar result in elementary homological algebra.

**3.3 Lemma**
*Let $E$, $F$ and $G$ be Banach spaces, $i \in \mathcal{L}(E,G)$, $p \in \mathcal{L}(G,F)$ and assume that the sequence*

$$0 \longrightarrow E \xrightarrow{i} G \xrightarrow{p} F \longrightarrow 0$$

*is exact, that is $i$ is injective, $p$ surjective and $\ker p = \operatorname{im} i$. Furthermore, let $s \in \mathcal{L}(F,G)$ be such that $p \circ s = \mathbb{1}_F$. Then $G$ is topologically isomorphic to $E \times F$ and $f \colon E \times F \to G$ defined by*

$$f(x,y) := i(x) + s(y)$$

*for all $(x,y) \in E \times F$ is an isomorphism.*
**Proof**
Obviously $f \in \mathcal{L}(E \times F, G)$ such that by the open mapping theorem it suffices to prove that $f$ is bijective.

First we show that $f$ is injective. Suppose that $f(x,y) = i(x) + s(y) = 0$ for some $(x,y) \in E \times F$. Then $\operatorname{im} i = \ker p$ and $p \circ s = \mathbb{1}_F$ imply that $0 = p \circ f(x,y) = y$. Now, $x = 0$ follows from the injectivity of $i$ and thus $f$ is injective.

Let now $z \in G$ be arbitrary. Put $y := p(z)$ and observe that by $p \circ s = \mathbb{1}_F$ we get $p\bigl(z - s(y)\bigr) = 0$, which means that $z - s(y) \in \ker p$. Since by assumption $\ker p = \operatorname{im} i$, there exists an $x \in E$ such that $i(x) = z - s(y)$. A simple calculation shows that $f(x,y) = z$ and thus $f$ is onto and the assertion follows. $\square$

Now, we are ready to prove the following proposition:

**3.4 Proposition**
*Let $\overline{E} = (E_0, E_1)$ and $\overline{F} = (F_0, F_1)$ be Banach couples and $\mathfrak{F}$ an interpolation method. Then $\overline{E \times F} := (E_0 \times F_0, E_1 \times F_1)$ is a Banach couple and*

$$\mathfrak{F}(\overline{E \times F}) \doteq \mathfrak{F}(\overline{E}) \times \mathfrak{F}(\overline{F}).$$

**Proof**
Define $i \in \mathcal{L}(\overline{E}, \overline{E \times F})$ and $p \in \mathcal{L}(\overline{E \times F}, \overline{F})$ for all $(x,y) \in E_0 \times F_0$ by $i(x) := (x,0)$ and $p(x,y) := y$, respectively. Moreover, define $s \in \mathcal{L}(\overline{F}, \overline{E \times F})$ by $s(y) := (0,y)$ for all $y \in F_0$. Then the first and the last row of the following diagram are exact sequences as defined in the above Lemma. Furthermore, $s$ satisfies the assumptions required there.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_1 & \xrightarrow{i} & E_1 \times F_1 & \underset{s}{\overset{p}{\rightleftarrows}} & F_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{F}(\overline{E}) & \xrightarrow{i} & \mathfrak{F}(\overline{E \times F}) & \underset{s}{\overset{p}{\rightleftarrows}} & \mathfrak{F}(\overline{F}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_0 & \xrightarrow{i} & E_0 \times F_0 & \underset{s}{\overset{p}{\rightleftarrows}} & F_0 & \longrightarrow & 0
\end{array}$$

By property $(F1)$ of $\mathfrak{F}$, the above diagram is commutative, the middle row is also exact and the map $s$ induced on $\mathfrak{F}(\overline{F})$ satisfies $p \circ s = \mathbb{1}_{\mathfrak{F}(\overline{F})}$. Therefore, the assertion follows from Lemma 3.3. □

If $\mathfrak{F}$ is an interpolation method and we take $X_0$ and $X_1$ as in Subsection A, we see that $\overline{X} = (X_0, X_1)$ is a Banach couple and the Banach space $Z := \mathfrak{F}(\overline{X})$ satisfies the requirements (1) and (2) of that subsection. Requirement (3), however, is not met since, so far, we have no information whatsoever on the norm of an operator $T: \overline{E} \to \overline{F}$ – between Banach couples $\overline{E}$ and $\overline{F}$ – when viewed as an operator $T \in \mathcal{L}\big(\mathfrak{F}(\overline{E}), \mathfrak{F}(\overline{F})\big)$. To remedy this situation we need the following definition.

**3.5 Definition**

An interpolation method $\mathfrak{F}: \mathfrak{B}_2 \to \mathfrak{B}_1$ is called *of exponent* $\theta \in (0,1)$ if there exists a positive constant $c(\mathfrak{F})$ such that

$$(F2) \quad \|T\|_{\mathcal{L}(\mathfrak{F}(\overline{E}),\mathfrak{F}(\overline{F}))} \leq c(\mathfrak{F}) \|T\|^{1-\theta}_{\mathcal{L}(E_0,F_0)} \|T\|^{\theta}_{\mathcal{L}(E_1,F_1)}$$

holds for all $\overline{E}$, $\overline{F} \in \mathfrak{B}_2$ and $T \in \mathcal{L}(\overline{E}, \overline{F})$.

If we can choose $c(\mathfrak{F}) = 1$ then $\mathfrak{F}$ is called *exact of exponent* $\theta$. □

A simple consequence of $(F2)$ is that we can estimate the norm in $\mathfrak{F}(\overline{E})$ by the norms in $E_0$ and $E_1$.

**3.6 Proposition**

*Let $\mathfrak{F}: \mathfrak{B}_2 \to \mathfrak{B}_1$ be an interpolation method of exponent $\theta \in (0,1)$. Then there exists a positive constant $\tilde{c}(\mathfrak{F})$ such that*

$$(F3) \quad \|x\|_{\mathfrak{F}(\overline{E})} \leq \tilde{c}(\mathfrak{F}) \|x\|^{1-\theta}_{E_0} \|x\|^{\theta}_{E_1}$$

for all $\overline{E} \in \mathfrak{B}_2$ and $x \in E_1$.

**Proof**

Let $\overline{F} := (\mathbb{K}, \mathbb{K})$ and set $\mathbb{K}_\theta := \mathfrak{F}(\overline{F})$ and $E_\theta := \mathfrak{F}(\overline{E})$. Take an arbitrary $x \in E_1$ and define $T \in \mathcal{L}(\overline{F}, \overline{E})$ by setting $T\lambda := \lambda x$ for $\lambda \in \mathbb{K}$. We obviously have

$$\|T\|_{\mathcal{L}(\mathbb{K},E_i)} = \|x\|_{E_i} \tag{3.1}$$

for $i = 0, 1, \theta$. Furthermore, by Proposition 3.2 $(a)$, we have $\mathbb{K}_\theta \doteq \mathbb{K}$. Thus there exists a positive constant $\hat{c}(\mathfrak{F})$, such that

$$\|\lambda\|_{\mathbb{K}_\theta} \leq \hat{c}(\mathfrak{F}) |\lambda| \tag{3.2}$$

holds for all $\lambda \in \mathbb{K}$. It is easily verified that

$$\|T\|_{\mathcal{L}(\mathbb{K},E_\theta)} \leq \hat{c}(\mathfrak{F})\|T\|_{\mathcal{L}(\mathbb{K}_\theta,E_\theta)}$$

holds. So, this together with (3.1) and $(F2)$ gives

$$\|x\|_{E_\theta} \leq \hat{c}(\mathfrak{F})\|T\|_{\mathcal{L}(\mathbb{K}_\theta,E_\theta)} \leq \hat{c}(\mathfrak{F})c(\mathfrak{F})\|T\|^{1-\theta}_{\mathcal{L}(\mathbb{K},E_0)}\|T\|^{\theta}_{\mathcal{L}(\mathbb{K},E_1)} \leq \tilde{c}(\mathfrak{F})\|x\|^{1-\theta}_{E_0}\|x\|^{\theta}_{E_1}$$

where we have set $\tilde{c}(\mathfrak{F}) := \hat{c}(\mathfrak{F})c(\mathfrak{F})$. □

Observe that the constant $\tilde{c}(\mathfrak{F})$ in $(F3)$ can be chosen to be equal to the constant $c(\mathfrak{F})$ appearing in $(F2)$ if the imbedding constant $\hat{c}(\mathfrak{F})$ in (3.2) is smaller or equal to 1.

We return to the remark preceding Definition 3.5, taking $\mathfrak{F}$ to be an interpolation method of exponent $\theta \in (0,1)$. Note that the conditions (1)–(3) required in Subsection A are all met by the space $Z = \mathfrak{F}(\overline{X})$, obtained by interpolation between $X_1$ and $X_0$. Moreover, for any Banach couples $\overline{E}$ and $\overline{F}$, we have a fairly precise idea of how the norm of a bounded operator $T \in \mathcal{L}(\overline{E},\overline{F})$ changes when viewing $T$ as an operator in $\mathcal{L}\big(\mathfrak{F}(\overline{E}),\mathfrak{F}(\overline{F})\big)$. This is the reason why interpolation spaces seem to be predestined to play an important rôle in the treatment of semilinear problems.

We now introduce some notation which will be repeatedly used in the subsequent sections. If for each $\theta \in (0,1)$ we have a given exact interpolation method $\mathfrak{F}_\theta$, we will use the notations:

$$E_\theta := (E_0,E_1)_\theta := \mathfrak{F}_\theta(\overline{E})$$

and

$$\|\cdot\|_\theta := \|\cdot\|_{E_\theta}.$$

**C. Admissible families:** Suppose that for each $\theta \in (0,1)$ we have a given exact interpolation method $(\cdot\,,\cdot)_\theta$. The family $\big((\cdot\,,\cdot)_\theta\big)_{\theta\in(0,1)}$ is called *admissible* if the following three conditions are satisfied:

$(AF1)$ $E_1 \overset{d}{\hookrightarrow} E_{\theta_2} \overset{d}{\hookrightarrow} E_{\theta_1} \overset{d}{\hookrightarrow} E_0$
for $0 < \theta_1 < \theta_2 < 1$.

$(AF2)$ The above imbeddings are compact
whenever $E_1 \overset{d}{\hookrightarrow\hookrightarrow} E_0$.

$(AF3)$ $(E_{\theta_1},E_{\theta_2})_\nu \doteq E_\theta$,
where $0 \leq \theta_1 \leq \theta_2 \leq 1$, $\nu \in (0,1)$ and $\theta = (1-\nu)\theta_1 + \nu\theta_2$.

A result of the type $(AF3)$ is called a *reiteration theorem.* A direct consequence of it, is the fact that we obtain a *scale of Banach spaces* $(E_\theta)_{0\le\theta\le 1}$ in the sense of [**85**], i.e. a family of Banach spaces satisfying $(AF1)$ and

$$\|x\|_\theta \le c(\theta,\theta_1,\theta_2)\|x\|_{\theta_1}^{\frac{\theta_2-\theta}{\theta_2-\theta_1}}\|x\|_{\theta_2}^{\frac{\theta-\theta_1}{\theta_2-\theta_1}} \tag{3.3}$$

for all $x \in E_{\theta_2}$ and all $0 \le \theta_1 \le \theta \le \theta_2 \le 1$. Indeed, from the reiteration theorem we get

$$\|x\|_\theta \le c(\theta,\theta_1,\theta_2)\|x\|_{\theta_1}^{1-\nu}\|x\|_{\theta_2}^{\nu} \tag{3.4}$$

for all $0 \le \theta_1 \le \theta_2 \le 1$, $\nu \in (0,1)$ and $\theta$ defined as in $(AF3)$. Choosing now $\nu := \frac{\theta-\theta_1}{\theta_2-\theta_1}$ we immediately obtain (3.3).

Observe that to prove that a given method which has property $(AF1)$ and $(AF3)$ has property $(AF2)$ it suffices to prove that the imbeddings

$$E_1 \overset{d}{\hookrightarrow} E_\theta \overset{d}{\hookrightarrow} E_0 \tag{3.5}$$

are compact for each $\theta \in (0,1)$ whenever $E_1 \overset{d}{\hookrightarrow\hookrightarrow} E_0$. In this case we obtain $(AF2)$ immediately from $(AF3)$.

The reason we will choose to work with admissible families will be evident from the proofs in Section 5, where we will sometimes have to compare the norms of an operator in the different interpolation spaces. One could actually relax a bit the definition of an admissible family at the cost of clarity in the proofs. We will prefer not to do this.

Of course a central question now is: do any admissible families exist? In the next section we will describe the three most widely used interpolation methods: the real, complex and continuous interpolation methods. They supply us with examples of admissible families. We will also give examples of some concrete interpolation spaces associated to certain function spaces arising in the applications.

We emphasize that everything we need from abstract interpolation theory is contained above.

**Notes and references:** Some standard references in interpolation theory are Bergh and Löfström [**22**], Triebel [**122**] or Krein, Petunin and Semenov [**85**]. Further books dealing with certain aspects of interpolation theory are [**26**], [**31**].

Although the assertion of Proposition 3.4 is not at all surprising we were not able to find a proof in this generaltity, i.e. for arbitrary interpolation methods.

## 4. The real, complex and continuous interpolation methods

In this section we describe the three most widely used interpolation methods. These methods play an important rôle in the applications.

**A. The real interpolation method:** In the sequel $\overline{E} = (E_0, E_1)$ will stand for an arbitrary Banach couple over the fixed field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The norms on $E_0$ and $E_1$ will be denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. For any $x \in E_0$ we define the set of $\overline{E}$-*decompositions* by

$$D(x; \overline{E}) := \{(x_0, x_1) \in E_0 \times E_1;\ x_0 + x_1 = x\}. \tag{4.6}$$

We now define a function

$$K(\cdot\,, \cdot\,; \overline{E})\colon (0, \infty) \times E_0 \to \mathbb{R}_+$$

by setting

$$K(t, x; \overline{E}) := \inf\{\|x_0\|_0 + t\|x_1\|_1;\ (x_0, x_1) \in D(x; \overline{E})\}.$$

This function is called the $K$-*functional*. Furthermore, for each $x \in E_0$, the function $K(\cdot\,, x; \overline{E})$ is increasing and concave. It is not difficult to prove that $\big(K(t, \cdot\,; \overline{E})\big)_{t>0}$ is a family of norms on $E_0$ which are all equivalent to $\|\cdot\|_0$.

Let now $\theta \in (0, 1)$ and $1 \le p < \infty$ and define for each $x \in E_0$ the expressions

$$\|x\|_{\theta,p} := \Big(\int_0^\infty \big(t^{-\theta} K(t, x; \overline{E})\big)^p \frac{dt}{t}\Big)^{\frac{1}{p}},$$

and

$$\|x\|_{\theta,\infty} := \sup_{t>0} t^{-\theta} K(t, x; \overline{E}).$$

**4.1 Definition**
For each $\theta \in (0, 1)$ and $1 \le p \le \infty$ set

$$\mathfrak{F}^{\mathbb{R}}_{\theta,p}(\overline{E}) := (E_0, E_1)_{\theta,p} := \{x \in E_0;\ \|x\|_{\theta,p} < \infty\}.$$

With this definition we have the following important theorem.

**4.2 Theorem**
*Let $\theta \in (0, 1)$ and $1 \le p \le \infty$. Equipped with the norm $\|\cdot\|_{\theta,p}$, $(E_0, E_1)_{\theta,p}$ becomes a Banach space and*

$$\mathfrak{F}^{\mathbb{R}}_{\theta,p}\colon \mathfrak{B}_2 \to \mathfrak{B}_1$$

*is an exact interpolation method of exponent $\theta$.*

**Proof**
This follows from Theorem 3.1.2 and 3.4.2($a$) in [**22**]. □

We shall call $(\cdot\,, \cdot)_{\theta,p}$ the (*standard*) *real interpolation method with parameter $p$ and exponent $\theta$*. The following theorem shows that these interpolation methods provide us with examples of admissible families.

**4.3 Theorem**

*Let* $1 \leq p < \infty$. *The family* $\big((\cdot\,,\cdot)_{\theta,p}\big)_{0<\theta<1}$ *is an admissible family of interpolation methods.*

**Proof**

Theorem 3.4.1($d$) and 3.4.2($b$) in [**22**] give property ($AF1$). Corollary 3.8.2 and Theorem 3.5.3, also in [**22**], prove the remaining two properties. □

**4.4 Remarks**

(a) In Theorem 4.3 we exclude the case $p = \infty$ because $E_1$ needs not to be dense in $(E_0, E_1)_{\theta,\infty}$. By the same results in [**22**] we invoked in the case $p \neq \infty$, all other properties of an admissible family are also valid if we let $p = \infty$ . We shall return to this when we treat the continuous interpolation functor.

(b) We actually have imbeddings for different $p$'s (cf. [**22**], Theorem 3.4.1($b$)).

$$(E_0, E_1)_{\theta,p} \hookrightarrow (E_0, E_1)_{\theta,q}$$

for any $\theta \in (0, 1)$ and $1 \leq p \leq q \leq \infty$. Moreover, these imbeddings are dense if $q < \infty$ (cf. [**22**], Theorem 3.4.2($b$)). We also have the following imbeddings (cf. [**122**], Theorem 1.3.3($e$))

$$(E_0, E_1)_{\theta_1,p} \hookrightarrow (E_0, E_1)_{\theta_0,q}$$

for all $0 < \theta_0 < \theta_1 < 1$ and $1 \leq p, q \leq \infty$.

(c) A property of the real-interpolation method which could be appropriately called extremely interesting, is the *extremal property* (cf. [**22**] Theorem 3.9.1 and 3.3.1):

> *Suppose that* $\mathfrak{F}_\theta : \mathfrak{B}_2 \to \mathfrak{B}_1$ *is an interpolation method of exponent* $\theta$, $\theta \in (0, 1)$. *Then we have*
>
> $$(E_0, E_1)_{\theta,1} \hookrightarrow \mathfrak{F}_\theta(\overline{E}) \hookrightarrow (E_0, E_1)_{\theta,\infty}.$$

**4.5 Remark**

The real (and also the complex) interpolation methods can be actually defined for a class of pairs of Banach spaces (or even normed spaces) which is larger than $\mathfrak{B}_2$. This class comprises all pairs $\overline{E} = (E_0, E_1)$ of Banach spaces which are mutually imbedded in a Hausdorff topological vector space $\mathcal{X}$, and is called the class of *compatible pairs* ([**22**]) or *interpolation couples* ([**122**]). The reason for requiring that $E_0$ and $E_1$ are imbedded in a third space $\mathcal{X}$ is that we can then build the *intersection space*

$$\Delta(\overline{E}) := E_0 \cap E_1$$

and the *sum-space*

$$\Sigma(\overline{E}).$$

These spaces can be equipped with the norms

$$\|x\|_{\Delta(\bar{E})} := \max\{\|x\|_0, \|x\|_1\}$$

and

$$\|x\|_{\Sigma(\bar{E})} := \inf\{\|x\|_0 + \|x\|_1;\ (x_0, x_1) \in D(x; \overline{E})\},$$

where $D(x; \overline{E})$ is defined as in (4.1), which makes them into Banach spaces. Observe that if $\overline{E}$ is a Banach couple, we have $E_1 \doteq \Delta(\overline{E})$ and $E_0 \doteq \Sigma(\overline{E})$. In general the pair $\big(\Delta(\overline{E}), \Sigma(\overline{E})\big)$ will not be a Banach couple since the imbedding $\Delta(\overline{E}) \hookrightarrow \Sigma(\overline{E})$ is not necessarily dense. But we can nevertheless define the $K$-functional for this kind of pairs, and it turns out that we also get interpolation methods for this larger class of spaces provided we use the correct definition of the notion of an interpolation method in this class. Thus, interpolating between $E_0$ and $E_1$ will be nothing else than interpolating between $\Sigma(\overline{E})$ and $\Delta(\overline{E})$. But not all the properties of the real interpolation method we listed above hold in this general context. Some of them depend on the fact that we require a dense imbedding in the definition of a Banach couple.

The more general construction just described has indeed a justification. First of all, classic interpolation theorems like the Riesz-Thorin or the Marcinkiewicz Theorem ([**22**]) – where one interpolates between $L_p(\mathbb{R}^n)$ and $L_q(\mathbb{R}^n)$ for $p \neq q$, so that neither space is imbedded in the other – do not fit in the theory for Banach couples. Secondly, if $\overline{E}$ is a Banach couple we have that $E_0' \hookrightarrow E_1'$ so that $(E_0', E_1')$ is not a Banach couple. Then, a beautiful formula like

$$(E_0, E_1)_{\theta,p} \doteq (E_0', E_1')_{\theta,p'} \qquad \big(\tfrac{1}{p} + \tfrac{1}{p'} = 1\big)$$

would not even make sense in the context of Banach couples. In all books on interpolation theory the general theory is developed. We refrained from doing so in order to avoid the need for too much terminology and to be able to state the results we need in a concise form. □

**B. The complex interpolation method:** Suppose now that $\overline{E} = (E_0, E_1)$ is an arbitrary Banach couple over the field $\mathbb{C}$. Consider the following subsets of the complex-plane:

$$S := [0 \le \operatorname{Re}\lambda \le 1], \qquad S_0 := \operatorname{int}_{\mathbb{C}}(S)$$
$$\partial_0 S := [\operatorname{Re}\lambda = 0], \qquad \partial_1 S := [\operatorname{Re}\lambda = 1].$$

Thus, $S$ is the closed strip enclosed by the lines $\partial_0 S$ and $\partial_1 S$, and $S_0$ is the open strip enclosed by the same lines.

Define $\mathcal{A}(\overline{E})$ to be the vector space of all functions $f\colon S \to E_0$ having the following properties:

(1) $f: S \to E_0$ is bounded and continuous,

(2) $f: S_0 \to E_0$ is analytic,

(3) $f: \partial_0 S \to E_0$ and $f: \partial_1 S \to E_1$ are bounded and continuous functions vanishing at infinity.

The space $\mathcal{A}(\bar{E})$ can be made into a Banach space if we equip it with the norm

$$\|f\|_{\mathcal{A}(\bar{E})} := \max\{\|f\|_{BC(\partial_0 S, E_0)}, \|f\|_{BC(\partial_1 S, E_1)}\},$$

where we recall that the norm in the space of bounded and continuous functions is just the supremum-norm.

**4.6 Definition**
For each $\theta \in (0,1)$ define

$$\mathfrak{F}_\theta^{\mathbb{C}}(\bar{E}) := [E_0, E_1]_\theta := \{x \in E_0;\ \text{ there is an } f \in \mathcal{A}(\bar{E}) \text{ with } x = f(\theta)\}$$

and provide this space with the norm

$$\|x\|_\theta := \inf\{\|f\|_{\mathcal{A}(\bar{E})};\ f \in \mathcal{A}(\bar{E}) \text{ with } x = f(\theta)\}.$$

The following theorem will probably not surprise the reader.

**4.7 Theorem**
*For each $\theta \in (0,1)$ we have that $[E_0, E_1]_\theta$ is a Banach space. Moreover,*

$$\mathfrak{F}_\theta^{\mathbb{C}}: \mathfrak{B}_2 \to \mathfrak{B}_1$$

*is an exact interpolation method of exponent $\theta$.*

**Proof**
The result follows from Theorem 4.1.2 in [**22**]. □

The interpolation method $[\cdot\,,\cdot]_\theta$ is called the (*standard*) *complex interpolation method with exponent* $\theta$. The companion result to Theorem 4.3 for the real method is the following

**4.8 Theorem**
*The family $\big([\cdot\,,\cdot]_\theta\big)_{0<\theta<1}$ is an admissible family of interpolation methods.*

**Proof**
Property $(AF1)$ follows from Theorem 4.2.1$(b)$ in [**22**]. Property $(AF3)$ follows from Theorem 4.6.1 in [**22**]. To prove property $(AF2)$ note that by the extremal property of the real method (Remark 4.4(c)) we have that

$$E_1 \overset{d}{\hookrightarrow} (E_0, E_1)_{\theta,1} \hookrightarrow [E_0, E_1]_\theta \hookrightarrow (E_0, E_1)_{\theta,\infty} \overset{d}{\hookrightarrow} E_0. \tag{4.7}$$

If $E_1 \overset{d}{\hookrightarrow} E_0$ is compact we then get by the admissibility of $\big((\cdot\,,\cdot)_{\theta,1}\big)_{0<\theta<1}$ and Remark 4.4(a), that the left and right imbeddings in (4.2) are compact. By the observation preceding (3.5) we now obtain $(AF2)$. □

**4.9 Remark**
In case that $\mathbb{K} = \mathbb{R}$ we have to interpolate between the complexifications of $E_0$ and $E_1$ and put

$$[E_0, E_1]_\theta := [(E_0)_\mathbb{C}, (E_1)_\mathbb{C}]_\theta \cap E_0 \tag{4.8}$$

Then, $[E_0, E_1]_\theta$ is a real Banach space and defines an exact interpolation method of exponent $\theta$ for the class of Banach couples over $\mathbb{R}$. The family of methods obtained in this way is obviously admissible. □

**4.10 Remark**
As already mentioned in Remark 4.5 the complex method may be defined not only in $\mathfrak{B}_2$ but on the larger class of compatible pairs. □

**C. The continuous interpolation method:** In the sequel $\overline{E} = (E_0, E_1)$ will be as usual an arbitrary Banach couple over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The norms on $E_0$ and $E_1$ will be denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. In Remark 4.4 we mentioned that the family $\big((\cdot\,,\cdot)_{\theta,\infty}\big)_{0<\theta<1}$ of real interpolation methods enjoyed all the properties of an admissible family with the sole exception of the density of the imbedding $E_1 \hookrightarrow (E_0, E_1)_{\theta,\infty}$. We can remedy this flaw by considering the closure of $E_1$ in $(E_0, E_1)_{\theta,\infty}$ as the actual interpolation space.

**4.11 Definition**
For any $\theta \in (0, 1)$ we set

$$(E_0, E_1)^0_{\theta,\infty} := \mathrm{cl}_{(E_0,E_1)_{\theta,\infty}}(E_1)$$

and provide this space with the same norm as $(E_0, E_1)_{\theta,\infty}$. □

Since we have defined our new intermediate space just as a closed subspace of the space $(E_0, E_1)_{\theta,\infty}$ and, by definition, $E_1$ is dense in it, we immediately see that $\big((\cdot\,,\cdot)^0_{\theta,\infty}\big)_{0<\theta<1}$ is an admissible family of interpolation functors.

**4.12 Theorem**
*For each $\theta \in (0, 1)$ we have that*

$$(\cdot\,,\cdot)^0_{\theta,\infty} \colon \mathfrak{B}_2 \to \mathfrak{B}_1$$

*is an exact interpolation method of exponent $\theta$. Furthermore, the family $\big((\cdot\,,\cdot)^0_{\theta,\infty}\big)_{0<\theta<1}$ is admissible.*

The interpolation method described above will be called *continuous interpolation method with exponent* $\theta$. Actually, the continuous interpolation method was originally defined in Da Prato and Grisvard [**40**] in a completely different way. However, in Dore and Favini [**49**] it is shown that both definitions lead to the same interpolation spaces up to equivalent norms. We will now give some descriptions of the continuous interpolation spaces in case that the smaller space $E_1$ is given – up to equivalent norms – by the domain of definition of the infinitesimal generator of a $C_0$-semigroup of operators on $E_0$. This is the case which interests us most. We follow quite closely the presentation of S. Angenent in [**31**].

Suppose now that $-A$ is the infinitesimal generator of a $C_0$-semigroup on $E_0$ and that $E_1 \doteq D(A)$, where, as usual, we equip $D(A)$ with the graph norm. Furthermore, let $\omega_0$ be a positive number such that $(\omega_0, \infty)$ is contained in $\varrho(-A)$, the resolvent set of $-A$. Such an $\omega_0$ exists by the Hille-Yosida Theorem (see e.g. [**100**]).

**4.13 Definitions**
(a) We define a subspace of $E_0$ by

$$Z_\theta(A) := \big\{x \in E_0;\ \lim_{\lambda \searrow 0} \lambda^\theta \|(\lambda + A)^{-1}x\|_1 = 0\big\}$$

and provide it with the norm

$$\|x\|_{Z_\theta(A)} := \sup_{\omega_0 \le \lambda < \infty} \lambda^\theta \|(\lambda + A)^{-1}x\|_1,$$

which makes it into a Banach space.

(b) Another subspace of $E_0$ may be defined by setting

$$D_\theta(A) := \big\{x \in E_0;\ \lim_{t \searrow 0} t^{-\theta} \|e^{-tA}x - x\|_0 = 0\big\}.$$

Equipped with the norm

$$\|x\|_{D_\theta(A)} := \|x\|_0 + \sup_{0<t\le 1} t^{-\theta} \|e^{-tA}x - x\|_0$$

$D_\theta(A)$ is a Banach space. □

The fundamental result is then:

**4.14 Theorem**
*For all* $\theta \in (0,1)$ *the spaces*

$$(E_0, E_1)^0_{\theta,\infty}\,, \qquad Z_\theta(A)\,, \qquad D_\theta(A)$$

*coincide up to equivalent norms.*

**Proof**
In [**31**] Theorem 6.5 and 6.6, as well as in Lemma 6.7, it is shown that $Z_\theta(A)$ and $D_\theta(A)$ coincide with the continuous interpolation space as defined by Da Prato and Grisvard. In [**49**], Theorem 2.6, it is shown that $(E_0, E_1)^0_{\theta,\infty}$ also coincides with the interpolation space of Da Prato and Grisvard, proving the assertion. $\square$

**4.15 Remark**
The relationship between the interpolation spaces obtained by real, complex and continuous interpolation is depicted in the following diagram:

$$(4.4)\qquad \begin{aligned}(E_0, E_1)_{\beta,p} &\overset{d}{\hookrightarrow} (E_0, E_1)^0_{\beta,\infty} \overset{d}{\hookrightarrow} (E_0, E_1)_{\theta,1}\\ &\overset{d}{\hookrightarrow} [E_0, E_1]_\theta \overset{d}{\hookrightarrow} (E_0, E_1)^0_{\theta,\infty} \overset{d}{\hookrightarrow} (E_0, E_1)_{\alpha,p}\,,\end{aligned}$$

whenever $0 < \alpha < \theta < \beta < 1$ and $1 \leq p < \infty$. Indeed, this follows from (4.2), Remark 4.4(b) and the definition of the continuous interpolation method. $\square$

**D. Examples:** We shall now give a description of those interpolation spaces which arise in connection with second order elliptic operators. In the bounded domain case we shall only be concerned with the $L_p$-formulation and the real and complex interpolation scales. For equations on $\mathbb{R}^n$ we shall work in the setting of $BUC(\mathbb{R}^n)$ or $C_0(\mathbb{R}^n)$ and consider the continuous interpolation scale.

1) We start with the bounded domain case as described in Section 1.C. Let $\big(\Omega, \mathcal{A}, \mathcal{B}\big)$ be a second order elliptic boundary value problem of class $C^0$. Let $1 < p < \infty$ and set

$$X_0 := L_p(\Omega) \qquad \text{and} \qquad X_1 := W^2_{p,\mathcal{B}}(\Omega).$$

Consider the $L_p$-realization

$$A\colon D(A) \subset X_0 \to X_0$$

of $\big(\Omega, \mathcal{A}, \mathcal{B}\big)$, where $D(A) = X_1$. In Appendix 3 and 4 we introduce the spaces $H^s_p(\Omega)$, $W^s_p(\Omega)$ and $W^s_p(\partial\Omega)$ for $0 \leq s \leq 2$. The $H^s_p$-spaces are called the Bessel-potential spaces and the $W^s_p$-spaces the Sobolev-Slobodeckii spaces. In Appendix 4 we also mentioned the existence of the trace operators

$$\gamma \in \mathcal{L}\big(W^s_p(\Omega), W^{s-1/p}_p(\partial\Omega)\big) \cap \mathcal{L}\big(H^s_p(\Omega), W^{s-1/p}_p(\partial\Omega)\big)$$

whenever $1/p < s \leq 2$. Hence, in a completely analogous way to Section 1.C, one may define the abstract version $B_p$ of the boundary operator $\mathcal{B}$. If $\mathcal{B}$ is the Dirichlet boundary operator, we have

$$(4.5)\qquad B_p \in \mathcal{L}\big(W^s_p(\Omega), W^{s-1/p}_p(\partial\Omega)\big) \cap \mathcal{L}\big(H^s_p(\Omega), W^{s-1/p}_p(\partial\Omega)\big) \qquad (1/p < s \leq 2)$$

and we set

$$\text{(4.6)}\qquad H^s_{p,\mathcal{B}} := \begin{cases} \{u \in H^s_p(\Omega);\ B_p u = 0\} & \text{if } \ 1/p < s \le 2 \\ H^s_p(\Omega) & \text{if } \ 0 \le s < 1/p \end{cases}$$

and

$$\text{(4.7)}\qquad W^s_{p,\mathcal{B}} := \begin{cases} \{u \in W^s_p(\Omega);\ B_p u = 0\} & \text{if } \ 1/p < s \le 2 \\ W^s_p(\Omega) & \text{if } \ 0 \le s < 1/p. \end{cases}$$

If $\mathcal{B}$ is either the Neumann or the Robin boundary operator we have

$$\text{(4.8)}\ \ B_p \in \mathcal{L}\big(W^s_p(\Omega), W^{s-1-1/p}_p(\partial\Omega)\big) \cap \mathcal{L}\big(H^s_p(\Omega), W^{s-1-1/p}_p(\partial\Omega)\big) \qquad (1+1/p < s \le 2)$$

and we set

$$\text{(4.9)}\qquad H^s_{p,\mathcal{B}} := \begin{cases} \{u \in H^s_p(\Omega);\ B_p u = 0\} & \text{if } \ 1+1/p < s \le 2 \\ H^s_p(\Omega) & \text{if } \ 0 \le s < 1+1/p \end{cases}$$

and

$$\text{(4.10)}\qquad W^s_{p,\mathcal{B}} := \begin{cases} \{u \in W^s_p(\Omega);\ B_p u = 0\} & \text{if } \ 1+1/p < s \le 2 \\ W^s_p(\Omega) & \text{if } \ 0 \le s < 1+1/p. \end{cases}$$

Observe that $H^s_p(\Omega)$ and $W^s_p(\Omega)$ is not defined if $s = 1/p$ and $1 + 1/p$, respectively. We have now the the following characterization of the interpolation spaces between the domain of the $L_p$-realization of an elliptic boundary value problem and $L_p(\Omega)$.

**4.16 Theorem**
*Let the hypotheses from above be satisfied. Then,*

$(i)\quad [X_0, X_1]_\alpha \doteq H^{2\alpha}_{p,\mathcal{B}}(\Omega)$ *and*

$(ii)\quad (X_0, X_1)_{\alpha,p} \doteq W^{2\alpha}_{p,\mathcal{B}}(\Omega) \qquad (\alpha \ne 1/2)$

*hold for all* $\alpha \in [0,1] \setminus \{\frac{1}{2p}, \frac{1}{2} + \frac{1}{2p}\}$ *and* $1 < p < \infty$.

**Proof**
For a proof of $(i)$ see [**111**], Theorem 4.1 (for more general systems see [**15**], Lemma 5.1). Assertion $(ii)$ is easily deduced from diagram (4.4), part $(i)$, Proposition 3.2$(b)$ as well as Proposition A3.11 in the Appendix. □

It is worthwhile to observe that from the above theorem and the definition of the spaces $H^s_{p,\mathcal{B}}(\Omega)$ and $W^s_{p,\mathcal{B}}(\Omega)$ it follows that the spaces $X_\alpha$ do not depend on the boundary conditions for $\alpha$ sufficiently small.

Finally, we want to give some imbedding properties of the spaces $X_\alpha$ considered above.

**4.17 Corollary**
*Let $X_\alpha$ be defined by $[X_0, X_1]_\alpha$ or $(X_0, X_1)_{\alpha,p}$. Then we have the following imbeddings:*

(i) $X_\alpha \hookrightarrow C^t(\overline{\Omega})$ *whenever* $1 \geq \alpha > \frac{1}{2}\big(t + \frac{n}{p}\big)$.

(ii) *Let $\mathcal{B}$ be the Dirichlet boundary operator. Then, if* $0 \leq \alpha < \frac{1}{2p}$ *and* $t > 2\alpha$, *it holds that* $C^t(\overline{\Omega}) \hookrightarrow X_\alpha$.

(iii) *Let $\mathcal{B}$ be the Neumann or Robin boundary operator. Then, if* $0 \leq \alpha < \frac{1}{2}\big(1+\frac{1}{p}\big)$ *and* $t > 2\alpha$, *it holds that* $C^t(\overline{\Omega}) \hookrightarrow X_\alpha$.

**Proof**
Note that $H^s_{p,\mathcal{B}}(\Omega) \hookrightarrow H^s_p(\Omega)$ and $W^s_{p,\mathcal{B}}(\Omega) \hookrightarrow W^s_p(\Omega)$. The assertions then follow from the imbedding theorems A3.12 and A3.13 in Appendix 3 and the definitions of $H^s_{p,\mathcal{B}}(\Omega)$ and $W^s_{p,\mathcal{B}}(\Omega)$. □

Let us now turn to the continuous setting concerned with reaction-diffusion problems in $\mathbb{R}^n$.

2) Consider the Laplacian $\Delta$ on $\mathbb{R}^n$ and set

$$X_0 := BUC(\mathbb{R}^n) \qquad \text{or} \qquad C_0(\mathbb{R}^n).$$

Then the domain of definition of the $X_0$-realization of $\Delta$ is denoted by $X_1$, i.e.

$$X_1 := \{u \in X_0;\ \Delta u \in X_0\}.$$

The following result characterizes the continuous interpolation spaces $(X_0, X_1)^0_{\alpha,\infty}$. The little Hölder spaces $buc^s(\mathbb{R}^n)$ and $c^s_0(\mathbb{R}^n)$ are introduced in Appendix 1.

**4.18 Theorem**
*Let $\alpha \in (0,1) \setminus \{1/2\}$. If $X_0 = BUC(\mathbb{R}^n)$, we have*

$$(X_0, X_1)^0_{\alpha,\infty} \doteq buc^{2\alpha}(\mathbb{R}^n).$$

*If $X_0 = C_0(\mathbb{R}^n)$, then*

$$(X_0, X_1)^0_{\alpha,\infty} \doteq c^{2\alpha}_0(\mathbb{R}^n).$$

**Proof**
In [**91**], Proposition 2.15, the assertion is proved for $X_0 = C_0(\mathbb{R}^n)$ and $\alpha \in (0, 1/2)$. The same proof also workes for $X_0 = BUC(\mathbb{R}^n)$. The proof relies on Theorem 2.10 of that paper. Since in the case of the Laplacian this theorem remains valid for $\alpha \in (1/2, 1)$, we see that the asserion is also true for such $\alpha$'s. □

The above interpolation spaces have the following imbedding properties:

**4.19 Corollary**
*Let* $\alpha \in (0,1) \setminus \{1/2\}$*. Then for any* $t > \alpha$ *we have that*

$$X_\alpha \hookrightarrow BUC^t(\mathbb{R}^n$$

*if* $X_0 = BUC(\mathbb{R}^n)$ *and*

$$X_\alpha \hookrightarrow BUC^t(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$$

*if* $X_0 = C_0(\mathbb{R}^n)$

**Proof**

The assertions are an easy consequence of the above theorem and Proposition A1.4 in the appendix. □

**E. Interpolation and fractional powers:** In the introduction we mentioned the fact that it has been common practice to work in the so called fractional power spaces rather than in the interpolation space setting we shall adopt in this work. In this subsection, we shall briefly explain the relationship between these two types of spaces.

We start with some basic definitions. Let $(X_0, X_1)$ be a Banach couple and

$$A\colon D(A) \subset X_0 \to X_0$$

be a closed operator such that $-A$ is the generator of a stronly continuous analytic semigroup such that $D(A) \doteq X_1$ and $[\operatorname{Re}\mu \geq 0] \subset \varrho(-A)$. For any $\alpha > 0$ we define

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-tA}\, dt\,.$$

Then it can be shown (e.g. [**66**], [**100**]) that $A^{-\alpha} \in \mathcal{L}(X_0)$ and that it is invertible. Hence, the inverse is a closed operator on $X_0$ which we denote by $A^\alpha$. The domain of this operator is $D(A^\alpha)$, which becomes a Banach space when endowed with the graph norm induced by $A^\alpha$. Note that $A^0 = 1\!\!1_{X_0}$ and $A^1 = A$. For any $\alpha \in [0,1]$ we put

$$X^\alpha := D(A^\alpha).$$

The space $X^\alpha$ is then called the *$\alpha$-th fractional power space associated to $A$*. It is now possible to show (e.g. [**66**]) that $(X^\alpha)_{0 \leq \alpha \leq 1}$ is a family of Banach spaces satisfying

$$X_1 \overset{d}{\hookrightarrow} X^\beta \overset{d}{\hookrightarrow} X^\alpha \overset{d}{\hookrightarrow} X_0$$

for $0 \leq \alpha \leq \beta \leq 1$. The following diagram relates the fractional power spaces to the real interpolation method (see [**122**], Theorem 1.15.2):

$$(X_0, X_1)_{\theta,1} \overset{d}{\hookrightarrow} X^\theta \overset{d}{\hookrightarrow} (X_0, X_1)_{\theta,\infty} \tag{4.11}$$

if $\theta \in (0,1)$. Using Remark 4.15, we thus obtain

$$\begin{aligned}[X_0,X_1]_\beta &\overset{d}{\hookrightarrow} (X_0,X_1)_{\zeta,p} \overset{d}{\hookrightarrow} (X_0,X_1)^0_{\zeta,\infty} \overset{d}{\hookrightarrow} X^\theta \\ &\overset{d}{\hookrightarrow} (X_0,X_1)_{\eta,p} \overset{d}{\hookrightarrow} (X_0,X_1)^0_{\eta,\infty} \overset{d}{\hookrightarrow} [X_0,X_1]_\alpha\,, \end{aligned} \tag{4.12}$$

whenever $0<\alpha<\eta<\theta<\zeta<\beta<1$. If $(X_\alpha)$ is any of the standard interpolation scales we deduce from the above diagram that

$$X_{\alpha-\varepsilon} \overset{d}{\hookrightarrow} X^\alpha \overset{d}{\hookrightarrow} X_{\alpha-\varepsilon}$$

holds for all $\alpha \in (0,1)$ and $\varepsilon > 0$ arbitrarily small. In other words, we have almost $X_\alpha \doteq X^\alpha$. It is thus a fairly natural question to ask under what conditions on $A$ this equality holds.

One possible sufficient condition involves the purely imaginary powers $A^{is}$ for $s \in \mathbb{R}$. A definition may be found in [**122**], Section 1.15.1.

**4.20 Theorem**
*Assume that for all $\varepsilon \in [-1,1]$ we have that*

$$A^{i\varepsilon} \in \mathcal{L}(X_0) \qquad \textit{and} \qquad \|A^{i\varepsilon}\| \le c \tag{4.13}$$

*for a suitable constant $c > 0$. Then*

$$X^\alpha \doteq [X_0,X_1]_\alpha$$

*holds for all $\alpha \in (0,1)$.*

The proof of the above theorem is rather easy and may be found in [**12**], Theorem 3.3. Much more difficult is the verification of (4.13) for concrete operators. For operators on $L_p$ induced by a second order elliptic boundary value problem this has been carried out by R. Seeley [**110**] for $C^\infty$-coefficients and by J. Prüß and H. Sohr [**102**] for weaker regularity conditions (consult also the bibliograpy given there).

**Notes and references:** The books on interpolation theory listed at the end of Section 4 contain also the construction and further properties of the concrete interpolation methods we have presented in this section. For the continuous interpolation method see [**31**]. We refer also to G. Simonett [**112**].

The results in Theorem 4.16 are due to Seeley [**111**] and those of Theorem 4.18 to Lunardi [**91**].

## 5. The evolution operator in interpolation spaces

We are now ready to derive the crucial estimates for the evolution operator in interpolation spaces. The advantage of prefering to work in these spaces rather than in fractional power spaces, will be evident not only from the simplicity of the proofs, but also from the fact that most of the estimates for the evolution operator are simpler than the corresponding estimates in fractional power spaces, allowing a more natural and elegant treatment of semilinear equations.

We use the same notation as in Section 2. Furthermore, we assume that $\big((\cdot\,,\cdot)_\theta\big)_{0<\theta<1}$ is an admissible family of interpolation methods. We set for each $\alpha \in (0,1)$:

$$X_\alpha := (X_0, X_1)_\alpha$$

By $\|\cdot\|_\alpha$ and $\|\cdot\|_{\alpha,\beta}$, we denote the norms in $X_\alpha$ and $\mathcal{L}(X_\alpha, X_\beta)$, respectively ($\alpha, \beta \in [0,1]$).

For any $0 \le \alpha < \beta \le 1$ we have:

$$X_\beta \overset{d}{\hookrightarrow} X_\alpha. \tag{5.1}$$

Let $U(\cdot\,,\cdot)$ be the evolution operator associated to the family $\big(A(t)\big)_{0\le t\le T}$. By $(U1)$ we have $U(t,s) \in \mathcal{L}(X_0, X_1)$, for $(t,s) \in \dot{\Delta}_T$. Therefore we get by (5.1):

$$U(t,s) \in \mathcal{L}(X_\alpha, X_\beta) \qquad \text{for each } \alpha, \beta \in [0,1],\ (t,s) \in \dot{\Delta}_T.$$

Observe that we denote by the same symbol $U(t,s)$ the various operators obtained by restricting the domain of definition and the range of $U(t,s)$. This should not lead to any difficulties.

**5.1 Remark**

If $A(0)$ has a compact inverse the imbedding $X_1 \hookrightarrow X_0$ is compact, since we can write $A(0)^{-1}A(0)$ for the inclusion mapping and this is a compact operator. Furthermore, because $U(t,s) \in \mathcal{L}(X_\alpha, X_\beta)$ we obtain that $U(t,s)$ is compact whenever $\alpha \neq \beta$ and $(\alpha,\beta) \neq (0,1)$ (compare Section 0.D). □

**A. Estimates and continuity properties of the evolution operator:** In our first lemma we collect some results on the evolution operator $U$ as a function $\Delta_T \to \mathcal{L}(X_\alpha, X_\beta)$.

**5.2 Lemma**

(*a*) *For any* $0 \le \alpha \le 1$ *we have:*

$$U \in C\big(\Delta_T, \mathcal{L}_s(X_\alpha)\big).$$

(*b*) *For* $0 \le \alpha \le \beta \le 1$ *and* $(t,s) \in \Delta_T$*, we have:*

$$\|U(t,s)\|_{\beta,\alpha} \le c(\alpha,\beta). \tag{5.2}$$

(*c*) *For* $0 \le \alpha < \beta \le 1$ *and* $(t,s) \in \dot{\Delta}_T$*, we have:*

$$\|U(t,s)\|_{\alpha,\beta} \le c(\alpha,\beta)(t-s)^{\alpha-\beta}. \tag{5.3}$$

*Moreover, the constants appearing in* (5.2) *and* (5.3) *depend only on* $\alpha$*,* $\beta$*,* $M$ *(in* (*A*2)*),* $\rho$ *(in* (*A*3)*), the Hölder norm of* $A(\cdot)$ *and a bound for* $\|A(t)A^{-1}(s)\|$*.*

**Proof**

(*b*) Using $(U1)$, the compactness of $\Delta_T$, and the uniform boundedness principle we get $\|U(t,s)\|_{i,i} \le c$, $i = 0,1$ for some constant $c > 0$. From this and inequality $(F2)$ we immediately obtain:

$$\|U(t,s)\|_{\alpha,\alpha} \le c.$$

(5.2) now follows from the imbedding $X_\beta \hookrightarrow X_\alpha$.

(*a*) It easily follows from $(U1)$ that for each $x \in X_1$ the mapping $(t,s) \mapsto U(t,s)x$, is continuous from $\Delta_T$ to $X_\alpha$. Furthermore, $\{U(t,s);\ (t,s) \in \Delta_T\}$ is bounded in $\mathcal{L}(X_\alpha)$ by (*b*). The density of $X_1$ in $X_\alpha$ now implies that $U$ is strongly continuous from $\Delta_T$ to $\mathcal{L}(X_\alpha)$ as claimed (see Section 1.B).

(*c*) By Remark 2.4 we have the estimate:

$$\|U(t,s)\|_{0,1} \le c(t-s)^{-1} \tag{5.4}$$

for $(t,s) \in \dot{\Delta}_T$. By $(F2)$ we obtain for $\gamma \in (0,1)$:

$$\|U(t,s)\|_{0,\gamma} \le \|U(t,s)\|_{0,0}^{1-\gamma} \|U(t,s)\|_{0,1}^{\gamma}$$

which together with (5.4) and (*b*) implies

$$\|U(t,s)\|_{0,\gamma} \le c(\gamma)(t-s)^{-\gamma} \tag{5.5}$$

for $(t,s) \in \dot{\Delta}_T$.

Setting $\gamma := \frac{\beta-\alpha}{1-\alpha}$ the reiteration Theorem $(AF3)$ gives:

$$(X_\gamma, X_1)_\alpha \doteq X_\beta.$$

This situation is depicted in the following diagram:

$$\begin{array}{ccccc} X_1 & \hookrightarrow & X_\beta = (X_\gamma, X_1)_\alpha & \hookrightarrow & X_\gamma \\ \uparrow{\scriptstyle U(t,s)} & & \uparrow{\scriptstyle U(t,s)} & & \uparrow{\scriptstyle U(t,s)} \\ X_1 & \hookrightarrow & X_\alpha & \hookrightarrow & X_0 \end{array}$$

We now obtain from $(F2)$:

$$\|U(t,s)\|_{\alpha,\beta} \leq c(\alpha,\beta,\gamma)\|U(t,s)\|_{0,\gamma}^{1-\alpha}\|U(t,s)\|_{1,1}^{\alpha} \tag{5.6}$$

for $(t,s) \in \dot{\Delta}_T$. The estimate (5.3) now follows from (5.6), (5.5) and $(b)$. This completes the proof of the lemma. □

Lemma 5.2 will be the basis of all subsequent results. We will often use it without further mention, specially in later chapters, so the reader is advised to become familiar with its statements. Our next result concerns the regularity of the solution of the homogeneous Cauchy-problem (2.1), i.e. $f \equiv 0$:

**5.3 Lemma**

*Let $s \in [0,T)$ be fixed. Then the following assertion hold:*

$(a)$ *If $\alpha \in [0,1]$ is arbitrary, we have that*

$$U(\cdot\,,s) \in C\big([s,T],\mathcal{L}_s(X_\alpha)\big)$$

$(b)$ *If $0 \leq \alpha < \beta \leq 1$, then*

$$U(\cdot\,,s) \in C^{\beta-\alpha}\big([s,T],\mathcal{L}(X_\beta,X_\alpha)\big),$$

*where* $1-0 := 1-$.

$(c)$ *We have that*

$$U(\cdot\,,s) \in C^1\big([s,T],\mathcal{L}_s(X_1,X_0)\big).$$

**Proof**

$(a)$ is an immediate consequence of Lemma 5.2$(a)$.

$(b)$ For the case $\alpha = 0$ and $0 < \beta \leq 1$ we observe that by $(U4)$ we have that $U(\cdot\,,s) \in C^1\big((s,T],\mathcal{L}(X_0)\big)$ and

$$\partial_1 U(t,s) = -A(t)U(t,s) \quad t \in (s,T].$$

Let now $s \leq r < t \leq T$. Then the following estimates hold:

$$\begin{aligned}\|\partial_1 U(t,s)\|_{\beta,0} = \|A(t)U(t,s)\|_{\beta,0} &\leq c\|U(t,s)\|_{\beta,1}\\ &\leq c\|U(t,r)\|_{\beta,1}\|U(r,s)\|_{\beta,\beta}\\ &\leq c(t-r)^{\beta-1},\end{aligned}$$

where we used Remark 2.1(c) and $(A2)$ for the first and Lemma 5.2 for the last inequality. We thus obtain for any small $\varepsilon > 0$ and $x \in X_\beta$:

$$\begin{aligned}\|U(t,s)x - U(r+\varepsilon,s)x\|_0 &\leq \int_{r+\varepsilon}^t \|\partial_1 U(\tau,s)\|_{\beta,0}(\tau-r)^{-(\beta-1)}(\tau-r)^{\beta-1}d\tau\|x\|_\beta\\ &\leq c\int_{r+\varepsilon}^t (\tau-r)^{\beta-1}d\tau\|x\|_\beta\\ &\leq c(t-r)^\beta\|x\|_\beta.\end{aligned}$$

Letting $\varepsilon \to 0$ we easily get:

$$\|U(t,s)x - U(r,s)x\|_0 \le c(t-r)^\beta \|x\|_\beta$$

and thus

$$\|U(t,s) - U(r,s)\|_{\beta,0} \le c|t-r|^\beta. \qquad \text{for } t,r \in [s,T]. \tag{5.7}$$

This implies the asserted regularity of $U(\cdot\,,s)$. (The dummy variable $\varepsilon$ was needed for the case $r = s$).

Consider now the case $0 < \alpha < \beta \le 1$. For this let $t,r \in [s,T]$. Then

$$\|U(t,s) - U(r,s)\|_{\beta,\beta} \le c$$

holds. Furthermore, from the reiteration theorem $(AF3)$ we easily obtain:

$$\|L\|_{\beta,\alpha} \le c\|L\|_{\beta,0}^{\frac{\beta-\alpha}{\beta}} \|L\|_{\beta,\beta}^{\frac{\alpha}{\beta}} \qquad \text{for } L \in \mathcal{L}(X_\beta).$$

To see this we just have to look at the diagram:

$$\begin{array}{ccccc}
X_\beta & \hookrightarrow & X_\alpha = (X_\beta, X_0)_{\frac{\alpha}{\beta}} & \hookrightarrow & X_0 \\
\uparrow L & & \uparrow L & & \uparrow L \\
X_\beta & \hookrightarrow & X_\beta & \hookrightarrow & X_\beta
\end{array}$$

and use $(F2)$.

Using the facts above and (5.7) we get that

$$\begin{aligned}
\|U(t,s) - U(r,s)\|_{\beta,\alpha} &\le c\|U(t,s) - U(r,s)\|_{\beta,0}^{\frac{\beta-\alpha}{\beta}} \|U(t,s) - U(r,s)\|_{\beta,\beta}^{\frac{\alpha}{\beta}} \\
&\le c|t-r|^{\beta-\alpha}
\end{aligned}$$

holds. This obviously implies the assertion in the case $0 < \alpha < \beta \le 1$, proving $(b)$ in its full generality.

$(c)$ To prove the last assertion let $x \in X_1$. We then have by $(U4)$ and $(U1)$

$$U(\cdot\,,s)x \in C^1\big((s,T],X_0\big) \cap C\big([s,T],X_1\big),$$

and

$$\partial_1 U(t,s)x = -A(t)U(t,s)x.$$

But then we have by $(A3)$

$$A(\cdot)U(\cdot\,,s)x \in C\big([s,T],X_0\big),$$

which immediately implies

$$U(\cdot\,,s)x \in C^1\big([s,T],X_0\big). \qquad \square$$

For greater clarity we reformulate Lemma 5.3 as a result on the solution of the homogeneous Cauchy-problem (2.1).

**5.4 Corollary**

*Let $\beta \in [0,1]$, $(s,x) \in [0,T) \times X_\beta$ be given and $f \equiv 0$. Then the unique solution $u := U(\cdot\,,s)x$ of (2.1) lies in $C^1\big((s,T],X_0\big) \cap C^{\beta-\alpha}\big([s,T],X_\alpha\big)$ for any $0 \le \alpha \le \beta$. If $x \in X_1$ we even have $u \in C^1\big([s,T],X_0\big)$.*

**B. Regularity of integral operators:** In order to prove a similar regularity result for the solution of the inhomogeneous Cauchy-problem we need the following lemma on the regularity of certain integral operators:

**5.5 Lemma**

*Let $s \in [0,T)$. For $g \in L_\infty\big((s,T),X_0\big)$ and $t \in [s,T]$ set*

$$H_s(g)(t) := \int_s^t U(t,\tau)g(\tau)d\tau.$$

*Then for any $0 \le \alpha < 1$*

$$H_s \in \mathcal{L}\big(L_\infty\big((s,T),X_0\big), C^{\beta-\alpha}\big([s,T],X_\alpha\big)\big),$$

*for arbitrary $\beta \in [\alpha,1)$. Moreover, if $\gamma \in (0,1]$, we have that*

$$H_s \in \mathcal{L}\big(L_\infty\big((s,T),X_\gamma\big), C^{\beta-\alpha}\big([s,T],X_1\big)\big).$$

**Proof**

$(i)$ We first consider the case where $0 \le \alpha < \beta < 1$. Let $\tau < r < t \le T$. Then we obtain from Lemma 5.2 and 5.3:

$$\begin{aligned}\|U(t,\tau)-U(r,\tau)\|_{0,\alpha} &\le \|U(t,r)-U(r,r)\|_{\beta,\alpha}\|U(r,\tau)\|_{0,\beta}\\ &\le c(t-r)^{\beta-\alpha}(r-\tau)^{-\beta}.\end{aligned}$$

This inequality gives

$$\begin{aligned}\int_s^r \|U(t,\tau)-U(r,\tau)\|_{0,\alpha}d\tau &\le c(t-r)^{\beta-\alpha}\int_s^r (r-\tau)^{-\beta}d\tau\\ &\le c(t-r)^{\beta-\alpha}(r-s)^{1-\beta} \le cT^{1-\beta}(t-r)^{\beta-\alpha}.\end{aligned} \tag{5.8}$$

We also have the estimate

$$\int_r^t \|U(t,\tau)\|_{0,\alpha} d\tau \le \int_r^t (t-\tau)^{-\alpha} d\tau = c(t-r)^{1-\alpha} \le cT^{1-\beta}(t-r)^{\beta-\alpha}, \tag{5.9}$$

where we used that $(t-r)^{1-\alpha} = (t-r)^{\beta-\alpha}(t-r)^{1-\beta} \le T^{1-\beta}(t-r)^{\beta-\alpha}$. From (5.8) and (5.9) we now obtain

$$\begin{aligned}\|H_s(g)(t) - H_s(g)(r)\|_\alpha \le & \int_s^r \|U(t,\tau) - U(r,\tau)\|_{0,\alpha} \|g(\tau)\|_0 d\tau \\ & + \int_r^t \|U(t,\tau)\|_{0,\alpha} \|g(\tau)\|_0 d\tau \\ \le & c(t-r)^{\beta-\alpha} \|g\|_\infty ,\end{aligned}$$

which easily implies

$$\|H_s(g)(t) - H_s(g)(r)\|_\alpha \le c|t-r|^{\beta-\alpha} \|g\|_\infty \qquad \text{for all } t,r \in [s,T]. \tag{5.10}$$

This gives $H_s(g) \in C^{\beta-\alpha}\big([s,T], X_\alpha\big)$.

By Lemma 5.2 we have

$$\|H_s(g)(t)\|_\alpha = \| \int_s^t U(t,\tau) g(\tau) d\tau \|_\alpha \le c \int_s^t (t-\tau)^{-\alpha} d\tau \|g\|_\infty .$$

This together with (5.10) proves the lemma in the case $\alpha < \beta < 1$.

(*ii*) Let $\alpha = \beta < 1$ and $0 \le s < \tau < r < t \le T$. From Lemma 5.2 and 5.3 we obtain that

$$\begin{aligned}&\|\big(U(t,r) - U(r,r)\big)U(r,\tau)g(\tau)\| \\ &\qquad \le \|U(t,r) - U(r,r)\|_{\alpha,\alpha} \|U(r,\tau)\|_{0,\alpha} \|g(\tau)\| \le c(r-\tau)^{-\alpha}.\end{aligned}$$

On the other hand, for fixed $\tau \in (s,r]$, we have that

$$\lim_{t\to r} \big(U(t,r) - U(r,r)\big)U(r,\tau)g(\tau) = 0.$$

Indeed, this follows from the strong continuity of $U(\cdot\,,\tau)$ as a map from $[\tau,T]$ to $\mathcal{L}(X_\alpha)$. Hence, by Lebesgue's theorem on dominated convergence we get that

$$\lim_{t\to r} \int_s^r \big(U(t,r) - U(r,r)\big)U(r,\tau)g(\tau)\, d\tau = 0.$$

Arguments completely analogous as in (*i*) yield now the assertion in the case $\alpha = \beta < 1$.

The case $\alpha = \beta = 1$ is handled similar. The details are left to the reader. □

From Lemma 5.4 and 5.5 we immediately obtain the following result:

**5.6 Corollary**
*Let $s \in [0,T)$. For $x \in X_0$, $g \in L_\infty\big((s,T),X_0\big)$ and $t \in [s,T]$ set*

$$K_s(x,g)(t) := U(t,s)x + \int_s^t U(t,\tau)g(\tau)d\tau.$$

*Then for any $0 \le \alpha \le \beta < 1$*

$$K_s \in \mathcal{L}\big(X_\beta \times L_\infty\big((s,T),X_0\big), C^{\beta-\alpha}\big([s,T],X_\alpha\big)\big).$$

*Moreover, it holds that*

$$K_s \in \mathcal{L}\big(X_1 \times L_\infty\big((s,T),X_\gamma\big), C\big([s,T],X_1\big)\big),$$

*whenever $\gamma \in (0,1]$.*

In terms of solutions of the inhomogeneous Cauchy-problem the above result reads:

**5.7 Corollary**
*Let $\beta \in [0,1)$ and $(s,x,f) \in [0,T) \times X_\beta \times C\big([s,T],X_0\big)$ be given. Then, if the inhomogeneous Cauchy-problem* (2.1) *has a* (*necessarily unique*) *solution $u$, it lies in the space $C^{\beta-\alpha}\big([s,T],X_\alpha\big)$ for any $\alpha \in [0,\beta]$. If $(s,x,f) \in [0,T) \times X_1 \times C\big([s,T],X_\gamma\big)$ for some $\gamma \in (0,1]$, then the solution of* (2.1) *lies in $C\big([s,T],X_1\big)$.*

Observe that $\alpha = \beta = 1$ is only possible if the image of $g$ lies in a space $X_\gamma$ which is smaller than $X_0$. The reason for this is that the integral after the first inequality sign in (5.8) is not finite if $\beta = 1$.

**C. Existence of solutions for inhomogeneous Cauchy-problems:** We will now give an existence result for the inhomogeneous Cauchy-problem which complements the result of Sobolevskii and Tanabe in Section 2. But we first need a simple technical lemma.

**5.8 Lemma**
*Let $Y$ be a Banach space and $a \in C(\dot{\Delta}_T,Y)$. Suppose there exists an $\alpha \in [0,1)$ such that*

$$\|a(t,\tau)\|_Y \le c(t-\tau)^{-\alpha}$$

*holds for all $(t,\tau) \in \dot{\Delta}_T$ and some constant $c \ge 0$. Setting*

$$v(t,s) := \int_s^t a(t,\tau)\,d\tau$$

*for $(t,s) \in \Delta_T$, we have*

$$v \in C(\Delta_T,Y).$$

**Proof**

For each $\varepsilon > 0$ consider the closed subset $\Delta_T^\varepsilon$ of $\Delta_T$ defined by

$$\Delta_T^\varepsilon := \{(t,s) \in \Delta_T\,;\; t - s \geq \varepsilon\}.$$

Define now for $(t,s) \in \Delta_T$

$$v_\varepsilon(t,s) := \begin{cases} \displaystyle\int_s^{t-\varepsilon} a(t,\tau)\,d\tau & \text{for } (t,s) \in \Delta_T^\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$\|v_\varepsilon(t,s) - v(t,s)\|_Y \leq \begin{cases} \displaystyle\int_{t-\varepsilon}^{t} \|a(t,\tau)\|_Y\,d\tau \leq \frac{1}{1-\alpha}\varepsilon^{1-\alpha} & \text{for } (t,s) \in \Delta_T^\varepsilon \\ \displaystyle\int_s^t \|a(t,\tau)\|_Y\,d\tau \leq \frac{1}{1-\alpha}\varepsilon^{1-\alpha} & \text{otherwise,} \end{cases}$$

which implies that $v_\varepsilon \to v$ uniformly on $\Delta_T$ as $\varepsilon \to 0$. It therefore suffices to prove the continuity of $v_\varepsilon\colon \Delta_T \to Y$ for each $\varepsilon > 0$.

To do this fix an $\varepsilon > 0$ and let $(t_n, s_n) \in \Delta_T^\varepsilon$, $n \in \mathbb{N}$, be such that $(t_n, s_n) \to (\tilde{t}, \tilde{s})$ as $n \to \infty$. Define now $g_n, \tilde{g}\colon [0,T] \to Y$ by

$$g_n(\tau) := \begin{cases} a(t_n,\tau) & \text{for } s_n \leq \tau \leq t_n \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \tilde{g}(\tau) := \begin{cases} a(\tilde{t},\tau) & \text{for } \tilde{s} \leq \tau \leq \tilde{t} \\ 0 & \text{otherwise,} \end{cases}$$

respectively. We obviously have

$$g_n \to \tilde{g} \quad \text{almost everywhere}$$

and

$$\|g_n(\tau)\| \leq (t_n - \tau)^{-\alpha} \leq \varepsilon^{-\alpha}$$

for any $\tau \in [s_n, t_n]$. This gives

$$\|g_n(\tau)\| \leq \varepsilon^{-\alpha}$$

for any $\tau \in [0,T]$. By Lebesgue's dominated convergence theorem we obtain $g_n \to \tilde{g}$ in $L_1\big((0,T),Y\big)$. In particular

$$v_\varepsilon(t_n,s_n) = \int_0^T g_n(\tau)\,d\tau \to \int_0^T \tilde{g}(\tau)\,d\tau = v_\varepsilon(\tilde{t},\tilde{s}).$$

This gives the continuity of $v_\varepsilon$ on $\Delta_T^\varepsilon$. To get continuity on $\Delta_T$ observe that the boundary of $\Delta_T^\varepsilon$ in $\Delta_T$ consists of the points $(t, t-\varepsilon)$, $t \in [0, T-\varepsilon]$, and on this points $v_\varepsilon$ takes the value 0. But $v_\varepsilon$ also vanishes in $\Delta_T \setminus \Delta_T^\varepsilon$. This implies the continuity of $v_\varepsilon$ on $\Delta_T$ and proves the lemma. □

The main theorem on the solvability of the inhomogeneous Cauchy-problem (2.1) is the following one.

**5.9 Theorem**
*Assume that the function $f\colon [0,T] \to X_0$ satisfies one of the following conditions:*

(1) $f \in C^\eta\big([0,T], X_0\big)$ *for some* $\eta \in (0,1]$,

(2) $f \in C\big([0,T], X_\gamma\big)$ *for some* $\gamma \in (0,1]$.

*Then, for any $(s,x) \in [0,T) \times X_0$ the Cauchy-problem* (2.1) *has a unique solution $u(\cdot\,; s,x) \in C\big([s,T], X_0\big) \cap C^1\big((s,T], X_0\big)$. Moreover, we have for an arbitrary $\beta \in [0,1)$ that*

$$u(\cdot\,;\cdot\,,x) \in C(\dot{\Delta}_T, X_\beta) \tag{5.11}$$

*holds, and if $x \in X_\beta$ we even have*

$$u(\cdot\,;\cdot\,,x) \in C(\Delta_T, X_\beta). \tag{5.12}$$

*If* (2) *holds, we can also admit $\beta = 1$ in* (5.11) *and* (5.12).

**Proof**
We first prove that (5.11) and (5.12) hold assuming that we have already shown existence. Define now

$$v(t,s) := \int_s^t U(t,\tau) f(\tau)\, d\tau$$

for $(t,s) \in \Delta_T$. Then, by the variation-of-constants formula, we get that

$$u(t;s,x) = U(t,s)x + v(t,s) \quad \text{for all } (t,s) \in \Delta_T. \tag{5.13}$$

Observe that, by $(U1)$, for any $\beta \in [0,1]$ it holds that

$$U(\cdot\,,\cdot)x \in C(\dot{\Delta}_T, X_\beta), \tag{5.14}$$

and even

$$U(\cdot\,,\cdot)x \in C(\Delta, X_\beta), \tag{5.15}$$

provided $x \in X_\beta$. Therefore, we only have to worry about the integral term $v$ in (5.13).

Fix now $0 \le \beta < 1$. Note that by continuity $\{\|f(t)\|;\ t \in [0,T]\}$ is a bounded set. From Lemma 5.2 we thus obtain

$$\|U(t,\tau)f(\tau)\|_\beta \le \|U(t,\tau)\|_{0,\beta}\|f(\tau)\|_0 \le c(t-\tau)^{-\beta} \tag{5.16}$$

for all $(t,\tau) \in \dot{\Delta}_T$. The preceding lemma with $a(t,\tau) = U(t,\tau)f(\tau)$ then shows that

$$v \in C(\Delta_T, X_\beta). \tag{5.17}$$

This gives (5.11) and (5.12) for $\beta \in [0,1)$.

Assume now that (2) holds and that $\beta = 1$. Again by continuity of the function $f$, the set $\{\|f(t)\|_\gamma;\ t \in [0,T]\}$ is bounded. Consequently, by Lemma 5.2

$$\|U(t,\tau)f(\tau)\|_1 \le \|U(t,\tau)\|_{\gamma,1}\|f(\tau)\|_1 \le c(t-\tau)^{\gamma-1}. \tag{5.18}$$

and by Lemma 5.8

$$v \in C(\Delta_T, X_1). \tag{5.19}$$

Hence, if (2) holds we obtain (5.11) and (5.12) with $\beta = 1$.

We now turn to the question of existence. In case that (1) holds existence is ensured by Theorem 2.8. Suppose that (2) holds. By Remark 2.7(a) we only have to prove that

$$v(\cdot\,,s) \in C^1\big((s,T], X_0\big) \tag{5.20}$$

and

$$\partial_t v(t,s) = -A(t)v(t,s) + f(t) \quad \text{for } t \in (s,T] \tag{5.21}$$

hold.

To this end let $h > 0$ be such that $s < t < t+h \le T$ and observe that

$$h^{-1}\big(v(t+h,s) - v(t,s)\big) = h^{-1}\big(a(h) + b(h)\big), \tag{5.22}$$

where we have set

$$a(h) := \int_s^t \big(U(t+h,\tau) - U(t,\tau)\big)f(\tau)\,d\tau$$

and

$$b(h) := \int_t^{t+h} U(t+h,\tau)f(\tau)\,d\tau.$$

We then have

$$\|h^{-1}b(h) - f(t)\|_\gamma \le h^{-1}\int_t^{t+h} \|U(t+h,\tau)f(\tau) - U(t,t)f(t)\|_\gamma d\tau.$$

But the righthand side of the above inequality tends to 0 as $h \to 0$, because

$$[(t,\tau) \mapsto U(t,\tau)f(\tau)] \in C(\Delta_T, X_\gamma)$$

as easily follows from Lemma 5.2 $(a)$. Thus

$$h^{-1}b(h) \to f(t) \quad \text{in } X_\gamma \text{ as } h \to 0. \tag{5.23}$$

To deal with $a(h)$ note that by $(U4)$

$$U(t+h,\tau) - U(t,\tau) = -\int_t^{t+h} A(\sigma)U(\sigma,\tau)\,d\sigma$$

for any $s \le \tau < t$, and thus, by $(A3)$ and Lemma 5.2,

$$\begin{aligned}\|U(t+h,\tau) - U(t,\tau)\|_0 &\le c\int_t^{t+h} \|A(\sigma)\|_{1,0}\|U(\sigma,\tau)\|_{\gamma,1}d\tau \\ &\le ch(t-\tau)^{\gamma-1}.\end{aligned}$$

Note that, by $(U4)$, for each $\tau \in [s,t)$

$$h^{-1}\Big(U(t+h,\tau) - U(t,\tau)\Big)f(\tau) \to -A(t)U(t,\tau)f(\tau)$$

as $h$ tends to 0. This together with the preceding inequality allow us to apply Lebesgue's dominated convergence theorem to obtain

$$h^{-1}a(h) \to \int_s^t A(t)U(t,\tau)f(\tau)\,d\tau.$$

We have seen in (5.18) that $U(t,\cdot)f(\cdot) \in L_1\big((s,t),X_1\big)$. Furthermore, $A(t) \in \mathcal{L}(X_1,X_0)$ so that (see [**88**], Chapter 11, Theorem 5.1)

$$\int_s^t A(t)U(t,\tau)f(\tau)\,d\tau = A(t)\int_s^t U(t,\tau)f(\tau)\,d\tau = A(t)v(t,s).$$

Thus

$$h^{-1}a(h) \to A(t)v(t,s)$$

as $h \searrow 0$, and we obtain the right differentiability of $v(\cdot\,,s)$ on $(s,T)$ and that the right derivative $\partial_t^+ v(\cdot\,,s)$ of $v(\cdot\,,s)$ satisfies

$$\partial_t^+ v(t,s) = -A(t)v(t,s) + f(t)$$

for all $t \in (s,T)$. But since, by (5.19), $v(\cdot\,,s) \in C\big([s,T],X_1\big)$ we obtain from $(A3)$

$$\partial_t^+ v(\cdot\,,s) \in C\big([s,T],X_0\big).$$

This implies that $v(\cdot\,,s)$ is continuously differentiable on $(s,T]$, and that (5.21) is satisfied. This concludes the proof of this theorem. □

**D. Perturbation Theorems:** We shall prove a perturbation theorem similar to Theorem 1.3.

**5.10 Theorem**

*Let $\big(A(t)\big)_{0\leq t\leq T}$ satisfy conditions $(A1)$–$(A3)$. If either*

$(a)$ $B(\cdot)\in C^\rho\big([0,T],\mathcal{L}(X_\alpha,X_0)\big)$ *for some* $\alpha\in[0,1)$

*or in the case* $\alpha=1$

$(b)$ $B(\cdot)\in C^\rho\big([0,T],\mathcal{K}(X_1,X_0)\big)$

*then the family $\big(A(t)+B(t)\big)_{0\leq t\leq T}$ satisfies conditions $(A1)$, $(A2')$ and $(A3)$.*

**Proof**

It is clear that $\big(A(t)+B(t)\big)_{0\leq t\leq T}$ satisfies conditions $(A1)$ and $(A3)$. It remains to show $(A2')$.

$(a)$ In virtue of Theorem 1.3($a$), we have to show that for any $a>0$ there exists a constant $b>0$ such that

$$\|B(t)x\|\leq a\|A(t)x\|+b\|x\|$$

holds for all $x\in X_1$ and $t\in[0,T]$. Taking into account Remark 2.1(c) and the fact that $\|B(t)\|_{\alpha,0}$ is uniformly bounded with respect to $t\in[0,T]$ it is sufficient to show that for any $a>0$ there exists a constant $b>0$ such that

$$\|x\|_\alpha\leq a\|x\|_1+b\|x\|_0 \tag{5.24}$$

holds for all $x\in X_1$. Note that by the interpolation inequality (see Proposition 3.6) there exists a constant $c(\alpha)$ such that for all $x\in X_1$

$$\|x\|_\alpha\leq c(\alpha)\|x\|_1^{1-\alpha}\|x\|_0^\alpha\,. \tag{5.25}$$

Applying Young's inequality (see e.g. [**60**], Corollary 2.2.3) to the right hand side we obtain

$$\|x\|_1^{1-\alpha}\|x\|_0^\alpha=\varepsilon\|x\|_1^{1-\alpha}\varepsilon^{-1}\|x\|_0^\alpha\leq(1-\alpha)\varepsilon^{\frac{1}{1-\alpha}}\|x\|_1+\alpha\varepsilon^{-\frac{1}{\alpha}}\|x\|_0$$

for all $x\in X_1$ and $\varepsilon>0$. Together with (5.25) assertion (5.24) follows and the proof of the theorem in case $(a)$ is complete.

$(b)$ It is not hard to see that the set $\big\{B(t)x;\ t\in[0,T],\|x\|_1\leq1\big\}$ is relatively compact in $X_0$. An application of the precise statement of Theorem 1.3($b$) yields the assertion. □

**E. Diagonal systems:** Of course, the above theory is applicable to diagonal systems of the form we described in Example 2.9(e). Suppose that all the conditions made in this example hold. Using Proposition 3.4 it is then possible to describe the spaces $X_\alpha$ via decoupling. More precisely, it holds that

$$X_\alpha \doteq \prod_{i=1}^{N} X_\alpha^i,$$

where $X_\alpha^i := (X_0^i, X_1^i)$ for all $i = 1, \dots, N$. In certain applications this allows us to give a complete characterization of the interpolation spaces $X_\alpha$.

As an example we consider the following system

$$\begin{cases} \partial_t u + \mathcal{A}_{11}(t)u + \mathcal{A}_{12}(t)v = h_1(x,t) & \text{in } \Omega \times (0,\infty) \\ \partial_t v + \mathcal{A}_{22}(t)v + \mathcal{A}_{21}(t)u = h_2(x,t) & \text{in } \Omega \times (0,\infty) \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \partial\Omega \times (0,\infty) \\ \big(u(0), v(0)\big) = (u_0, v_0) & \text{on } \Omega, \end{cases} \tag{5.26}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ of class $C^\infty$. Here, we assume that $\mathcal{A}_{ii}(t)$ and $\mathcal{B}_i(t)$ $(i = 1, 2)$ satisfy the conditions of Example 2.9(d) and that $\mathcal{A}_{12}(t)$ and $\mathcal{A}_{21}$ have the form

$$\mathcal{A}_{12}(t)v = \sum_{i=1}^{n} a_i^1(x,t)\partial_i v + a_0^1 v \qquad \text{and} \qquad \mathcal{A}_{21}(t)v = \sum_{i=1}^{n} a_i^2(x,t)\partial_i u + a_0^2 u\,,$$

where $a_i^k \in C^{\eta \frac{\eta}{2}}\big(\overline{\Omega} \times \mathbb{R}_+\big)$ for $i = 0, \dots n$ and $k = 1, 2$.

Let $A_i(t)$ be the $L_p$-realization of the elliptic boundary value problems $(\Omega, \mathcal{A}_i, \mathcal{B}_i)$ for $i = 1, 2$ and some $p \in (1, \infty)$,

$$X_0 := L_p(\Omega, \mathbb{R}^2) \qquad \text{and} \qquad X_1 := W^2_{p,\mathcal{B}_1}(\Omega) \times W^2_{p,\mathcal{B}_2}(\Omega).$$

Define $\boldsymbol{A}(t)$ by $\boldsymbol{A}(t)(u,v) :=\big(A_1(t)u, A_2(t)v\big)$ for all $(u,v) \in X_1$. Then, $\big(\boldsymbol{A}(t)\big)_{0\le t\le T}$ satisfies the conditions $(A1)$–$(A3)$.

Take now the complex interpolation method and set $X_\alpha := [X_0, X_1]_\alpha$ for any $\alpha \in (0,1)$. By definition of $X_0$ and $X_1$ as well as Proposition 3.4 and Theorem 4.16 we have that

$$X_\alpha \doteq H^{2\alpha}_{p,\mathcal{B}_1}(\Omega) \times H^{2\alpha}_{p,\mathcal{B}_2}(\Omega) \tag{5.27}$$

for all $\alpha \in [0,1] \setminus \{1/p, 1 + 1/p\}$.

Set now

$$\boldsymbol{B}(t)(u,v) := \begin{bmatrix} 0 & \mathcal{A}_{12}(t) \\ \mathcal{A}_{21}(t) & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

for all $u, v \in X_\alpha$ with $\alpha > 1/2$. Then it is obvoius that

$$\boldsymbol{B}(\cdot) \in C^{\eta/2}\big([0,T], \mathcal{L}(X_\alpha, X_0)\big)$$

for some $\alpha \in (1/2, 1)$. Applying the perturbation Theorem 5.10($a$) we get that the family $\big(-(\boldsymbol{A}(t)+\boldsymbol{B}(t))\big)_{0\le t\le T}$ satisfies the conditions ($A1$)–($A3$). So we have found a complete description of the interpolation spaces corresponding to the $L_p$-realization of a system of the form (5.26).

A similar result holds if we replace the complex by the real interpolation method. Of course, the same considerations may be carried out for an arbitrary system, where the 'principal part' is in diagonal form.

Another example of a system is treated in Section 26. For more general systems we refer to [**15**] or [**18**].

**Notes and references:** The evolution operator on fractional power spaces was considered, for instance, by Amann [**7**], Friedman [**59**], Henry [**66**] or Pazy [**100**]. The idea to consider evolution equations on interpolation spaces goes back to Da Prato and Grisvard [**39**]–[**41**]. These ideas were adopted by the italian school to establish results under various assumptions on the family $\big(A(t)\big)_{0\le t\le T}$. We refer to Acquistapace and Terreni [**1**], Lunardi [**91**], [**92**], Sinestrari and Vernole [**113**] to name just a few references. They deal mostly with the continuous interpolation method and the so called maximal regularity theory and their approach does not always involve the construction of an evolution operator (for an overview see [**31**] or [**112**]).

The interpolation space approach as we present it here is due to Amann [**10**], where he also admits time-dependent domains of definition $D(A(t))$. The idea is to mimic the theory of abstract semilinear parabolic evolution equations in fractional power spaces as it is used in the literature listed above. A more satisfactory treatment of such evolution equations in the case where $D(A(t))$ depends on time was accomplished in Amann [**13**]. The underlying idea may be described as follows.

Consider abstract linear evolution equations of the type

$$(*) \qquad \partial_t u + A(t)u = f(t), \qquad t \in (0,T],$$

on the Banach space $X_0$, where the domains of definition, $X_1(t) := D(A(t))$, of the operators $A(t)$ may depend on $t \in [0,T]$. The main point is to construct for each $t \in [0,T]$ a scale $(X_\alpha(t))_{\alpha\in[-1,1]}$ of Banach spaces , such that

$$X_0(t) = X_0 \qquad \text{and} \qquad X_1(t) \overset{d}{\hookrightarrow} X_\beta(t) \overset{d}{\hookrightarrow} X_\alpha(t) \overset{d}{\hookrightarrow} X_{-1}(t)$$

hold for each $-1 \le \alpha \le \beta \le 1$, and a family $\{A_\alpha(t);\ \alpha \in [-1,1]\}$ of operators, such that for each $\alpha \in [-1,0]$ we have

$$A_\alpha(t) \in \mathcal{L}\big(X_{\alpha+1}(t), X_\alpha(t)\big),$$

$$A_0(t) = A(t) \qquad \text{and} \qquad A_\alpha(t) = A_{-1}(t) \restriction_{X_{\alpha+1}(t)} .$$

Assuming that for some $\beta \in [-1, 0]$ we have that $X_\beta(t)$ and $X_{\beta+1}(t)$ are independent of $t$ we may then put

$$X_0 := X_\beta := X_\beta(0) \qquad \text{and} \qquad X_1 := X_{\beta+1} := X_{\beta+1}(0)$$

and look at the Cauchy-problem

$$(*)_\beta \qquad \partial_t u + A_\beta(t)u = f(t) \qquad t \in (0, T]$$

in the Banach space $X_0$. The spaces $X_\beta(t)$ are called *extrapolation spaces*. After verifying the validity of assumptions $(A1)-(A3)$ of Section 2, we may apply the Sobolevskii-Tanabe theory to construct the evolution operator for $(*)_\beta$. One then has two alternatives: Either we work in the weaker setting of the Banach space $X_\beta$, or, under suitable assumptions, we 'lift' the evolution operator on $X_\beta$ to an evolution operator on $X_0$.

Note that the above mentioned generalization for time-dependent domains of definition is by no means of exclusively academic interest. Indeed, this theory may be applied to parabolic initial-boundary value problems, where the boundary conditions depend on time (see Amann [**15**] or [**18**]).

# II. Linear periodic evolution equations

In this chapter we investigate in some depth the evolution operator in the case that the time-dependency is periodic. In particular, we derive some estimates related to spectral decompositions of the period-map. We also prove some results for the inhomogeneous problem. In the last section we establish a Floquet representation for linear inhomogeneous nonautonomous parabolic equations.

## 6. The evolution operator

We start this section giving some basic properties of the evolution operator and its spectrum in the case the family $\big(A(t)\big)_{t\in\mathbb{R}}$ depends periodically on $t$ with some period $T>0$. Then we investigate stability properties of the solutions of a periodic homogeneous Cauchy problem. In order to prove stability in weaker norms in Section 7 we provide here the required estimates for the evolution operator. In the final subsection we give two simple criteria for the existence of periodic solutions for inhomogeneous Cauchy problems.

**A. Basic features:** As usual $X_0$ and $X_1$ denote Banach spaces, such that $X_1 \overset{d}{\hookrightarrow} X_0$. Let $T>0$. We now consider a family $\big(A(t)\big)_{t\in\mathbb{R}}$ of closed linear operators in $X_0$ satisfying

$(A0)\quad A(t+T)=A(t)$ for any $t\in\mathbb{R}$,

as well as $(A1)$–$(A3)$ from Section 2.

Setting

$$\Delta_\infty := \{(t,s);0\le s\le t<\infty\} \qquad \text{and} \qquad \dot{\Delta}_\infty := \{(t,s);0\le s< t<\infty\}$$

it is clear by the results of Section 2 that there is a unique function

$$U\colon\Delta_\infty\to\mathcal{L}(X_0),$$

satisfying $(U1)$–$(U4)$ on $\Delta_{nT}$ for each $n\in\mathbb{N}$. This function will be called the *evolution operator* for the $T$-periodic family $(A(t))_{t\in\mathbb{R}}$

The $T$-periodicity of the problem is captivated by the following simple lemma:

**6.1 Lemma**

*For any $(t,s) \in \Delta_\infty$ the following identities hold:*

(*a*) $U(t+T, s+T) = U(t,s)$.

(*b*) $U(t+nT, s) = U(t+T,t)^n U(t,s) = U(t,s)U(s+T,s)^n$ *for any* $n \in \mathbb{N}$.

**Proof**

(*a*) Let $0 \le s < nT$ for some $n \in \mathbb{N}$. Take $x \in X_0$ and set $v(t) := U(t+T, s+T)x - U(t,s)x$ for $t \in [s, nT]$. Then, by $(U4)$ and the $T$-periodicity of $A(\cdot)$,

$$\partial_t v(t) + A(t)v(t) = 0 \quad \text{and} \quad v(s) = 0$$

holds for every $t \in (s, nT]$. By the uniqueness of the solution to this problem we obtain $v(t) = 0$ for all $t \in [s, nT]$, which proves the assertion.

(*b*) By $(U2)$ and part $(a)$ we get for any $s \ge 0$ and $n \in \mathbb{N}$

$$U(s+T,s)^n = \prod_{k=0}^{n-1} U\bigl(s+(n-k)T, s+(n-k-1)T\bigr) = U(s+nT, s),$$

where this product is to be taken in the order of appearance of the index, since the operators do not necessarily commute. Again by $(U2)$ and part $(a)$ we have for any $(t,s) \in \Delta_\infty$

$$U(t+nT,t)U(t,s) = U(t+nT,s) = U(t+nT,s+nT)U(s+nT,s) = U(t,s)U(s+nT,s).$$

These two observations prove $(b)$. □

Set now

$$\Delta := \{\,(t,s) \in \mathbb{R}^2\,;\ s \le t\,\} \qquad \text{and} \qquad \dot{\Delta} := \{\,(t,s) \in \mathbb{R}^2\,;\ s < t\}.$$

By part $(a)$ of the preceding lemma we can define $U(t,s)$ for any $(t,s) \in \Delta$ by setting

$$U(t,s) := U(t+nT, s+nT)$$

for any $n \in \mathbb{N}$, such that $(t+nT, s+nT) \in \Delta_\infty$. Furthermore, the function $U\colon \Delta \to \mathcal{L}(X_0)$ satisfies $(U1)$–$(U4)$ on $\Delta \cap [-nT, nT]^2$ for any $n \in \mathbb{N}$. Of course Lemma 6.1 holds also on $\Delta$.

We consider now the *$T$-periodic inhomogeneous Cauchy-problem*:

$$(6.1) \qquad \begin{cases} \partial_t u + A(t)u = f(t) & \text{for } t > s \\ u(s) = x, \end{cases}$$

where $(s,x) \in \mathbb{R} \times X_0$ and $f \in C(\mathbb{R}_+, X_0)$ . Observe that just now we do not require that $f$ be $T$-periodic.

**6.2 Definition**

By a *solution* of (6.1) we mean a function

$$u \in C\big([s,\infty), X_0\big) \cap C^1\big((s,\infty), X_0\big)$$

such that $u(t) \in X_1$ for $t > s$, $u(s) = x$, and $\partial_t u(t) + A(t)u(t) = f(t)$ for $t > s$. □

By the variation-of-constants formula in Section 2 the solution $u$, if existent, is given by

$$u(t) = U(t,s)x + \int_s^t U(t,\tau)f(\tau)\,d\tau$$

for every $t \geq s$. Furthermore, by Theorem 5.9, if either $f$ is locally Hölder-continuous, or takes values in an interpolation space $X_\alpha$ and is continuous with respect to this stronger topology, then the solution exists.

We now proceed to study the homogeneous Cauchy-problem (6.1), that is $f \equiv 0$. Of particular importance in the investigation of periodic problems is the *shift-operator*, or *period-map* associated with (6.1). This operator is defined by:

$$S(s) := U(s+T, s) \in \mathcal{L}(X_0).$$

In the following proposition we collect some important properties of the period-map:

**6.3 Proposition**

($a$) $S(s+T) = S(s)$ *for all* $s \geq 0$.

($b$) $U(t+nT, s) = S(t)^n U(t,s) = U(t,s)S(s)^n$ *for all* $(t,s) \in \Delta$ *and* $n \in \mathbb{N}$.

($c$) $\sigma(S(s)) \setminus \{0\}$ *is independent of* $s \in \mathbb{R}$.

($d$) $\sigma_p(S(s)) \setminus \{0\}$ *is independent of* $s \in \mathbb{R}$.

**Proof**

Observe that it suffices to prove the statements for $t, s \geq 0$, and that assertions ($a$) and ($b$) are basically a restatement of Lemma 6.1.

($d$) Let $\mu$ be a non-zero eigenvalue of $S(s)$ and $t \in [s, s+T]$. We will show that $\mu$ is an eigenvalue of $S(t)$. From this the assertion follows by the periodicity of $S(\cdot)$. Take $x \in X_0 \setminus \{0\}$, such that $S(s)x = \mu x$ and set $y := U(t,s)x$. By ($b$) we obtain:

$$S(t)y = U(t,s)S(s)x = \mu U(t,s)x = \mu y.$$

Furthermore, we have

$$U(s+T,t)y = S(s)x = \mu x \neq 0,$$

so that $y \neq 0$, which implies that $\mu \in \sigma_p(S(t))$.

($c$) Let now $\mu \in \varrho(S(s)) \setminus \{0\}$ and $t \in [s, s+T]$. To prove that $\mu \in \varrho(S(t))$, we note that $\mu - S(t)$ is injective by ($d$). We show that $\mu - S(t)$ is surjective, and thus continuously invertible by the open mapping theorem. The assertion follows then by the periodicity of $S(\cdot)$.

For any $y \in X_0$ set:

$$x := \mu^{-1}\Big(y + U(t,s)\big(\mu - S(s)\big)^{-1}U(s+T,t)y\Big).$$

Then

$$S(t)x = \mu^{-1}\Big(S(t)y + S(t)U(t,s)\big(\mu - S(s)\big)^{-1}U(s+T,t)y\Big) = \mu x - y,$$

where we used that

$$\begin{aligned} S(t)U(t,s)\big(\mu - S(s)\big)^{-1}U(s+T,t)y &= U(t,s)S(s)\big(\mu - S(s)\big)^{-1}U(s+T,t)y \\ &= -U(t,s)\Big(-\mu + \big(\mu - S(s)\big)\Big)\big(\mu - S(s)\big)^{-1}U(s+T,t)y \\ &= \mu U(t,s)\big(\mu - S(s)\big)^{-1}U(s+T,t)y - U(t,s)U(s+T,t)y \\ &= \mu U(t,s)\big(\mu - S(s)\big)^{-1}U(s+T,t)y - S(t)y. \end{aligned}$$

In the above manipulations we took advantage of the fact that $S(s)$ commutes with its resolvent $\big(\mu - S(s)\big)^{-1}$ and that, by Lemma 6.1,

$$U(t,s)U(s+T,t) = U(t+T,s+T)U(s+T,t) = S(t).$$

This gives $\big(\mu - S(t)\big)x = y$, i.e. the surjectivity of $\mu - S(t)$, and completes the proof of the lemma. □

**6.4 Remarks**

(a) For each $\alpha \in [0,1]$ and $s \in \mathbb{R}$, we can consider $S(s)$ as an operator in $\mathcal{L}(X_\alpha)$. Noting that in the proof of Proposition 6.3($c$) we have $x \in X_\alpha$ if $y \in X_\alpha$, we see that the non-zero part of the spectrum of $S(s)$ depends neither on $s \in \mathbb{R}$ nor on $\alpha \in [0,1]$.

(b) By Remark 5.1, if $A(0)$ has compact inverse and $0 \leq \alpha < 1$, then $S(s)$ is a compact operator on $X_\alpha$. This implies that in this case the non-zero part of the spectrum of $S(s)$ consists entirely of eigenvalues, so that statements ($c$) and ($d$) in the preceding proposition are actually equivalent.

(c) We also remark that the eigenvalues of $S(s)$, which by Proposition 6.3 are independent of $s \in \mathbb{R}$, are called *Floquet* or *characteristic multipliers* of the homogeneous $T$-periodic equation

$$\partial_t u(t) + A(t)u(t) = 0 \qquad \text{for } t \in \mathbb{R}. \tag{6.2}$$

(d) Observe that (6.2) has a non-trivial $T$-periodic solution – i.e. a $T$-periodic function $u \in C^1(\mathbb{R}, X_0)$ satisfying (6.2) – if and only if 1 is a Floquet multiplier. This follows immediately from Proposition 6.3($a$). Indeed, if $u$ is a non-trivial $T$-periodic solution, and we put $x = u(0)$, we obtain

$$x = u(0) = u(T) = U(T, 0)x = S(0)x,$$

so that 1 is a Floquet multiplier. The opposite implication is clear. □

**B. Stability of the zero solution:** We now turn to the study of the asymptotic behaviour of the solution $u(\cdot) = U(\cdot\,, s)x$ of the homogeneous Cauchy-problem (6.1).

By Proposition 6.3 the spectral radius $r(S(s))$ does not depend either on $s \in \mathbb{R}$ or on the special choice of the underlying space $X_\alpha$, $\alpha \in [0, 1]$. Therefore, either there exists a unique real number $\omega_0$ such that

$$r(S(s)) = e^{-T\omega_0} \tag{6.3}$$

holds for all $s \in \mathbb{R}$, or $r(S(s)) = 0$ for all $s \in \mathbb{R}$. In this last case we put $\omega_0 = \infty$.

**6.5 Lemma**
*Let $s \in [0, T]$, $\alpha \in [0, 1]$ and $\omega < \omega_0$. Then there exists a constant $M := M(\alpha, s, \omega) \geq 1$ such that*

$$\|U(t, s)\|_{\alpha,\alpha} \leq M e^{-(t-s)\omega}$$

*holds for all $t \geq s$.*

**Proof**
To simplify the notation we write $\|\cdot\|$ for $\|\cdot\|_{\alpha,\alpha}$ and set $m_\alpha := \sup_{0 \leq t-r \leq T} \|U(t, r)\|$. By Lemma 5.2($b$) and Lemma 6.1($a$) we have that $m_\alpha < \infty$.

We first show that

$$e^{(t-s)\omega}\|U(t, s)\| \to 0 \qquad \text{for } t \to \infty. \tag{6.4}$$

Assuming that (6.4) does not hold we find an $\varepsilon > 0$ and an increasing sequence $t_k \nearrow \infty$, satisfying:

$$\varepsilon \leq e^{(t_k - s)\omega}\|U(t_k, s)\| \qquad \textit{for all } k \in \mathbb{N}. \tag{6.5}$$

Take now $n_k \in \mathbb{N}$ and $r_k \in [0, T)$ with: $t_k - s = n_k T + r_k$. Then by (6.5)

$$\varepsilon e^{-r_k \omega} \le e^{n_k T \omega} \|U(s + n_k T + r_k, s)\|.$$

Observing that, by Proposition 6.3, $U(s + n_k T + r_k, s) = U(s + r_k, s)S(s)^{n_k}$ holds, we obtain

$$\varepsilon e^{-r_k \omega} \le e^{n_k T \omega} \|U(s + r_k, s)\| \, \|S(s)^{n_k}\|.$$

Taking $n_k$-th roots on both sides of this inequality we get:

$$\varepsilon^{\frac{1}{n_k}} e^{\frac{-r_k \omega}{n_k}} \le e^{T\omega} m_\alpha^{\frac{1}{n_k}} \|S(s)^{n_k}\|^{\frac{1}{n_k}}$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$ we find

$$1 \le e^{T\omega} e^{-T\omega_0} = e^{-T(\omega_o - \omega)} < 1,$$

which is absurd. This shows that (6.4) holds.

By (6.4) there exists a $t_0 > 0$ such that $\|U(t,s)\| \le e^{-(t-s)\omega}$ for all $t \ge t_0$. Taking

$$M := \max\Big\{1, \big(\sup_{s \le t \le t_0} \|U(t,s)\|\big) \cdot \big(\min_{0 \le t \le t_0} e^{-(t-s)\omega}\big)^{-1}\Big\},$$

the assertion follows. □

We are now able to prove the following important estimate:

**6.6 Theorem**
*Let $\alpha \in [0,1]$ and $\omega < \omega_0$. Then there exists a constant $M := M(\alpha, \omega) \ge 1$, such that*

$$\|U(t,s)\|_{\alpha,\alpha} \le M e^{-(t-s)\omega} \tag{6.6}$$

*holds for all $(t,s) \in \Delta$.*

**Proof**
We need only consider the case $(t,s) \in \Delta_\infty$. As in the proof of Lemma 6.5, we set $\|\cdot\| := \|\cdot\|_{\alpha,\alpha}$ and $m_\alpha := \sup_{0 \le t-s \le T} \|U(t,s)\|$.

$(i)$ Take first $s \in [0,T]$ and $t \ge T$. By Lemma 6.5 we get

$$\begin{aligned} \|U(t,s)\| &\le \|U(t,T)\| \, \|U(T,s)\| \\ &\le c(\alpha, T) m_\alpha e^{-(t-T)\omega} \le M_1(\alpha, \omega) e^{-(t-s)\omega}. \end{aligned}$$

This implies by Lemma 6.1$(a)$ that

$$\|U(t,s)\| \le M_1(\alpha, \omega) e^{-(t-s)}$$

whenever $t - s \geq 0$.

$(ii)$ Let now $(t, s) \in \Delta_\infty$ satisfy $0 \leq t - s \leq T$. Take $n \in \mathbb{N}$ such that $t - nT,\ s - nT \in [0, T]$. Then

$$\|U(t,s)\| = \|U(t - nT, s - nT)\| \leq m_\alpha$$

holds, so that

$$\|U(t,s)\| \leq M_2(\alpha, \omega) e^{-(t-s)\omega}$$

where $M_2(\alpha, \omega) := m_\alpha / \min\limits_{0 \leq r \leq T} e^{-r\omega}$.

This, together with $(i)$, implies inequality (6.6), proving the proposition. □

**6.7 Remark**

A consequence of Theorem 6.6 is that, if $\omega_0 > 0$ holds, the trivial solution (i.e. the zero solution) of (6.2) is *exponentially asymptotically stable,* i.e., there exists $\omega > 0$ and $M > 0$, such that

$$\|u(t)\|_\alpha \leq M e^{-t\omega} \qquad \text{for } t > 0$$

for any solution $u$ of (6.1), i.e. for any $u(\cdot) = U(\cdot, 0)x$ for some $x \in X_0$. Of course this estimate does not provide much information if $\omega \leq 0$. In the next section we shall give a more careful analysis of the asymptotics of the solutions of (6.1). □

**C. Some more estimates for the evolution operator:** We shall prove some estimates for the evolution operator when considered as an operator between different interpolation spaces. They resemble those in Lemma 5.2(b). Recall that $\omega_0 \in \mathbb{R}$ is defined as in (6.3). This estimates turn out to be useful in Section 22 where we prove that solutions of semilinear time-periodic equations that are bounded with respect to a norm which is weaker than the 'phase-space' norm are also bounded with respect to this latter one.

**6.8 Proposition**

*Let* $0 \leq \alpha < \beta \leq 1$ *and* $\omega < \omega_0$*. Then, there exists a constant* $N := N(\alpha, \beta, \omega) \geq 1$*, such that*

$$\|U(t,s)\|_{\alpha,\beta} \leq N e^{-(t-s)\omega}(t-s)^{\alpha-\beta} \tag{6.7}$$

*holds for all* $(t, s) \in \Delta$.

**Proof**

Without loss of generality we may assume that $\omega_0 > 0$, since, otherwise, we consider the family $(A(t) - k)_{t \in \mathbb{R}}$ for $k$ large enough and use Remark 2.7(b) to obtain the assertion for $(A(t))_{t \in \mathbb{R}}$.

Choose an $\varepsilon > 0$ such that $0 < \omega < \omega_0$. Since $\lim_{t\to\infty} t^{\beta-\alpha}e^{-t\varepsilon} = 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that

$$t^{\beta-\alpha}e^{-t\varepsilon} < 1 \tag{6.8}$$

whenever $t \geq n(\varepsilon)T$ holds.

$(i)$ We first prove the assertion for the case that $t - s \geq (n(\varepsilon) + 1)T$ holds. Take $n \geq n(\varepsilon)$ such that

$$(n+1)T \leq t - s \leq (n+2)T$$

is satisfied. Observe that (6.8) implies that

$$\begin{aligned} e^{-(n+2)T(\omega+\varepsilon)} &\leq e^{-(t-s)(\omega+\varepsilon)} \leq (t-s)^{\alpha-\beta}e^{-(t-s)\omega}(t-s)^{\beta-\alpha}e^{-(t-s)\varepsilon} \\ &\leq (t-s)^{\alpha-\beta}e^{-(t-s)\omega} \end{aligned} \tag{6.9}$$

holds.

Since $T \leq t - s - nT \leq 2T$ holds, we may apply Lemma 5.2(b) on $\Delta_{2T}$ to obtain

$$\|U(t, s + nT)\|_{\alpha,\beta} \leq c(\alpha, \beta)(t - s - nT)^{\alpha-\beta} \leq c(\alpha, \beta)T^{\alpha-\beta}. \tag{6.10}$$

This together with Lemma 6.6 and (6.9) now yield

$$\begin{aligned} \|U(t,s)\|_{\alpha,\beta} &\leq \|U(t, s+nT)\|_{\alpha,\beta}\|U(s+nT, s)\|_{\alpha,\alpha} \\ &\leq c(\alpha,\beta)M(\alpha,\omega+\varepsilon)T^{\alpha-\beta}e^{-nT(\omega+\varepsilon)} \\ &\leq c(\alpha,\beta)M(\alpha,\omega+\varepsilon)T^{\alpha-\beta}e^{2T}e^{-(n+2)T(\omega+\varepsilon)} \\ &\leq N(\alpha,\beta,\omega)(t-s)^{\alpha-\beta}e^{-(t-s)\omega} \end{aligned} \tag{6.11}$$

where we have set $N(\alpha, \beta, \omega) := c(\alpha, \beta)M(\alpha, \omega + \varepsilon)T^{\alpha-\beta}e^{2T}$. This proves (6.7) in case that $t - s \geq (n(\varepsilon) + 1)T$ holds.

$(ii)$ In case that $t-s \leq (n(\varepsilon+1)T$ holds the assertion follows easily from Lemma 5.2$(b)$ applied on $\Delta_{(n(\varepsilon)+1)T}$. □

**D. The inhomogeneous equation:** We close this section with some results on $T$-periodic solutions of the $T$-periodic inhomogeneous equation

$$\partial_t u(t) + A(t)u(t) = f(t) \quad \text{for } t \in \mathbb{R}, \tag{6.12}$$

where $f\colon \mathbb{R} \to X_0$ is now taken to be a $T$-periodic function. Here a $T$*-periodic solution* of (6.12) is a function $u \in C^1(\mathbb{R}, X_0)$ which is $T$-periodic, i.e. $u(t + T) = u(t)$ for all $t \in \mathbb{R}$, and such that $u(t) \in X_1$ for all $t \in \mathbb{R}$ and (6.12) is satisfied.

**6.9 Proposition**

*Assume that* $f\colon \mathbb{R} \to X_0$ *lies either in* $C^\eta(\mathbb{R}, X_0)$, *for some* $\eta \in (0,1)$, *or in* $C(\mathbb{R}, X_\alpha)$, *for some* $n\alpha \in (0,1]$. *Furthermore let* $f$ *be* $T$*-periodic.*

*Then if* $1 \in \varrho(S(0))$, (6.12) *has a unique* $T$*-periodic solution* $u$ *given by*

$$u(t) = U(t,0)x + \int_0^t U(t,\tau)f(\tau)\,d\tau \tag{6.13}$$

*for all* $t \geq 0$, *where we have set*

$$x := \big(1 - S(0)\big)^{-1} \int_0^T U(T,\tau)f(\tau)\,d\tau. \tag{6.14}$$

**Proof**

Suppose $u$ is a $T$-periodic solution of (6.12). Set $x := u(0)$. Then

$$\big(1 - S(0)\big)x = u(T) - U(T,0)x = \int_0^T U(T,\tau)f(\tau)\,d\tau$$

holds by the variation-of-constants formula. Thus (6.14) holds and the solution is unique.

On the other hand, if $x$ is defined by (6.14) we see that $u(T) = x$. Using $(a)$ and $(b)$ in Proposition 6.3 we thus obtain

$$\begin{aligned} u(t+T) &= U(t+T,0)x + \int_0^{t+T} U(t+T,\tau)f(\tau)\,d\tau \\ &= U(t+T,T)\Big(U(T,0)x + \int_0^T U(T,\tau)f(\tau)\,d\tau + \int_T^{t+T} U(t+T,\tau)f(\tau)\,d\tau\Big) \\ &= U(t,0)x + \int_0^t U(t,\tau)f(\tau)\,d\tau \\ &= u(t) \end{aligned}$$

for all $t \geq 0$, and $u$ is $T$-periodic. □

**6.10 Remark**

We have showed in the proof of the proposition that there is a one-to-one correspondence between the $T$-periodic solutions of (6.12) and the fixed-points of the mapping $S_f\colon X_0 \to X_0$ defined by

$$S_f(x) := U(T,0)x + \int_0^T U(T,\tau)f(\tau)\,d\tau.$$

$S_f$ is called the *period-map* corresponding to (6.12). □

The next theorem gives a condition which is necessary and sufficient for the existence of a $T$-periodic solution of (6.12) provided $A(0)$ has a compact inverse.

**6.11 Theorem**

*Assume that $A(0)$ has a compact inverse. Then, the $T$-periodic equation* (6.12) *has a $T$-periodic solution if and only if there exists an $x \in X_0$ such that the solution of* (6.1) *is bounded.*

**Proof**

If $u$ is a $T$-periodic solution it is clear that it is bounded and that it is the solution of $(6.1)_{(0,u(0),f)}$.

Assume now that there is no $x \in X_0$, such that the solution of (6.1) is $T$-periodic. This means

$$\big(1 - S(0)\big)x \neq y := \int_0^T U(T,\tau)f(\tau)\,d\tau \quad \text{for all } x \in X_0.$$

Observe now that since $A(0)$ has a compact inverse and, hence, $S(0)$ is compact, the operator $\big(1 - S(0)\big)$ is a Fredholm operator of index 0 ([**88**]). In particular $\operatorname{im}\big(1 - S(0)\big)$ is a closed subspace of $X_0$. This and the fact that $\operatorname{im}\big(1 - S(0)\big)^{\perp} = \ker\big(1 - S(0)'\big)$ holds – compare Section 0.E – imply that for each $x' \in \ker\big((1 - S(0))'\big)$ we have

$$<x', y> \neq 0 \quad \text{and} \quad S(0)'x' = x'.$$

Let now $x_0 \in X_0$ be arbitrary and set for $k \in \mathbb{N}$

$$x_k := U(kT,0)x_0 + \int_0^{kT} U(kT,\tau)f(\tau)\,d\tau.$$

It is easy to see that

$$x_k := S(0)^k x_0 + \sum_{j=0}^{k-1} S(0)^j y,$$

so that we have

$$\begin{aligned} <x', x_k> &= <\big(S(0)'\big)^k x', x_0> + \sum_{j=0}^{k-1} <\big(S(0)^j\big)' x', y> \\ &= <x', x_0> + \sum_{j=0}^{k-1} <x', y>. \end{aligned}$$

As $<x', y> \neq 0$ holds, we obtain the unboundedness of the sequence $(x_k)$, i.e. the unboundedness of the solution of (6.1). This proves the theorem. □

**Notes and references:** The results of this section are a rather straight forward generalization of the material in Chapter 8 of Henry's lecture notes [**66**]. Theorem 6.10 is the infinite dimesional version of a well known result for ordinary (finite dimensional) differential equations (see e.g. [**17**], Theorem 20.3). Other references on the evolution operator in the periodic case are Lunardi [**93**] and Fuhrmann [**61**]. There, they also investigate the existence of bounded solutions of inhomogeneous equations, which is interesting in view of Theorem 6.10. The results on bounded solutions are used in Lunardi [**95**] to treat some control theoretic problems. Consult also Daleckiĭ and Krein [**33**] and Krasnoselskiĭ [**82**] for the case of bounded operators.

## 7. Spectral decompositions

We continue to investigate the evolution operator $U\colon \Delta \to \mathcal{L}(X_0)$ associated to the periodic family $\big(A(t)\big)_{t\in\mathbb{R}}$ considered in the preceding section and derive some estimates which will be of use in the study of the inhomogeneous problem. Recall that the period-map with initial time $s \in \mathbb{R}$ was given by $S(s) := U(s+T, s)$ and that $\sigma(S(s)) \setminus \{0\}$ is independent of $s$.

**A. Separation of the spectrum:** Suppose that $\sigma_2 \subset \mathbb{C} \setminus \{0\}$ is a spectral set of $S(s)$ for one (and thus for all) $s \in \mathbb{R}$, i.e. $\sigma_2$ is a subset of $\sigma(S(s))$ which is both open and closed in $\sigma(S(s))$. We set $\sigma_1(s) := \sigma(S(s)) \setminus \sigma_2$. Note that $\sigma_1(s) \setminus \{0\}$ is independent of $s \in \mathbb{R}$. We will sometimes write $\sigma_2(s)$ instead of $\sigma_2$ if notational convenience dictates. We have then:

$$\sigma(S(s)) = \sigma_1(s) \,\dot\cup\, \sigma_2. \tag{7.1}$$

Let $U \subset \mathbb{C}$ be a bounded domain, such that

$$\sigma_2 \subset U, \quad \sigma_1(s) \subset \mathbb{C} \setminus \bar U \text{ for all } s \in \mathbb{R}, \text{ and}$$
$$\partial U = \bigcup_{j=1}^{m} \Gamma_j,$$

where $(\Gamma_j)_{1\le j\le m}$ is a family of disjoint smooth Jordan-curves which are positively oriented with respect to $U$. We can then define a family of projections on $X_0$ by the following Dunford-type integral:

$$P_2(s) := \frac{1}{2\pi i} \int_{\partial U} \big(\lambda - S(s)\big)^{-1}\, d\lambda. \tag{7.2}$$

$P_2(s)$ is called the *spectral projection with respect to the spectral set $\sigma_2$ of $S(s)$*. Setting

$$P_1(s) := \mathbb{1} - P_2(s) \quad \text{and} \quad X_0^i(s) := P_i(s)(X_0) \tag{7.3}$$

for $s \in \mathbb{R}$ and $i = 1, 2$, we obtain a decomposition of $X_0$ in $S(s)$-invariant subspaces

$$X_0 = X_0^1(s) \oplus X_0^2(s), \tag{7.4}$$

such that

$$\sigma(S_i(s)) = \sigma_i(s), \quad i = 1, 2, \tag{7.5}$$

where for each $s \in \mathbb{R}$ we have defined $S_i(s) \colon X_0^i(s) \to X_0^i(s)$ by $S_i(s)x := S(s)x$ for $x \in X_0^i(s)$. Thus, with obvious notation,

$$S(s) = S_1(s) \oplus S_2(s).$$

**7.1 Lemma**
*Let* $(t, s) \in \Delta$. *Then:*
(*a*) $U(t, s) \in \mathcal{L}\big(X_0^1(s), X_0^1(t)\big)$
(*b*) $U(t, s) \in \mathrm{Isom}\big(X_0^2(s), X_0^2(t)\big)$

**Proof**
By Proposition 6.3 we have

$$U(t, s)\big(\lambda - S(s)\big)^{-1} = \big(\lambda - S(t)\big)^{-1} U(t, s)$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$ lying in the resolvent set of $S(r)$ for one and thus for all $r \geq 0$. From (7.2) it follows that

$$U(t, s)P_2(s) = P_2(t)U(t, s).$$

This implies $U(t, s) \in \mathcal{L}\big(X_i(s), X_i(t)\big)$ for $i = 1, 2$.

Assume now that there exists an $x \in X_2(s) \setminus \{0\}$, such that $U(t, s)x = 0$. Then by Proposition 6.3 we obtain

$$S_2(s)^n x = U(s + nT, s)x = U(s + nT, t)U(t, s)x = 0$$

for any $n \in \mathbb{N}$, such that $s \leq t \leq s + nT$. This means $0 \in \sigma(S_2(s)) = \sigma_2$, a contradiction. Thus $U(t, s) \restriction X_2(s)$ is injective.

Let $y \in X_2(t)$ be arbitrary and let $n \in \mathbb{N}$ be such that $s \leq t \leq s + nT$. Then $S_2(t)^n$ is invertible and we find a $z \in X_2(t)$, satisfying $S_2(t)^n z = y$. Set $x := U(s + nT, t)z$. Then

$$U(t, s)x = U(t + nT, s + nT)U(s + nT, t)z = U(t + nT, t)z = S_2(t)^n z = y,$$

which shows that $U(t, s)\big(X_2(s)\big) = X_2(t)$, proving $(b)$. □

We can thus define for $i = 1, 2$ and $(t, s) \in \Delta$

$$U_i(t, s)\colon X_0^i(s) \to X_0^i(t)$$

by setting $U_i(t, s)x := U(t, s)x$ for any $x \in X_0^i(s)$, i.e.

$$U(t, s) = U_1(t, s) \oplus U_2(t, s)$$

holds. By the preceding lemma we can also define

$$U_2(s, t) := U_2(t, s)^{-1}$$

for any $(t, s) \in \Delta$.

**7.2 Remarks**

(*a*) It is evident that the following identity holds:

$$U_2(t, s) = U_2(t, r)U_2(r, s) \quad \text{for any } r, s, t \in \mathbb{R}$$

(*b*) By the spectral theorem we have $\sigma(S_2(s)^{-1}) = \{\frac{1}{\lambda}\,;\ \lambda \in \sigma_2\}$, so that

$$\mathrm{r}\big(S_2(0)^{-1}\big) = \inf\{|\lambda|\,;\ \lambda \in \sigma_2\}$$

holds. □

As $\sigma_1(s) \setminus \{0\}$ is independent of $s \in \mathbb{R}$, the same is true of $\mathrm{r}\big(S_1(s)\big)$. Let now $\omega_1$ and $\omega_2$ be real numbers, such that

$$\mathrm{r}\big(S_1(s)\big) = e^{-T\omega_1}$$

and

$$\mathrm{r}\big(S_2(s)^{-1}\big) = e^{-T\omega_2}$$

hold for every $s \in \mathbb{R}$. The subsequent estimates for $U_1$ and $U_2$ will prove to be extremely useful:

**7.3 Theorem**

*Let $\alpha \in [0, 1]$.*

(*a*) *For any $\omega < \omega_1$ there exists a constant $M_1 := M_1(\alpha, \omega) \geq 1$, such that*

$$\|U_1(t, s)x\|_\alpha \leq M_1 e^{-(t-s)\omega}\|x\|_\alpha \tag{7.6}$$

*holds for any $(t, s) \in \Delta$ and $x \in X_0^1(s) \cap X_\alpha$.*

(*b*) *For any* $\omega > \omega_2$ *there exists a constant* $M_2 := M_2(\alpha, \omega)$*, such that*

$$\|U_2(s,t)x\|_\alpha \le M_2 e^{-(t-s)\omega}\|x\|_\alpha \tag{7.7}$$

*holds for any* $(t,s) \in \Delta$ *and* $x \in X_0^2(t) \cap X_\alpha$.

**Proof**

The proof of (*a*) is identical to the proof of Theorem 6.6 and (*b*) is proved in a very similar way. □

**B. The hyperbolic case:** We consider the case where $S(s)$ is an hyperbolic endomorphism of $X_0$ – i.e. $\sigma(S(s)) \cap \mathbb{S} = \emptyset$ – for one (and thus for all) $s \in \mathbb{R}$. We set for each $s \in \mathbb{R}$

$$\sigma_1(s) := \sigma(S(s)) \cap [|\mu| < 1]$$

and

$$\sigma_2 := \sigma_2(s) := \sigma(S(s)) \setminus \sigma_1(s).$$

We are now in the situation of Subsection A and have

$$e^{-T\omega_1} < 1 < e^{-T\omega_2},$$

i.e. $\omega_1 > 0 > \omega_2$.

**7.4 Proposition**

*Let* $\alpha \in [0,1]$ *and* $\omega_2 < -\gamma_2 < 0 < \gamma_1 < \omega_1$. *Then*

$$\|U_1(t,s)x\|_\alpha \le M_1(\gamma_1,\alpha)e^{-(t-s)\gamma_1}\|x\|_\alpha$$

*and*

$$M_2(-\gamma_2,\alpha)e^{(t-s)\gamma_2}\|x\|_\alpha \le \|U(t,s)x\|_\alpha$$

*hold for every* $(t,s) \in \Delta$ *and every* $x \in X_0^1(s) \cap X_\alpha$ *and* $x \in X_0^2(s) \cap X_\alpha$ *respectively. The constants appearing are the same constants as in Theorem* 7.3.

**Proof**

The assertions follows immediately from Theorem 7.3 by observing that

$$\|x\|_\alpha \le \|U_2(s,t)\|_{\alpha,\alpha}\|U(t,s)x\|_\alpha$$

holds for every $(t,s) \in \Delta$ and $x \in X_0^2(s) \cap X_\alpha$. □

We thus obtain a splitting of the space of initial values $X_0$ into two invariant subspaces $X_0^1(0)$ and $X_0^2(0)$ such that the solution of $(6.1)_{(0,x,0)}$ is exponentially attracted (exponentially repelled) by the zero solution of (6.2) if and only if $x$ belongs to $X_0^1(0)$ ($X_0^2(0)$).

For this reason $X_0^1(0)$, resp. $X_0^2(0)$, is called the stable, resp. unstable, manifold of (6.2). Linear homogeneous evolution equations for which a splitting in a stable and an unstable manifold can be found are said to admit an exponential dichotomy. More on exponential dichotomies can be found in [**66**].

**Notes and references:** Again, the results on spectral decompositions are essentially contained in Chapter 8 of Henry [**66**]. They play an essential rôle in the next section and have been used to study bounded solutions of inhomogeneous equations in Lunardi [**93**] and Fuhrmann [**61**]. In Krein [**84**] some somewhat more general ideas are expounded on evolution operators beeing decomposed by a time dependent splitting of the space of initial values.

## 8. Floquet representations

We make the same assumptions as in Section 6.

**A. Motivation:** Suppose, as in the previous section, that $\sigma_2$ is a spectral set for the period map $S(0)$ which does not contain 0. Therefore, we have an spectral decomposition $\sigma\big(S(s)\big) = \sigma_1(s) \mathbin{\dot\cup} \sigma_2$ for all $s \in \mathbb{R}$. In Section 7 we obtained a time-dependent (in fact $T$-periodic) splitting, $X_1(t) \oplus X_2(t)$, of $X_0$, which is invariant with respect to the evolution operator $U\colon \Delta \to \mathcal{L}(X_0)$, i.e. we have a decomposition

$$U(t,s) = U_1(t,s) \oplus U_2(t,s)\colon X_1(s) \oplus X_2(s) \to X_1(t) \oplus X_2(t)$$

for $(t,s) \in \Delta$. Moreover, $\sigma(S_2(s)) = \sigma_2$ for all $s \in \mathbb{R}$, where $S_i(s) := U_i(s+T,s)$, for $i = 1,2$.

Now, the ideal situation would be that such a spectral decomposition as above should induce a decoupling of (6.2) into two evolution equations: one in $X_1(0)$ and the other on $X_2(0)$. Up to now, if we take an initial value in $X_i(s)$, for some $s \in \mathbb{R}$ and $i = 1,2$, we are only able to follow the solution, $t \mapsto U(t,s)x$, along the time-dependent spaces $X_i(s+t)$ as time evolves, but we are not able to write down an equation on a fixed space for it. In fact, this does not seem to be possible for both $U_1(\cdot,s)$ and $U_2(\cdot,s)$. Nevertheless, under suitable assumptions on $\sigma_2$, we shall be able to find a $T$-periodic family of transformations

$$Q(t) \in \mathcal{GL}\big(X_2(0), X_2(t)\big),$$

for $t \in \mathbb{R}$, and a bounded operator $B \in \mathcal{L}\big(X_2(0)\big)$, such that the *Floquet representation*

$$U_2(t,s) = Q(t)e^{(t-s)B}Q(s)^{-1}$$

holds for $(t,s) \in \Delta$. This means that $U_2$ may be transformed in such a way as to render the evolution operator of the time-independent evolution equation

$$\dot{v} + Bv = 0$$

on $X_2(0)$.

It is well-known that in the case of linear time-periodic finite-dimensional equations one can give a Floquet representation of the full evolution operator. This means that the full original equation is equivalent to an equation with time-independent principal part (cf. [**17**]). This result does not hold in this generality even in the case of bounded $A(t)$'s. In [**33**] it is proved that this is only possible if one is able to define the logarithm of the period-map $S(0)$, thus imposing certain restrictions on its spectrum.

**B. Autonomizing one part of the equation:** As we have already said we continue to work under the same hypotheses as in the previous section but make the following additional assumption:

$\sigma_2$ does not surround the origin of the complex plane.

This is the case, for instance, if there is a ray starting at the origin which is not intercepted by $\sigma_2$. Observe that we can always find such a ray if we assume that $A(0)$ has compact resolvent, since then the period maps $S(s)$ are compact operators and $\sigma_2$ consists of a finite number of points (see Remark 6.4 (b)).

The condition above ensures that we can find a branch of the logarithm defined on a neighbourhood of $\sigma_u$. This allows us to define a logarithm of the operator $V_2(0) \in \mathcal{L}(X_2(0))$ by using the well-known functional calculus given by the Dunford-integral (cf. [**127**]). So we have the following

**8.1 Lemma**
*There exists an operator $B \in \mathcal{L}\big(X_2(0)\big)$, namely $B := -\frac{1}{T}\log V_2(0)$, satisfying*

$$e^{-TB} = V_2(0) = U_2(T,0).$$

The operator $B$ is not necessarily uniquely determined since, as a rule, different branches of the logarithm will produce different $B$'s.

We now set for each $t \in \mathbb{R}$

$$Q(t) := U_2(t,0)e^{tB} \in \mathcal{L}\big(X_2(0), X_2(t)\big).$$

We then have the following important result:

**8.2 Proposition**

*(i)* $Q(t+T) = Q(t)$ *for each* $t \in \mathbb{R}$ *and* $Q(0) = \mathbb{1}_{X_u(0)}$.

*(ii)* $Q(t) \in \mathrm{Isom}(X_2(0), X_2(t))$ *for each* $t \in \mathbb{R}$.

*(iii)* $U_2(t,s) = Q(t)e^{-(t-s)B}Q(s)^{-1}$ *for each* $t, s \in \mathbb{R}$.

**Proof**

$(i)$ By definition of $B$ and the results of Section 6 we see that

$$Q(t+T) = U_2(t+T,0)e^{(t+T)B} = U_2(t,0)U_2(T,0)e^{TB}e^{tB} = U_2(t,0)e^{tB} = Q(t)$$

$(ii)$ This is clear since both $U_2(t,0)$ and $e^{tB}$ are isomorphisms.

$(iii)$ This follows from the following simple calculation

$$\begin{aligned} Q(t)e^{-(t-s)B}Q(s)^{-1} &= U_2(t,0)e^{tB}e^{-(t-s)B}e^{-sB}U_2(0,s) \\ &= U_2(t,0)U_2(0,s) = U_2(t,s) \end{aligned}$$

where we used Remark 7.2(a). □

The representation of $U_2(t,s)$ given in part $(iii)$ of the proposition above is called the *Floquet representation* of the $X_2(t)$-part of the evolution operator.

Now, suppose that $u$ is the mild solution of the inhomogeneous Cauchy-problem

$$(8.1) \qquad \begin{cases} \partial_t u + A(t)u = f(t) & \text{for } t > s \\ u(s) = x, & \end{cases}$$

where $(s,x) \in \mathbb{R} \times X_0$ and $f \in C([s,\infty), X_0)$ are given. We set for every $t \geq s$:

$$x_1 := P_1(s)x, \quad x_2 := P_2(s)x, \qquad u_1(t) := P_1(t)u(t) \qquad \text{and} \qquad u_2(t) := P_2(t)u(t).$$

Using (7.2) we see that $U(t,s)P_i(s) = P_i(t)U(t,s)$ holds for every $(t,s) \in \Delta$. From this together with the fact that $u$ is given by the variation-of-constants formula we readily obtain:

$$(8.2) \qquad u_1(t) = U_1(t,s)x_1 + \int_s^t U_1(t,\tau)P_1(\tau)f(\tau)\,d\tau$$

and also

$$(8.3) \qquad u_2(t) = U_2(t,s)x_2 + \int_s^t U_2(t,\tau)P_2(\tau)f(\tau)\,d\tau.$$

We can now show that (8.3) is equivalent to an equation in $X_2(0)$ with time-independent and bounded principal part.

**8.3 Proposition**
*If we set*

$$v(t) := Q(t)^{-1}u_2(t) \qquad \text{and} \qquad y := Q(s)^{-1}x_2,$$

*then $v$ lies in $C^1\big((s,\infty), X_2(0)\big)$ and is the unique solution of*

$$\begin{cases} \partial_t v + Bv = Q(t)^{-1}P_2(t)f(t) & \text{for } t > s \\ \qquad v(s) = y \end{cases} \tag{8.4}$$

**Proof**
We apply $Q(t)^{-1}$ to both sides of (8.3) and obtain

$$\begin{aligned} v(t) &= Q(t)^{-1}U_2(t,s)Q(s)Q(s)^{-1}x_2 + \int_s^t Q(t)^{-1}U_2(t,\tau)Q(\tau)Q(\tau)^{-1}P_2(\tau)f(\tau)\,d\tau \\ &= e^{-(t-s)B}y + \int_s^t e^{-(t-\tau)B}Q(\tau)^{-1}P_2(\tau)f(\tau)\,d\tau. \end{aligned}$$

Thus, $v$ is a mild solution of (8.4). But since $B$ is a bounded operator, every mild solution is also a classical solution. This proves the proposition. □

**Notes and references:** The idea of autonomizing just a portion of the equation was taken from Henry's monograph [**66**], and seems to go back to a paper by Stokes [**117**] on functional differential equations. In [**30**] Chow, Lu and Mallet-Paret obtain a full Floquet representation for a class of time-periodic 1-dimensional parabolic equations, but they do not work in an abstract setting. For related topics consult the bibliography in [**30**].

In [**79**] the second author applied the present autonomization to some problems in control theory. Using Floquet representations he was able to give necessary and sufficient conditions for the stabilizability of an abstract evolution equation by means of a feedback mechanism. He also aplied the abstract results to second order linear and semilinear initial-boundary value problems on bounded domains.

# III. Miscellaneous

In this chapter we shall describe further properties of the evolution operator such as approximations, parameter dependence and superconvexity. Because of the technical nature of the subject the reader is advised to just browse over the matherial at first reading.

## 9. Abstract Volterra integral equations

In this section we investigate abstract integral equations. For several reasons it is important to know how to solve such an integral equations. On the one hand, the theory is used to solve equation (2.5) which arises in the construction of the evolution operator. On the other hand, it shall be used when considering Yosida approximations or parameter dependence in the next sections.

We start by a subsection containing the general theory. In a second subsection, we consider integral equations which are concerned with the evolution operator. The final subsection is devoted to the construction of the evolution operator for a family of bounded operators.

**A. Abstract Volterra integral equations:** Let $X$ be a Banach space and $T > 0$. We consider the integral equations

$$u(t,s) = a(t,s) + \int_s^t u(t,\tau)k_1(\tau,s)\,d\tau \tag{9.1}$$

and

$$v(t,s) = a(t,s) + \int_s^t h_1(t,\tau)v(\tau,s)\,d\tau \tag{9.2}$$

in the space $C\big(\Delta_T, \mathcal{L}_s(X)\big)$. We assume that $a(\cdot\,,\cdot)$ and $b(\cdot\,,\cdot)$ belong to $C\big(\Delta_T, \mathcal{L}_s(X)\big)$, and $k_1(\cdot\,,\cdot)$ and $h_1(\cdot\,,\cdot)$ to $C\big(\dot{\Delta}_T, \mathcal{L}(X)\big)$. Moreover, assume that there exist constants $c_0 > 0$ and $\rho \in (0,1]$ such that

$$\|k_1(t,s)\| \le c_0(t-s)^{\rho-1} \qquad \text{and} \qquad \|h_1(t,s)\| \le c_0(t-s)^{\rho-1} \tag{9.3}$$

holds for all $(t,s) \in \dot{\Delta}_T$. Note, that in all our considerations, $\mathcal{L}_s(X)$ can be replaced by $\mathcal{L}(X)$.

We give now the construction of a solution of (9.1). Equation (9.2) is solved in a completely analogous manner.

The idea is to consider the integral term as a linear operator and to show that its spectral radius is zero. More precisely, for any $v \in C\big(\Delta_T, \mathcal{L}_s(X)\big)$ we define

$$[Qv](t,s) := \int_s^t v(t,\tau)k_1(\tau,s)\,d\tau \tag{9.4}$$

for all $(t,s) \in \Delta_T$. Lemma 5.8 implies that

$$Q \in \mathcal{L}\big(C(\Delta_T, \mathcal{L}_s(X))\big) \tag{9.5}$$

holds. Using these notations, (9.8) takes the form

$$(\mathbb{1} - Q)u = a \tag{9.6}$$

in $C\big(\Delta_T, \mathcal{L}_s(X)\big)$. If we are able to show that $Q$ has spectral radius zero, the solution of (9.6) can be represented by Neumann's series. In the theory of integral equations, the operator $(\mathbb{1}-Q)^{-1} - Q = Q(\mathbb{1}-Q)^{-1}$ is called the *resolvent of* $Q$. The resolvent of $Q$ – as we show in a moment – is an integral operator, too. Its kernel is called the *resolvent kernel of* $Q$. It can be constructed by successive approximation as follows:

Note that $Q^m$ is an integral operator, whose kernel is defined inductively by

$$k_m(t,s) := \int_s^t k_{m-1}(t,\tau)k_1(\tau,s)\,d\tau \tag{9.7}$$

for all $m \geq 2$ and $(t,s) \in \Delta_T$. By induction, it can be seen that

$$\|k_m(t,s)\| \leq \frac{\big[c_0\,\Gamma(\rho)\big]^m}{\Gamma(m\rho)}\,(t-s)^{\rho m-1}, \tag{9.8}$$

holds for all $m \geq 1$ and $(t,s) \in \dot{\Delta}_T$. To prove this we used (9.3) and the fact that for all $\alpha, \beta > 0$

$$\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{9.9}$$

and

$$\int_s^t (t-\tau)^{\alpha-1}(\tau-s)^{\beta-1}\,d\tau = (t-s)^{\alpha+\beta-1}\,\mathrm{B}(\alpha,\beta) \tag{9.10}$$

for all $(t, s) \in \Delta_T$, where $\mathrm{B}(\cdot\,,\cdot)$ and $\Gamma(\cdot)$ denote the Beta and the Gamma function respectively.

In particular, estimate (9.8) implies that the spectral radius of $Q$ – which is given by $\lim_{m\to\infty} \|Q^m\|^{1/m}$ – is zero. Due to (9.8) it is clear the the series

$$k(t,s) := \sum_{m=1}^{\infty} k_m(t,s) \tag{9.11}$$

converges in $\mathcal{L}(X)$ and that it can be estimated by

$$\|k(t,s)\| \le (t-s)^{\rho-1} \sum_{m=1}^{\infty} \frac{\big[c_0\,\Gamma(\rho)\big]^m}{\Gamma(m\rho)}\, T^{\rho(m-1)} =: C(t-s)^{\rho-1} \tag{9.12}$$

for all $(t, s) \in \dot{\Delta}_T$. The constant $C > 0$ depends only on $c_0$, $\rho$ and $T$. Moreover, the convergence of (9.11) is uniform with respect to $(t, s) \in \Delta_T^{\varepsilon}$ for all $\varepsilon > 0$, where

$$\Delta_T^{\varepsilon} := \{(t,s) \in \Delta_t;\ t - s \ge \varepsilon\}\,. \tag{9.13}$$

Thus, $k(\cdot\,,\cdot)$ is an element of $C\big(\dot{\Delta}_T, \mathcal{L}(X)\big)$.

In the following theorem, we collect the results obtained by the considerations above.

**9.1 Theorem**

*Suppose that the hypotheses from above are satisfied. Then, the equations* (9.1) *and* (9.2) *have a unique solution in* $C\big(\Delta_T, \mathcal{L}_s(X)\big)$. *By means of the resovent kernels* $k(\cdot\,,\cdot)$ *and* $h(\cdot\,,\cdot)$, *this solution can be represented by*

$$u(t,s) = a(t,s) + \int_s^t a(t,\tau)k(\tau,s)\,d\tau \tag{9.14}$$

*and*

$$u(t,s) = b(t,s) + \int_s^t h(t,\tau)b(\tau,s)\,d\tau \tag{9.15}$$

*respectively. If* $a(\cdot\,,\cdot)$ *and* $b(\cdot\,,\cdot)$ *are in* $C\big(\Delta_T, \mathcal{L}(X)\big)$, *so are* $u(\cdot\,,\cdot)$ *and* $v(\cdot\,,\cdot)$. *Moreover, the resolvent kernels can be estimated by*

$$\|k(t,s)\| \le C(t-s)^{\rho-1} \qquad \text{and} \qquad \|h(\cdot\,,\cdot)\| \le C(t-s)^{\rho-1} \tag{9.16}$$

*for all* $(t, s) \in \dot{\Delta}_T$, *where* $C$ *is a constant depending only on* $c_0$, $\rho$ *and* $T$.

We turn now to some integral equations concerning the evolution operator for a family of closed operators.

**B. Integral equations for the evolution operator:** As usual we assume that $X_0$ and $X_1$ are Banach spaces with $X_1 \overset{d}{\hookrightarrow} X_0$ and that $\big(A(t)\big)_{0\le t\le T}$ is a family of closed operators on $X_0$ satisfying conditions $(A1)$–$(A3)$ of Section 2.

We shall consider here two integral equations which are satisfied by the evolution operator $U(\cdot\,,\cdot)$ associated to the family $\big(A(t)\big)_{0\le t\le T}$.

In order to construct the evolution operator in the proof of Theorem 2.6 we have seen, that it must satisfy the integral equation

$$U(t,s) = e^{-(t-s)A(s)} + \int_s^t U(t,\tau)R_1(\tau,s)\,d\tau$$

in $C\big(\Delta_T, \mathcal{L}_s(X_0)\big)$, where the kernel $R_1(\cdot\,,\cdot)$ is given by

$$R_1(t,s) = -\big(A(t)-A(s)\big)A^{-1}(s)\,A(s)e^{-(t-s)A(s)} \tag{9.17}$$

for all $(t,s)\in\dot{\Delta}_T$. It turns out later, that the assumptions of the previous subsection are fulfilled, which shows that $U(\cdot\,,\cdot)$ lies in $C\big(\Delta_T, \mathcal{L}(X_0)\big)$. It is the purpose of the following considerations to construct a similar integral equation, which is solved by $A(t)U(t,s)A^{-1}(s)$. In particular, this shows, that $U(\cdot\,,\cdot)$ lies in $C\big(\Delta_T, \mathcal{L}(X_1)\big)$.

Using the properties of the evolution operator, we get that for all $(t,s)\in\dot{\Delta}_T$ and $t\ge\tau>s$:

$$\partial_\tau\big[e^{-(t-\tau)A(t)}U(\tau,s)\big] = e^{-(t-\tau)A(t)}\big(A(t)-A(\tau)\big)U(\tau,s).$$

Applying $A(t)$ from the left and $A^{-1}(s)$ from the right and integrating both sides, we obtain

$$A(t)U(t,s)A^{-1}(s) = e^{-(t-s)A(t)}A(t)A^{-1}(s) + \int_s^t H_1(t,\tau)A(\tau)U(\tau,s)A^{-1}(s)\,d\tau \tag{9.18}$$

for all $(t,s)\in\Delta_T$. The kernel $H_1(\cdot\,,\cdot)$ is for every $(t,s)\in\dot{\Delta}_T$ given by

$$H_1(t,s) = A(t)e^{-(t-\tau)A(t)}\big(A(t)-A(\tau)\big)A^{-1}(s)\,. \tag{9.19}$$

The kernels (9.17) and (9.19) are very similar. We deduce now an estimate of the form (9.3) for them.

By $(A3)$ we find a constant $c_1>0$ depending only on the Hölder norm of $A(\cdot)$ and a bound for $\|A(s)\|_{1,0}$, such that

$$\|\big(A(t)-A(\tau)\big)A^{-1}(s)\| \le c_1(t-s)^\rho \tag{9.20}$$

holds for all $(t,s)\in\dot{\Delta}_T$. Furthermore, it follows from $(A2)$, Remark 1.2 and formula (1.18), that there exists a constant $c_2>0$ depending only on $M$ (in $(A2)$), such that

$$\|A(t)e^{-\tau A(t)}\| \le c_2\tau^{-1} \tag{9.21}$$

holds for all $t \in [0,T]$ and $\tau > 0$. By the definition of $R_1(\cdot\,,\cdot)$ and $H_1(\cdot\,,\cdot)$, it follows now immediately that for any $(t,s) \in \dot{\Delta}_T$

$$\|R_1(t,s)\| \le c_0(t-s)^{\rho-1} \qquad \text{and} \qquad \|H_1(t,s)\| \le c_0(t-s)^{\rho-1}, \tag{9.22}$$

where $c_0 := c_1c_2$. Moreover, from Remark 1.2, the representation formula (1.17) as well as $(A2)$ and $(A3)$ it can be seen that $[(t,s) \mapsto e^{-(t-s)A(s)}] \in C\big(\Delta_T, \mathcal{L}_s(X_0)\big)$ and $[(t,s) \mapsto A(s)e^{-(t-s)A(s)}] \in C\big(\dot{\Delta}_T, \mathcal{L}(X_0)\big)$. The same smoothness conditions hold for $(t,s) \mapsto e^{-(t-s)A(t)}$ and $(t,s) \mapsto A(t)e^{-(t-s)A(t)}$ respectively. In particular, this ensures, that $R_1(\cdot\,,\cdot)$ and $H_1(\cdot\,,\cdot)$ are in $C\big(\dot{\Delta}_T, \mathcal{L}(X_0)\big)$.

The following theorem is now a direct consequence of the results obtained in the previous subsection:

**9.2 Theorem**
*Let the assumptions from above be satisfied. Then the following representation formulas hold:*

$$U(t,s) = e^{-(t-s)A(s)} + \int_s^t e^{-(t-\tau)A(\tau)} R(\tau,s)\, d\tau \tag{9.23}$$

*and*

$$A(t)U(t,s)A^{-1}(s) = e^{-(t-\tau)A(t)}A(t)A^{-1}(s) + \int_s^t H(t,\tau)e^{-(\tau-s)A(\tau)}A(\tau)A^{-1}(s)\, d\tau\,, \tag{9.24}$$

*where $R(\cdot\,,\cdot)$ and $H(\cdot\,,\cdot)$ is are the resolvent kernels associated to the integral equations* (2.5) *and* (9.18) *respectively.*

*Moreover, the resolvent kernels can be estimated by*

$$\|R(t,s)\| \le C(t-s)^{\rho-1} \qquad \textit{and} \qquad \|H(t,s)\| \le C(t-s)^{\rho-1} \tag{9.25}$$

*for all $(t,s) \in \dot{\Delta}_T$, where $C > 0$ depends only on $M$ (in $(A2)$ ), the Hölder norm of $A(\cdot)$, a bond for $\|A(t)\|$ as well as $\rho$ (in $(A3)$ ) and $T$.*

**C. The evolution operator for a family of bounded operators:** For later reference and as a further application of the theory presented in Subsection A, we shall give here the construction of an evolution operator for a family of bounded operators. We assume that $B(\cdot) \in C\big([0,T], \mathcal{L}(X)\big)$ and consider the abstract Cauchy problem

$$\begin{cases} \dot{u} + B(t)u = 0 & 0 \le s < t \le T \\ u(s) = x & \end{cases} \tag{9.26}$$

in the Banach space $X$. It is now clear, that equation (9.26) is equivalent to the integral equation

$$u(t) = x - \int_s^t B(\tau)u(\tau)\,d\tau\,. \tag{9.27}$$

This fact makes the construction of the evolution operator much easier than in the case of unbounded operators. Suppose now that there exists an evolution operator $U(\cdot\,,\cdot)\colon \Delta_T \to \mathcal{L}(X)$. Then it must satisfy the integral equation

$$U(t,s) = \mathbb{1} - \int_s^t B(\tau)U(\tau,s)\,d\tau \tag{9.28}$$

in $C\big(\Delta_T, \mathcal{L}(X)\big)$. Note that this equation holds not only in the strong operator topology but also in the uniform operator topology. This integral equation is of the form (9.2) with $k_1(t,s) = B(s)$ and $\rho = 1$. The integral operator $Q \in \mathcal{L}\big(C(\Delta_T, \mathcal{L}(X))\big)$ takes the form

$$QV(t,s) = -\int_s^t B(\tau)V(\tau,s)\,d\tau$$

for all $V \in C\big(\Delta_T, \mathcal{L}(X)\big)$. As shown in Subsection A, $\mathbb{1} - Q$ is invertible and (9.28) has a unique solution which is given by Neumann's series. We can reformulate this assertion as follows. For any $(t,s) \in \Delta_T$ and $m \in \mathbb{N}$ set

$$U^{(0)}(t,s) := \mathbb{1} \qquad \text{and} \qquad U^{(m+1)}(t,s) := \mathbb{1} - \int_s^t B(\tau)U^{(m)}(\tau,s)\,d\tau\,. \tag{9.29}$$

Then we have

$$U(t,s) = \lim_{m\to\infty} U^{(m)}(t,s) \tag{9.30}$$

in $\mathcal{L}(X)$ uniformly in $(t,s) \in \Delta_T$. The proof of the following theorem is now almost complete.

**9.3 Theorem**

*Let $B(\cdot) \in C\big([0,T], \mathcal{L}(X)\big)$. Then there exists a unique evolution operator $U(\cdot\,,\cdot) \in C^1\big(\Delta_T, \mathcal{L}(X)\big)$ for the family $\big(B(t)\big)_{0\le t\le T}$. Furthermore, $U$ enjoys the following properties:*

(*a*) *$U(t,s) = U(t,\tau)U(\tau,s)$ for all $0 \le s \le \tau \le t \le T$.*

(*b*) *$\partial_1 U(t,s) = -B(t)U(t,s)$ and $\partial_2 U(t,s) = U(t,s)B(s)$ for all $(t,s) \in \Delta_T$.*

**Proof**

We have already shown the existence and uniqueness of a solution of the integral equation (9.28) above. Assertion (*a*) follows from the uniqueness of this solution. It remains to show $(b)$. By differentiating equation (9.28) with respect to $t$ we easily obtain $\partial_1 U(t,s) = -B(t)U(t,s)$.

Taking now the derivative with respect to $s$ (formally), we obtain

$$\partial_2 U(t,s) = B(s) - \int_s^t B(\tau)\partial_2 U(\tau,s)\,d\tau\,.$$

This integral equation has a unique solution which is given by $(\mathbb{1} - Q)^{-1}B(t,s) = U(t,s)B(s)$. From this, it follows that $\partial_2 U(t,s)$ exists and equals $U(t,s)B(s)$.

Since the partial derivatives of $U(\cdot\,,\cdot)$ exist and are continuous, we have that $U(\cdot\,,\cdot) \in C^1\big(\Delta_T, \mathcal{L}(X)\big)$. $\square$

**Notes and references:** These different kinds of integral equations was used by Sobolevskiĭ [**114**] and Tanabe [**118**], [**119**]in order to construct the evolution operator. Continuity properties of and estimates for solutions of such integral equations in interpolation spaces may be found in Amann [**11**] and [**18**].

## 10. Yosida approximations of the evolution operator

It is a well known rule of thumb in semigroup theory, that a semigroup inherits many properties from the resolvent of its generator and vice versa. In most cases, this is a direct consequence of formulas (1.5) and (1.6) which relate the resolvent to the semigroup. When considering the evolution operator for a nonautonomous evolution equation, no such direct relationship exists. Nevertheless, it is still possible to transfer many properties of the resolvent of $A(t)$, $0 \le t \le T$, to the evolution operator generated by the family $\big(A(t)\big)_{0\le t\le T}$ by a different technique. The idea is to find approximations of the evolution operator which inherit properties of the resolvent of $A(t)$ for $0 \le t \le T$ and then taking limits. The procedure is as follows: We replace the family of operators $\big(A(t)\big)_{0\le t\le T}$ by the family $\big(A_n(t)\big)_{0\le t\le T}$ of Yosida approximations, which was defined in Section 1.A. Observe that the $A_n(t)$ are bounded operators. Instead of the abstract Cauchy problem

$$\text{(10.1)} \qquad \begin{cases} \dot u + A(t)u = 0 & 0 \le s < t \le T \\ u(s) = x \end{cases}$$

we consider the equation

$$\text{(10.2)} \qquad \begin{cases} \dot u + A_n(t)u = 0 & 0 \le s < t \le T \\ u(s) = x \end{cases}$$

for every $n \in \mathbb{N}^*$. By Theorem 9.3, there exists a unique evolution operator $U_n(\cdot\,,\cdot)$ for this problem. Alternatively we can consider the evolution operator $V_n(\cdot\,,\cdot)$ belonging to the Cauchy problem

$$\begin{cases} \dot{v} - n^2\bigl(n + A(t)\bigr)^{-1} v = 0 & 0 \le s < t \le T \\ v(s) = x\,. \end{cases} \tag{10.3}$$

Since by definition

$$A_n(t) = nA(t)\bigl(n + A(t)\bigr)^{-1} = n - n^2\bigl(n + A(t)\bigr)^{-1} \tag{10.4}$$

for all $n \in \mathbb{N}^*$ and $t \in [0, T]$, the relationship between the evolution operators $U_n(\cdot\,,\cdot)$ and $V_n(\cdot\,,\cdot)$ is given by

$$U_n(t, s) = e^{-n(t-s)} V_n(t, s) \tag{10.5}$$

for all $(t, s) \in \Delta_T$ and $n \in \mathbb{N}^*$ (compare Remark 2.7(b)). Since $\bigl(n + A(t)\bigr)^{-1}$ is a bounded operator depending continuously on $t \in [0, T]$, equation (10.3) is equivalent to the integral equation

$$V_n(t, s)x = x + n^2 \int_s^t \bigl(n + A(\tau)\bigr)^{-1} V_n(\tau, s)x\, d\tau \tag{10.6}$$

for all $(t, s) \in \Delta_T$ and $n \in \mathbb{N}^*$. This equation, together with (10.5), gives the connection between $\bigl(n + A(t)\bigr)^{-1}$ and $U_n(\cdot\,,\cdot)$. It is not surprising that one can show that

$$\operatorname*{s\text{-}lim}_{n\to\infty} U_n(t, s) = U(t, s) \quad \text{and} \quad \operatorname*{s\text{-}lim}_{n\to\infty} A_n(t)U_n(t, s)A_n^{-1}(s) = A(t)U(t, s)A^{-1}(s) \tag{10.7}$$

uniformly with respect to $(t, s) \in \Delta_T$. In semigroup theory this corresponds to formula (1.9). In fact, the rest of this section is devoted to the proof of this approximation formula.

In a first subsection we investigate properties of the Yosida approximations $A_n$ of $A$. In the second subsection we investigate the convergence of Yosida approximations of semigroups and resolvent kernels. In the final subsection, we prove our main result (9.7) and some consequences.

**A. Yosida approximations:** In this subsection we shall dwell on some technicalities concerning the Yosida approximation (10.4) which shall pave the way to the approximation result we strive to prove. It is the aim of these considerations to establish on which data the strong convergence of these approximation to $A(t)$ depends. This knowledge is crucial in obtaining (10.7) and in handling parameter dependent problems in the next section. Before we start our investigations we prove a simple technical lemma concerning the uniform convergence of sequences of operators.

**10.1 Lemma**

*Let $X$, $Y$, $Z$ be Banach spaces, $S$ a compact metric space and*

$$A_n(\cdot) \in C\big(S, \mathcal{L}_s(Y,Z)\big) \qquad \textit{and} \qquad B_n(\cdot) \in C\big(S, \mathcal{L}_s(X,Y)\big) \tag{10.8}$$

*for all $n \in \mathbb{N}$. Suppose that $\text{s-lim}_{n\to\infty} A_n(s) = A(s)$ and $\text{s-lim}_{n\to\infty} B_n(s) = B(s)$ uniformly with respect to $s \in S$ for some $A(s) \in \mathcal{L}(Y,Z)$ and $B(s) \in \mathcal{L}(X,Y)$. Then*

$$\text{s-lim}\, A_n(s)B_n(s) = A(s)B(s) \tag{10.9}$$

*uniformly with respect to $s \in S$.*

**Proof**

Since for any $x \in X$ the sequence $\big(A_n(s)x\big)$ is uniformly convergent and since $S$ is compact, there exists a constant $M_x > 0$ such that $\|A_n(s)x\| \le M_x$ holds for all $n \in \mathbb{N}$ and $s \in S$. By the uniform boundedness principle, we can thus find a constant $M_0$ such that

$$\|A_n(s)\| \le M_0 \tag{10.10}$$

holds for all $n \in \mathbb{N}$ and $s \in S$. Using (10.10), we have that for each $x \in X$

$$\begin{aligned} &\|A_n(s)B_n(s)x - A(s)B(s)x\| \\ &\quad \le \|A_n(s)\big(B_n(s) - B(s)\big)x\| + \|\big(A_n(s) - A(s)\big)B(s)x\| \\ &\quad \le M_0\|B_n(s)x - B(s)x\| + \|\big(A_n(s) - A(s)\big)B(s)x\|\,. \end{aligned} \tag{10.11}$$

By the uniform convergence of the sequence $\big(B_n(s)x\big)$ we can make the first term arbitrarily small uniformly in $s \in S$. For the second term we obtain the estimate

$$\begin{aligned} &\|\big(A_n(s) - A(s)\big)B(s)x\| \\ &\quad \le \|\big(A_n(s) - A(s)\big)B(s_0)x\| + \|\big(A_n(s) - A(s)\big)\big(B(s) - B(s_0)\big)x\| \\ &\quad \le \|\big(A_n(s) - A(s)\big)B(s_0)x\| + 2M_0\|\big(B(s) - B(s_0)\big)x\| \end{aligned} \tag{10.12}$$

for any fixed $s_0 \in S$.

Since $B(\cdot)$ is strongly continuous, to each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|B(s)x - B(s_0)x\| < \varepsilon/2M_0$ whenever $d(s, s_0) < \delta$ ($d$ being the metric on $S$). Take now $n \in \mathbb{N}$ so large, that $\|\big(A_n(s) - A(s)\big)B(s_0)x\| < \varepsilon$ for every $s \in S$ (which is possible since $A_n(s)$ is strongly convergent uniformly in $s \in S$). Then, for each $\varepsilon > 0$ and $s_0 \in S$, there exist $\delta = \delta(s_0, \varepsilon)$ and $n_0 = n_0(s_0, \varepsilon)$ such that

$$\|A_n(s)B_n(s)x - A(s)B(s)x\| < 3\varepsilon \qquad \text{for } n > n_0 \text{ and } d(s, s_0) < \delta. \tag{10.13}$$

Hence, for fixed $\varepsilon > 0$ and $s_0 \in S$, there exists a neighbourhood of $s_0$, such that (10.13) holds. Since $S$ is compact, it can be covered by a finite number of such neighbourhoods. Let $N$ be the largest of the (finitely many) corresponding $n_0$'s. Then the inequality above holds for all $s \in S$ provided $n > N$. This completes the proof. □

For the rest of this subsection we assume that the family $\big(A(t)\big)_{0\le t\le T}$ satisfies conditions $(A1)$–$(A3)$ of Section 2. We emphasize that for all the considerations in this subsection we could replace the family $\big(A(t)\big)_{0\le t\le T}$ by a family $\big(A(s)\big)_{s\in S}$, where $S$ is any compact metric space. It is important to keep this in mind when considering parameter dependent problems later on.

In a next step towards understanding the nature of the convergence s-lim $A_n(t) = A(t)$, we investigate a strong approximation of the identity. Put

$$J_n(t) := n\big(n + A(t)\big)^{-1} = \big(1 + \tfrac{1}{n}A(t)\big)^{-1} \tag{10.14}$$

for all $n \in \mathbb{N}^*$ and $t \in [0,T]$. Then it follows from the resolvent estimate in $(A2)$ that

$$\|J_n(t)\| \le M \tag{10.15}$$

for all $n \in \mathbb{N}^*$ and $t \in [0,T]$, where $M$ is the constant appearing in $(A2)$. Again by $(A2)$ we get for every $x \in X_1$:

$$\|J_n(t)x - x\| = \|\big(n + A(t)\big)^{-1}A(t)x\| \le \frac{M}{1+n}\|A(t)\|_{1,0}\|x\| \tag{10.16}$$

for all $n \in \mathbb{N}^*$ and $t \in [0,T]$. Since, by $(A3)$ and the compactness of $[0,T]$, the numbers $\|A(t)\|_{1,0}$ are uniformly bounded with respect to $t \in [0,T]$, we get the following

**10.2 Lemma**
*Let $\big(A(t)\big)_{0\le t\le T}$ satisfy conditions $(A1)$–$(A3)$. Then*

$$\lim_{n\to\infty} J_n(t)x = x \tag{10.17}$$

*for all $x \in X_0$ uniformly with respect to $t \in [0,T]$. The convergence depends only on a bound for $M$ (in $(A2)$) and $\|A(t)\|_{1,0}$.*

**Proof**
For $x \in X_1$, the assertion follows from (10.16). By (10.15) and since $X_1$ is dense in $X_0$, we get the assertion for any $x \in X_0$ (compare Section 0.H). □

**10.3 Corollary**

*Let* $\big(A(t)\big)_{0\le t\le T}$ *satisfy the same conditions as in the lemma above. Then*

(*a*) $\lim_{n\to\infty} A_n(t)x = A(t)x$ *for all* $x \in X_1$ *uniformly with respect to* $t \in [0,T]$.

(*b*) $\operatorname{s-lim}_{n\to\infty} A_n(t)A_n^{-1}(s) = A(t)A^{-1}(s)$ *uniformly with respect to* $s,t \in [0,T]$.

**Proof**

Assertion (*a*) follows immediately from the fact that $A_n(t)x = A(t)J_n(t)x = J_n(t)A(t)x$ for all $x \in X_1$ and Lemma 10.1 and 10.2.

To prove (*b*) observe first that $0 \in \varrho\big(A_n(s)\big)$ for all $s \in [0,T]$ and $n \in \mathbb{N}^*$. A simple calculation shows that

$$A_n(t)A_n^{-1}(s) = J_n(t)A(t)A^{-1}(s) + J_n(t) - \mathbb{1} \tag{10.18}$$

for all $s,t \in [0,T]$ and $n \in \mathbb{N}^*$. Applying Lemma 10.1 and 10.2, the assertion follows. □

Next we shall show that $A_n(t)$ satisfies essentially the same resolvent estimate as $A(t)$.

**10.4 Lemma**

*Let* $\big(A(t)\big)_{0\le t\le T}$ *satisfy conditions* $(A1)$–$(A3)$. *Then we have*

$$[\operatorname{Re}\mu \ge 0] \subset \varrho\big(-A_n(t)\big) \tag{10.19}$$

*and*

$$\|\big(\lambda + A_n(t)\big)^{-1}\| \le \frac{M+\sqrt{2}}{1+|\lambda|} \tag{10.20}$$

*for all* $\lambda \in [\operatorname{Re}\mu \ge 0]$ *and* $n \in \mathbb{N}^*$.

**Proof**

Note first that $\lambda \in [\operatorname{Re}\mu \ge 0]$ immediately implies that $n\lambda(n+\lambda)^{-1} \in [\operatorname{Re}\mu \ge 0]$ for all $n \in \mathbb{N}^*$. Some simple algebraic manipulations yield

$$\big(\lambda + A_n(t)\big)^{-1} = (n+\lambda)^{-1} + n^2(n+\lambda)^{-2}\big(n\lambda(n+\lambda)^{-1} + A(t)\big)^{-1} \tag{10.21}$$

for all $n \in \mathbb{N}^*$ and $\lambda \in [\operatorname{Re}\mu \ge 0]$. Using the fact that $n^2 + |\lambda|^2 \ge \frac{1}{2}\big(n+|\lambda|\big)^2$, we get that

$$|n+\lambda|^{-1} \le \frac{\sqrt{2}}{1+|\lambda|} \tag{10.22}$$

holds for all $n \in \mathbb{N}^*$ and $\lambda \in [\operatorname{Re}\mu \ge 0]$. Due to the resolvent estimate in $(A2)$ we obtain

$$\begin{aligned} \|n^2(n+\lambda)^{-2}\big(n\lambda(n+\lambda)^{-1} + A(t)\big)^{-1}\| &\le Mn^2|n+\lambda|^{-2}\big(1+n|\lambda|\,|n+\lambda|^{-1}\big)^{-1} \\ &= Mn|n+\lambda|^{-1}\big(n^{-1}|n+\lambda| + |\lambda|\big)^{-1} \le \frac{M}{1+|\lambda|} \end{aligned} \tag{10.23}$$

for all $n \in \mathbb{N}^*$ and $\lambda \in [\mathrm{Re}\,\mu \geq 0]$. Putting (10.21)–(10.23) together, the assertion follows. □

In the next subsection we apply the above results to prove (9.7) in the case of semigroups and give some technical lemmas which are necessary for the proof of (9.7) in the general case.

**B. Yosida approximations of semigroups and resolvent kernels:** In the sequel let $\big(A(t)\big)_{0\leq t\leq T}$ satisfy conditions $(A1)$–$(A3)$ of Section 2 and denote by $A_n(t)$ the $n$-th Yosida approximation of $A(t)$ for all $t \in [0,T]$ and $n \in \mathbb{N}^*$. Then, by Remark 2.1(b) and the definition of $A_n(\cdot)$, we have that for any $n \in \mathbb{N}^*$

$$A_n(\cdot) \in C^\rho\big([0,T], \mathcal{L}(X_0)\big)\,, \tag{10.24}$$

holds, where $\rho \in (0,1)$ is the Hölder exponent from $(A3)$.

By Theorem 9.3, there exists a unique evolution operator $U_n(\cdot\,,\cdot)$ for the family $\big(A(t)\big)_{0\leq t\leq T}$ of bounded operators. Alternatively one may construct $U_n(\cdot\,,\cdot)$ by means of the methods developped in Section 9.B. Replacing $A$ by $A_n$, we have that $U_n(t,s)$ and $A(t)U_n(t,s)A_n^{-1}(s)$ can be represented by (9.23) and (9.24) respectively. We denote the corresponding resolvent kernels by $R^{(n)}(\cdot\,,\cdot)$ and $H^{(n)}(\cdot\,,\cdot)$.

The first thing we have to show is that (10.7) holds in the case of semigroups. Although this is well known we include a proof in order to keep track of the data on which the convergence depends.

**10.5 Proposition**

*Let $\big(A(t)\big)_{0\leq t\leq T}$ satisfy conditions $(A1)$–$(A3)$ and denote by $A_n(t)$ the $n$-th Yosida approximation of $A(t)$ for all $n \in \mathbb{N}^*$. Then*

*(a)* $\operatorname*{s-lim}_{n\to\infty} e^{-\tau A_n(s)} = e^{-\tau A(s)}$ *holds uniformly with respect to $s \in [0,T]$ and $\tau \geq 0$.*

*(b)* $\lim_{n\to\infty} A_n^k(s)e^{-\tau A_n(s)} = A^k(s)e^{-\tau A(s)}$ *holds for all $k \in \mathbb{N}$ uniformly with respect to $s \in [0,T]$ and $\tau \geq \varepsilon$ for all $\varepsilon > 0$.*

*Moreover, the convergence depends only on $M$ (in $(A2)$) and an upper bound for $\|A(t)\|_{1,0}$.*

**Proof**

By the resolvent estimate (10.28), Remark 1.2 and the representation formula (1.17) there exists a constant $c(M)$ depending only on $M$ such that

$$\|e^{-\tau A(s)}\| \leq c(M) \tag{10.25}$$

holds for all $\tau \geq 0$, $s \in [0,T]$ and $n \in \mathbb{N}^*$. Observe now that

$$\begin{aligned}\big(\lambda + A_n(s)\big)^{-1} - \big(\lambda + A(s)\big)^{-1} &= -\big(\lambda + A_n(s)\big)^{-1}\big(A_n(s) - A(s)\big)\big(\lambda + A(s)\big)^{-1}\\ &= -n^{-1}A_n(s)\big(\lambda + A_n(s)\big)^{-1}A(s)\big(\lambda + A(s)\big)^{-1}\end{aligned} \tag{10.26}$$

holds for all $\lambda \in \varrho\big(A(s)\big) \cap \varrho\big(A_n(s)\big)$. The resolvent estimates in $(A2)$ and (10.20) together with Remark 1.2 imply the existence of a constant $\tilde{c}(M)$ depending only on $M$ such that

$$\|A(s)\big(\lambda + A(s)\big)^{-1}\| \leq \tilde{c}(M) \qquad \text{and} \qquad \|A_n(s)\big(\lambda + A_n(s)\big)^{-1}\| \leq \tilde{c}(M) \tag{10.27}$$

hold for all $n \in \mathbb{N}^*$, $s \in [0,T]$ and $\lambda$ in some sector $S_{\frac{\pi}{2}+\alpha}$ for a suitable choice of $\alpha \in (0, \frac{\pi}{2}]$. The representation formula (1.17), as well as (10.26) and (10.27) show that

$$\begin{aligned} &A_n^k(s)e^{-\tau A_n(s)} - A^k(s)e^{-\tau A(s)} \\ &\qquad = \frac{-1}{2\pi i\, n} \int_\Gamma \lambda^k e^{\lambda\tau} A_n(s)\big(\lambda + A_n(s)\big)^{-1} A(s)\big(\lambda + A(s)\big)^{-1}\, d\lambda \end{aligned} \tag{10.28}$$

tends to zero as $n \to \infty$ uniformly with respect to $s \in [0,T]$ and $\tau \geq \varepsilon > 0$. This proves $(b)$.

To prove $(a)$ note that since $A(s)$ commutes with its resolvent we get for any $x \in X_1$

$$e^{-\tau A_n(s)} - e^{-\tau A(s)} = \frac{-1}{2\pi i\, n} \int_\Gamma e^{\lambda\tau} A_n(s)\big(\lambda + A_n(s)\big)^{-1}\big(\lambda + A(s)\big)^{-1}\, d\lambda\, A(s)x \tag{10.29}$$

for all $n \in \mathbb{N}^*$, $s \in [0,T]$ and $\tau \geq 0$. By the resolvent estimate in $(A2)$ and (10.27) this integral is uniformly bounded. Hence, assertion $(a)$ holds for $x \in X_1$. Since $X_1$ is dense in $X_0$ estimate (10.25) yields $(a)$ for every $x \in X_0$, completing the proof. □

We are now able to prove the convergence of the resolvent kernels.

**10.6 Lemma**

*Let $R(\cdot\,,\cdot)$, $H(\cdot\,,\cdot)$ be the resolvent kernels from Theorem 9.2. Moreover, suppose that and $R^{(n)}(\cdot\,,\cdot)$, $H^{(n)}(\cdot\,,\cdot)$ are the corresponding resolvent kernels to the $n$-th Yosida approximations. Then*

$$\|R^{(n)}(t,s)\| \leq c(t-s)^{\rho-1} \qquad \text{and} \qquad \|H^{(n)}(t,s)\| \leq c(t-s)^{\rho-1} \tag{10.30}$$

*holds for a constant $c$ depending only on $M$, $\rho$, $T$ and on upper bounds for $\|A(t)\|_{1,0}$ and the Hölder norm of $A(\cdot)$. Moreover,*

$$\operatorname*{s-lim}_{n\to\infty} R^{(n)}(t,s) = R(t,s) \qquad \text{and} \qquad \operatorname*{s-lim}_{n\to\infty} H^{(n)}(t,s) = H(t,s) \tag{10.31}$$

*holds uniformly with respect to $(t,s) \in \Delta_T^\varepsilon$ for every $\varepsilon > 0$, where $\Delta_T^\varepsilon$ is defined as in (9.13).*

**Proof**

Note that the resolvent kernels $R^{(n)}(\cdot\,,\cdot)$ and $H^{(n)}(\cdot\,,\cdot)$ are given by

$$R^{(n)}(t,s) = \sum_{m=1}^{\infty} R_m^{(n)}(t,s) \qquad \text{and} \qquad H^{(n)}(t,s) = \sum_{m=1}^{\infty} H_m^{(n)}(t,s) \tag{9.32}$$

respectively. Here, $R_1^{(n)}(\cdot\,,\cdot)$ is given by (9.17) replacing $A$ by $A_n$ and $R_m^{(n)}(\cdot\,,\cdot)$ for $m \geq 2$ by (9.7) replacing $k_1$ by $R_1^{(n)}$. In a similar way, the kernels $H_m^{(n)}(\cdot\,,\cdot)$ ($m \geq 1$) are defined.

We first deduce a uniform estimate for $R_1^{(n)}(t,s)$. Using the identity

$$\big(n + A(t)\big)^{-1} - \big(n + A(s)\big)^{-1} = -\big(n + A(t)\big)^{-1}\big(A(t) - A(s)\big)\big(n + A(s)\big)^{-1},$$

as well as (10.4), (10.15) and (9.20) we get that

$$\|\big(A_n(t) - A_n(s)\big)A_n^{-1}(s)\| = \|J_n(t)\big(A(t) - A(s)\big)A^{-1}(s)\| \leq Mc_1(t-s)^\rho \tag{10.33}$$

holds for all $(t,s) \in \Delta_T$ and $n \in \mathbb{N}^*$. On the other hand, Lemma 10.4, Remark 1.2 and (1.18) show that the second factor of $R_1^{(n)}(\cdot\,,\cdot)$ can be estimated by $c_2(t-s)^{-1}$ for all $(t,s) \in \dot{\Delta}_T$, where $c_2$ is a constant depending on the same quantities as $c_1$. Putting $c_0 := Mc_1c_2$ we get that

$$\|R_1^{(n)}(t,s)\| \leq c_0(t-s)^{\rho-1} \tag{10.34}$$

holds for all $(t,s) \in \dot{\Delta}_T$ and $n \in \mathbb{N}^*$.

Analogous to (9.8), we obtain by induction that for any $n \in \mathbb{N}^*$ and $m \geq 1$

$$\|R_m^{(n)}(t,s)\| \leq \frac{\big[c_0\,\Gamma(\rho)\big]^m}{\Gamma(\rho m)}\,(t-s)^{m\rho-1} \tag{10.35}$$

holds for all $(t,s) \in \dot{\Delta}_T$. Assertion (10.30) now easily follows from (10.32) and (10.35).

We turn now to the strong convergence of $R^{(n)}(t,s)$. We start by showing strong convergence of $R_1^{(n)}(t,s)$. From Corollary 10.3($b$) it follows immediately that

$$\operatorname*{s-lim}_{n\to\infty}\big(A_n(t) - A_n(s)\big)A_n^{-1}(s) = \big(A(t) - A(s)\big)A^{-1}(s),$$

the convergence dependending only on the quantities listed in the lemma. Together with Proposition 10.5($b$) and Lemma 10.1 we obtain that

$$\operatorname*{s-lim}_{n\to\infty} R_1^{(n)}(t,s) = R_1(t,s)$$

holds uniformly with respect to $(t,s) \in \Delta_T^\varepsilon$ for all $\varepsilon > 0$. Using the definition of $R_m^{(n)}(t,s)$ and Lemma 10.1 we get inductively that the same holds for $R_m^{(n)}(t,s)$ and, due to the uniform majorant (10.35), also for $R^{(n)}(t,s)$.

By the same arguments the assertion follows also for the kernels $H^{(n)}(\cdot\,,\cdot)$ and the proof of the lemma is complete. □

**C. Yosida approximations of evolution equations:** We start with the main result of this section and then proceed to give some simple applications. We use the same assumptions and notations as in the previous subsection.

**10.7 Theorem**

*Let* $\big(A(t)\big)_{0\le t\le T}$ *satisfy conditions* (A1)–(A3) *and* $A_n(t)$ *the* $n$*-th Yosida approximation of* $A(t)$ *for all* $n\in\mathbb{N}^*$*. Furthermore, let* $U(\cdot\,,\cdot)$ *and* $U_n(\cdot\,,\cdot)$ *be the corresponding evolution operators. Then*

$$\operatorname*{s-lim}_{n\to\infty} U_n(t,s)=U(t,s) \qquad\text{and}\qquad \operatorname*{s-lim}_{n\to\infty} A_n(t)U_n(t,s)A_n^{-1}(s)=A(t)U(t,s)A^{-1}(s)$$

*uniformly with respect to* $(t,s)\in\Delta_T$*. Moreover, the convergence depends only on* $M$*,* $\rho$*,* $T$ *and on an upper bound for* $\|A(t)\|_{1,0}$ *and the Hölder norm of* $A(\cdot)$*.*

**Proof**

To prove the theorem we have only to show the uniform convergence of the terms appearing in the representation formulas (9.23) and (9.24) if we replace $A$ by $A_n$.

The first term converges strongly uniformly in $(t,s)\in\Delta_T$. by Proposition 10.5($b$). We split the second term into two parts, namely

$$\int_s^{t-\varepsilon} e^{-(t-\tau)A_n(\tau)}R^{(n)}(\tau,s)\,d\tau \quad\text{and}\qquad \int_{t-\varepsilon}^{t} e^{-(t-\tau)A_n(\tau)}R^{(n)}(\tau,s)\,d\tau\,.$$

The first part converges strongly uniformly with respect to $(t,s)\in\Delta_T$. To see this apply Lemmas 10.1, 10.5 and 10.6. Using (10.25) and (10.30) the second term can be estimated uniformly in $n\in\mathbb{N}^*$ by

$$\Big\|\int_{t-\varepsilon}^{t} e^{-(t-\tau)A_n(\tau)}R^{(n)}(\tau,s)\,d\tau\Big\|\le c\varepsilon^{\rho}$$

where $c$ is a constant depending only on the quantities mentioned in the theorem. Since $\varepsilon$ can be chosen arbitrarily, the assertion follows. □

**10.8 Corollary**

*Let the same assumptions as in the above theorem be satisfied. Moreover, suppose that* $f$ *and* $f_n\in C\big([0,T],X_0\big)$ *are such that* $\lim_{n\to\infty} f_n=f$ *in* $C\big([0,T],X_0\big)$ *and that* $x$ *and* $x_n\in X_0$ *are such that* $\lim_{n\to\infty}x_n=x$*. If* $u_n$ *is the (mild) solution of the inhomogeneous Cauchy problem*

$$\begin{cases}\dot u_n+A_n(t)u_n=f_n(t) & 0\le s<t\le T\\ u_n(s)=x_n\end{cases}\tag{10.36}$$

*and* $u$ *the (mild) solution of*

$$\begin{cases}\dot u+A(t)u=f(t) & 0\le s<t\le T\\ u(s)=x\end{cases}\tag{10.37}$$

*then*

$$\lim_{n\to\infty}u_n=u$$

*in* $C\big([s,T],X_0\big)$*.*

**Proof**
The solution of (10.36) is given by the variation-of-constants formula

$$u_n(t) = U_n(t,s)x_n + \int_s^t U_n(t,\tau)f_n(\tau)\,d\tau\,.$$

It is an immediate consequence of Theorem 10.7 that the first term converges uniformly in $t \in [s,T]$. The uniform convergence of the integrand is easily obtained by the same arguments as in the proof of Lemma 10.1. Hence the assertion of the corollary follows. □

As an application of Theorem 10.7 we prove the following result on the invariance of closed convex sets under the evolution operator, which, in the case of semigroups, is an easy consequence of formula (1.6).

**10.9 Proposition**
*Let $\big(A(t)\big)_{0\le t\le T}$ be a family of closed operators satisfying (A1)–(A3) of Section 2 and let $C$ be a closed convex subset of $X_0$. Suppose in addition that there exists an $\varepsilon_0 > 0$ such that $\big(1+\varepsilon A(t)\big)^{-1}C \subset C$ holds for all $\varepsilon \in (0,\varepsilon_0)$. Then*

$$U(t,s)C \subset C \tag{10.38}$$

*holds for all $(t,s) \in \Delta_T$.*

**Proof**
By Theorem 10.7 and the closedness of $C$ it is enough to prove the assertion for the Yosida approximations $U_n(\cdot\,,\cdot)$ of $U(\cdot\,,\cdot)$ for $n \in \mathbb{N}$ large.

By (10.5) and (10.6), $U_n(\cdot\,,\cdot)$ satisfies the integral equation
(10.39)

$$U_n(t,s)x = e^{-n(t-s)}x + n\int_s^t e^{-n(t-\tau)}\big(1+\tfrac{1}{n}A(\tau)\big)^{-1}U_n(\tau,s)x\,d\tau$$

$$= \frac{1}{\frac{1}{n}e^{-n(t-s)} + \int_s^t e^{-(nt-\tau)}\,d\tau}\Big(\frac{1}{n}e^{-n(t-s)}x + \int_s^t e^{-n(t-\tau)}\big(1+\tfrac{1}{n}A(\tau)\big)^{-1}U_n(\tau,s)x\,d\tau\Big)\,.$$

Approximating the integrals by Riemann sums we see that the last term is a convex combination of $x$ and $\big(1+\frac{1}{n}A(\tau)\big)^{-1}U_n(\tau,s)x$ $(s \le \tau \le t)$. Equation (10.39) is a fixed piont equation of the same type as (10.27) and can be solved in the same way. By our hypotheses, $\big(1+\frac{1}{n}A(\tau)\big)^{-1}C \subset C$ holds whenever $n$ is sufficiently large. Therefore, (10.38) holds for the Yosida approximations if $n$ is large enough and the proof of the proposition is complete. □

As a corollary to Proposition 10.9 we obtain the positivity of the evolution operator provided all the $-A(t)$ generate positive semigroups. For the required notions on ordered Banach spaces and positive operators we refer to Section 12.

**10.10 Corollary**

*Let $\big(A(t)\big)_{0\le t\le T}$ satisfy (A1)–(A3). Suppose in addition that $X_0$ is an ordered vector space and that $-A(t)$ is the generator of a positive semigroup for each $t \in [0,T]$. Then $U(t,s)$ is a positive operator for all $(t,s) \in \Delta_T$.*

**Proof**

It is well known that the semigroup $e^{-tA(s)}$ is positive if and only if $\big(\lambda + A(s)\big)^{-1}$ is positive for $\lambda > 0$. (This is actually an easy consequence of formulas (1.5) and (1.6).) This means that $\big(1+\varepsilon A(s)\big)^{-1}C \subset C$ for each $s \in [0,T]$ and $\varepsilon \in (0,1)$, where $C$ is the positive cone in $X_0$. Now the assertion follows from Proposition 10.9. □

**Notes and References:** The presentation of the material in this section essentially follows Kato [**74**]. It is worthwhile noting that Kato actually constructs the evolution operator by means of Yosida approximations. Lemma 10.1 is taken from Kato [**73**]. An alternative proof to show that $\lim_{n\to\infty} U_n(t,s)x = U(t,s)x$ may by found in Kato [**76**]. The method used there is much easier than the one used here, but does not give uniform convergence with respect to $(t,s) \in \Delta_T$ but only with respect to $t \in [s,T]$.

For the assertion of Proposition 10.9 we were not able to find a reference, although it is neither surprising nor hard to prove.

## 11. Parameter Dependence

In this section we investigate continuous and differential dependence ot the evolution operator on a real parameter $\zeta$. We consider the Cauchy problem

$$\text{(11.1)}\qquad \begin{cases} \dot u + A(\zeta,t)u = 0 & 0 \le s < t \le T \\ u(s) = x \end{cases}$$

and ask if the corresponding evolution operator depends as regularly on $\zeta$ as $A(\cdot,t)$ does. We only consider dependence on one real parameter. Moreover, since continuity or differentiability are local properties of a function we assume that the parameter $\zeta$ varies only over a bounded (open) interval $\Lambda$ of the real axis.

We organize this section as follows. First, we give the basic assumptions. In a second subsection, we investigate parameter dependence of autonomous problems. After having established certain smoothness properties of the resolvent kernels which appeared in Theorem 9.2, we proceed to prove differentiable dependence of a parameter of the evolution operator. In a final subsection we show analytic dependence provided $A(\cdot\,,t)$ has this property for each $t \in [0,T]$. This is in some respect much easier than the differentiable case since the fact that the uniform limits of (complex) analytic functions is again analytic allows us to make use of our approximation result in Theorem 10.7

**A. Basic assumptions:** Let us now introduce the basic assumptions for this section. As usual we shall assume that $X_0$ and $X_1$ are Banach spaces with $X_1 \overset{d}{\hookrightarrow} X_0$. Furthermore, let $\Lambda$ be a bounded interval in $\mathbb{R}$. We fix a number $T > 0$ and consider a family $\big(A(\zeta,t)\big)_{(\zeta,t)\in\overline{\Lambda}\times[0,T]}$ of closed operators on $X_0$ satisfying the following properties.

$(\tilde{A}1)$ $D(A(\zeta,t)) = X_1$ for all $t \in [0,T]$ and $\zeta \in \overline{\Lambda}$.

$(\tilde{A}2)$ For all $t \in [0,T]$ and $\zeta \in \overline{\Lambda}$ we have

$$[\operatorname{Re}\mu \geq 0] \subset \varrho\big(-A(\zeta,t)\big).$$

Furthermore, there exists a constant $M > 0$, independent of $t \in [0,T]$ and $\zeta \in \overline{\Lambda}$, such that

$$\|\big(\lambda + A(\zeta,t)\big)^{-1}\| \leq \frac{M}{1+|\lambda|}$$

for all $(\lambda,\zeta,t) \in [\operatorname{Re}\mu \geq 0] \times \overline{\Lambda} \times [0,T]$.

$(\tilde{A}3)$ There exists a constant $\rho \in (0,1)$ and $r \in \mathbb{N} \cup \{\infty\}$, such that

$$A(\cdot\,,\cdot) \in C^{r,\rho}\big(\overline{\Lambda}\times[0,T], \mathcal{L}(X_1,X_0)\big).$$

Here, $C^{r,\rho}$ means that the derivatives with respect to the first variable up to the order $r$ are Hölder continuous in $t \in [0,T]$ with exponent $\rho$, uniformly in $\zeta \in \overline{\Lambda}$.

In the analytic case we assume the following:

$(\tilde{A}3)_\omega$ There exists a constant $\rho \in (0,1)$ such that

$$A(\cdot\,,\cdot) \in C^{\omega,\rho}\big(\overline{\Lambda}\times[0,T], \mathcal{L}(X_1,X_0)\big).$$

Here, $C^{\omega,\rho}$ means that $A(\cdot\,,t)$ is analytic uniformly with respect to $t \in [0,T]$ and that $\partial_\zeta A(\zeta,\cdot)$ is Hölder continuous for all $k \in \mathbb{N}$ uniformly with respect to $\zeta \in \overline{\Lambda}$. More precisely, for any $\zeta_0 \in \Lambda$ there exists a neigbourhood $U \subset \Lambda$ such that $A(\zeta,t)$ has a power series representation of the form $\sum_{k=0}^{\infty} A_k(t)(\zeta-\zeta_0)^k$ for all $(\zeta,t) \in U \times [0,T]$.

**11.1 Remark**

(a) As in Remark 2.1 one obtains from $(\tilde{A}3)$ that

$$\big(\lambda + A(\cdot\,,\cdot)\big)^{-1} \in C^{r,\rho}\big(\overline{\Lambda}\times[0,T], \mathcal{L}(X_0,X_1)\big) \tag{11.2}$$

whenever $\lambda \in \varrho\big(-A(\zeta,t)\big)$ for all $t \in [0,T]$ and $\zeta \in \overline{\Lambda}$. For this reason,

$$\|\partial_\zeta^j A(\zeta,t)A^{-1}(\zeta,s)\| \leq c \tag{11.3}$$

for a constant $c > 0$ independent of $t$, $s \in [0,T]$, $\zeta \in \overline{\Lambda}$ and $0 \leq j \leq k$ with $k \in \mathbb{N}$, $k \leq r$.

(b) A typical case of this situation is a differential operator of the form (2.8) where the coefficients $a_{ik}$, $a_i$ and $a_0$ depend additionally on a parameter $\zeta \in \Lambda$. □

**B. The autonomous case:** In a first step we prove differentiable dependence of the semigroup in the autonomous case. In order to make the application to nonautonomous problems of the results below we shall stick the family $\big(A(\zeta,t)\big)$.

**11.2 Proposition**

*Let assumptions $(\tilde{A}1)$–$(\tilde{A}3)$ be satisfied. Then, for any $\ell \in \mathbb{N}$, $s \in [0,T]$ and $\tau > 0$ we have that*

$$\big[\zeta \mapsto A^\ell(\zeta,s)e^{-\tau A(\zeta,s)}\big] \in C^r\big(\overline{\Lambda}, \mathcal{L}(X_0)\big) \tag{11.4}$$

*holds and that all derivatives up to the order $r$ are continuous functions of $(\zeta,s,\tau) \in \overline{\Lambda} \times [0,T] \times (0,\infty)$. Moreover, if $k \in \mathbb{N}$ with $k \leq r$ there exists a constant $c > 0$ independent of $0 \leq j \leq k$, $\zeta \in \overline{\Lambda}$, $s \in [0,T]$ and $\tau > 0$ such that*

$$\big\|\partial_\zeta^j A^\ell(\zeta,s)e^{-\tau A(\zeta,s)}\big\| \leq c\,\tau^{-\ell} \tag{11.5}$$

**Proof**

By the representation formula (1.17) we may write

$$A^\ell(\zeta,s)e^{-\tau A(\zeta,s)} = \frac{1}{2\pi i}\int_\Gamma \lambda^\ell e^{\lambda\tau}\big(\lambda + A(\zeta,s)\big)^{-1} d\lambda\,. \tag{11.6}$$

If we can show that for any $0 \leq j \leq k$

$$\big\|\partial_\zeta^j\big(\lambda + A(\zeta,s)\big)^{-1}\big\| \leq \frac{c_0}{|\lambda|} \tag{11.7}$$

holds for some constant $c_0 > 0$ independent of $\lambda$ in some sector $S_{\frac{\pi}{2}+\alpha}$ – for $\alpha \in \big(0, \frac{\pi}{2}\big]$ suitable – and of the quantities mentioned in the proposition, assertion (11.4) will follow from the theorem on differentiation of parameter integrals. The estimate (11.5) is then obtained in the same way as (1.18).

Before we prove (11.7) we show that there exists a constant $c_1 > 0$ not depending on the quantities mentioned above such that

$$\big\|A(\zeta,s)\partial_\zeta^j\big(\lambda + A(\zeta,s)\big)^{-1}\big\| \leq c_1 \tag{11.8}$$

for all $0 \le j \le k$.

If $j = 0$ (11.8) is an easy consequence of $(\tilde{A}2)$ and Remark 1.2 writing $A(\lambda + A)^{-1} = \mathbb{1} - \lambda(\lambda + A)^{-1}$. In the sequel we omit the arguments and write $A$ instead of $A(\zeta, s)$. Observe that

$$\partial_\zeta(\lambda + A)^{-1} = -(\lambda + A)^{-1}\big(\partial_\zeta A\big)(\lambda + A)^{-1} \tag{11.9}$$

holds for all $\lambda \in S_{\frac{\pi}{2}+\alpha}$. Indeed, (11.9) is obtained using the chain rule and differentiating the map $\big[B \mapsto B^{-1}\big] \in C^\infty\big(\mathrm{Isom}(X_1, X_0), \mathcal{L}(X_0, X_1)\big)$. Therefore, we get that

$$\|A\, \partial_\zeta(\lambda + A)^{-1}\| \le \|A(\lambda + A)^{-1}\|\, \|\big(\partial_\zeta A\big)A^{-1}\|\, \|A(\lambda + A)^{-1}\|. \tag{11.10}$$

Hence, (11.8) for $j = 1$ follows from the case $j = 0$ and (11.3). For $2 \le j \le k$ we prove (11.8) by induction.

Assume that the assertion holds for $1 \le j \le p$. Taking the $p$-th derivative of (11.9) on both sides and applying $A$, we obtain

$$\begin{aligned} &A\partial_\zeta^{p+1}(\lambda + A)^{-1} = \\ &\sum_{j=0}^{p} \binom{p}{j} A\partial_\zeta^{p-j+1}(\lambda + A)^{-1} \sum_{q=0}^{j} \binom{j}{q} \big(\partial_\zeta^{j-q} A\big)A^{-1}A\partial_\zeta^{q}(\lambda + A)^{-1}. \end{aligned} \tag{11.11}$$

By the induction hypothesis and (11.3) we have that (11.8) also holds for $p + 1$ proving (11.8).

Similar calculations together with (11.8) and the resolvent estimate in $(\tilde{A}2)$ show that (11.7) holds and the proof of the proposition is complete. □

**C. Smoothness properties of the resolvent kernels:** To get the assertions corresponding to Proposition 11.2 in the nonautonomous case, we consider the representation formulas (9.23) and (9.24). If we are able to prove that the the resolvent kernels $R$ and $H$ are differentiable and satisfy an estimate such that the theorem on the differentiation of parameter integrals can be applied, the desired result will follow.

**11.3 Lemma**
*Let assumptions $(\tilde{A}1)$–$(\tilde{A}3)$ be satisfied and let the resolvent kernels $R(\zeta; \cdot, \cdot)$ and $H(\zeta; \cdot, \cdot)$ for fixed $\zeta \in \overline{\Lambda}$ be defined as in Theorem 9.2. Then*

$$\big[\zeta \to R(\zeta; \cdot, \cdot)\big] \in C^r\big(\overline{\Lambda}, C(\dot{\Delta}_T, \mathcal{L}(X_0))\big) \tag{11.12}$$

*Moreover, if $k \in \mathbb{N}$ with $k \le r$ there exists a constant $c > 0$ independent of $0 \le j \le k$, $\zeta \in \overline{\Lambda}$ and $(t, s) \in \dot{\Delta}_T$ such that*

$$\big\|\partial_\zeta^j R(\zeta; t, s)\big\| \le c(t - s)^{\rho-1}, \tag{11.13}$$

*where $\rho \in (0, 1)$ is the Hölder exponent from $(\tilde{A}3)$. The same assertions hold for $H(\cdot\,; \cdot, \cdot)$.*

**Proof**

In Section 9.A we learned how to construct the resolvent kernel to an integral equation. Let $R_1(\zeta;\cdot,\cdot)$ and $H_1(\zeta;\cdot,\cdot)$ be defined by (9.17) and (9.19) respectively, where $A(\cdot)$ is replaced by $A(\zeta,\cdot)$. Starting from $R_1(\zeta;\cdot,\cdot)$, we define $R_m(\zeta;\cdot,\cdot)$ inductively by means of (9.7) replacing $k_1(\cdot,\cdot)$ by $R_1(\zeta;\cdot,\cdot)$. Then,

$$R(\zeta;t,s) = \sum_{m=1}^{\infty} R_m(\zeta;t,s) \tag{11.14}$$

for all $(t,s) \in \dot{\Delta}_T$ and $\zeta \in \overline{\Lambda}$. In a very similar way, $H_m(\zeta;\cdot,\cdot)$ and $H(\zeta;\cdot,\cdot)$ are constructed.

We deduce now an estimate for the derivatives of $R_m(\zeta;t,s)$. Due to the similar structure, the same estimates hold also for the derivatives of $H_m(\zeta;t,s)$.

We start with the case $m=1$. Let $k \in \mathbb{N}$ with $k \le r$. Then it follows from ($\tilde{A}3$) that there exists a constant $c_0$ not depending on $\zeta \in \overline{\Lambda}$, $(t,s) \in \Delta_T$ and $0 \le j \le k$ such that

$$\begin{aligned} &\big\|\partial_\zeta^j\big[\big(A(\zeta,t)-A(\zeta,s)\big)A^{-1}(\zeta,s)\big]\big\| \\ &\le \sum_{p=0}^{j}\binom{j}{p}\big\|\partial_\zeta^p A(\zeta,t)-\partial_\zeta^p A(\zeta,s)\big\|_{1,0}\,\big\|\partial_\zeta^{j-p}A^{-1}(\zeta,s)\big\|_{0,1} \\ &\le c_0(t-s)^\rho \end{aligned} \tag{11.15}$$

holds for all $(t,s) \in \Delta_T$ and $\zeta \in \overline{\Lambda}$. Take now the derivative in (11.14). Using (11.15) and Proposition 11.2 we find a constant $c_1 > 0$ independent of $\zeta \in \overline{\Lambda}$, $(t,s) \in \Delta_T$ and $0 \le j \le k$ such that

$$\big\|\partial_\zeta^j R_1(\zeta;t,s)\big\| \le c_1(t-s)^{\rho-1}. \tag{11.16}$$

Moreover, for any $0 \le j \le k$ and $m \ge 1$ we have that

$$\big\|\partial_\zeta^j R_m(\zeta;t,s)\big\| \le 2^{j(m-1)}\frac{\big[c_1\,\Gamma(\rho)\big]^m}{\Gamma(\rho m)}\,(t-s)^{m\rho-1} \tag{11.17}$$

holds for all $(t,s) \in \dot{\Delta}_T$. We prove this by induction.

For $m=1$ this is exactly (11.16). Using the induction hypothesis as well as (9.8) and (9.9) we get that

$$\begin{aligned} &\big\|\partial_\zeta^j R_{m+1}(\zeta;t,s)\big\| \\ &\quad= \Big\|\int_s^t \sum_{p=0}^{j}\binom{j}{p}\partial_\zeta^p R_m(\zeta;t,\tau)\partial_\zeta^{j-p}R_1(\zeta;\tau,s)\,d\tau\Big\| \\ &\quad\le c_1 2^{j(m-1)}\frac{\big[c_1\,\Gamma(\rho)\big]^m}{\Gamma(\rho m)}\sum_{p=0}^{j}\binom{j}{p}\int_s^t (t-\tau)^{m\rho-1}(\tau-s)^{\rho-1}\,d\tau \\ &\quad= 2^{jm}\frac{\big[c_1\,\Gamma(\rho)\big]^{m+1}}{\Gamma\big(\rho(m+1)\big)}\,(t-s)^{(m+1)\rho-1} \end{aligned} \tag{11.18}$$

holds. Hence, the proof of (11.17) is complete.

Estimate (11.17) implies now that the series (11.14) is differentiable with respect to $\zeta$ and the derivatives are given by

$$\sum_{m=0}^{\infty} \partial_\zeta^j R_m(\zeta; t, s.)$$

Moreover, this series converges for each $0 \leq j \leq k$ in $\mathcal{L}(X_0)$ uniformly with respect to $\zeta \in \overline{\Lambda}$ and $(t,s) \in \Delta_T^\varepsilon$ for any $\varepsilon > 0$. Therefore, $R(\cdot\,; t, s)$ is differentiable and all the derivatives up to the order $k$ are continuous functions of $(\zeta, t, s)$ and satisfy the estimate (11.13), completing the proof of the lemma. □

**D. The nonautonomous case:** Now we are able to prove the result announced at the beginning of this section.

**11.4 Theorem**

*Let assumptions $(\tilde{A}1)$–$(\tilde{A}3)$ be satisfied and let $U_\zeta(\cdot\,,\cdot)$ be the evolution operator for the family $\big(A(\zeta,t)\big)_{0\leq t\leq T}$. Then for any $(t,s) \in \Delta_T$ we have that*

$$\big[\zeta \mapsto U_\zeta(t,s)\big] \in C^r\big(\overline{\Lambda}, \mathcal{L}_s(X_0)\big) \cap C^r\big(\overline{\Lambda}, \mathcal{L}_s(X_1)\big).$$

**Proof**

The assertion follows from Proposition 11.2, Lemma 11.3 and the representation formulas (9.23) and (9.24). □

We have now proved differentiable parameter dependence of the evolution operator $U_\zeta(t,s)$ in $\mathcal{L}(X_0)$ and $\mathcal{L}(X_1)$. By means of interpolation theory it is now possible to extend this result to interpolation spaces between $X_1$ and $X_0$. To do this let $\big((\cdot\,,\cdot)_\alpha\big)_{0<\alpha<1}$ an admissible interpolation method as described in Section 3 and set

$$X_\alpha := (X_0, X_1)_\alpha$$

for any $\alpha \in (0,1)$.

**11.5 Corollary**

*Let the same hypotheses as in the above theorem be satisfied. Then, for any $\alpha \in [0,1]$ we have that*

$$\big[\zeta \mapsto U_\zeta(t,s)\big] \in C^r\big(\overline{\Lambda}, \mathcal{L}_s(X_\alpha)\big).$$

**Proof**

Let $x \in X_1$. Then, by Theorem 11.4, it is clear that for any $\zeta_0 \in \Lambda$ the difference quotient

$$\frac{U_{\zeta-\zeta_0}(t,s)x - U_{\zeta_0}(t,s)x - (\zeta - \zeta 0)\partial_\zeta U_{\zeta_0}(t,s)x}{(\zeta_0 - \zeta)} \tag{11.19}$$

tends to zero in $X_0$ and in $X_1$ as $\zeta$ approaches $\zeta_0$. By the interpolation inequality $(F3)$ in Proposition 3.6, the same holds in $X_\alpha$. On the other hand, (11.19) is uniformly bounded with respect to $\zeta \in \Lambda \setminus \{\zeta_0\}$ in $X_0$ for all $x \in X_0$ and in $X_1$ for all $x \in X_1$. Thus, by the principle of uniform boundedness,

$$\left\| \frac{U_{\zeta-\zeta_0}(t,s) - U_{\zeta_0}(t,s) - (\zeta-\zeta_0)\partial_\zeta U_{\zeta_0}(t,s)}{(\zeta_0-\zeta)} \right\|_{i,i} \qquad (i = 0, 1)$$

is uniformly bounded with respect to $\zeta \in \Lambda \setminus \{\lambda_0\}$ in $\mathcal{L}(X_0)$ and in $\mathcal{L}(X_1)$. By the interpolation inequality $(F2)$ in Definition 3.5, the same holds in $\mathcal{L}(X_\alpha)$. Since $X_1 \overset{d}{\hookrightarrow} X_\alpha$, (11.19) converges to zero in $X_\alpha$ for all $x \in X_\alpha$.

By the same arguments, the assertion holds for the higher derivatives and the assertion of the corollary follows. □

**11.6 Corollary**

*Under the assumptions of the above corollary, it holds that for any $\alpha \in [0, T]$*

$$\big[(\zeta, x) \mapsto U_\zeta(\cdot,\cdot)x\big] \in C^r\big(\Lambda \times X_\alpha, C(\Delta_T, X_\alpha)\big) \tag{11.20}$$

**Proof**

The assertion follows from Corollary 11.5 and the properties of strongly differentiable functions (compare Section 0.H). □

**11.7 Remark**

The main difficulty in the proof of Theorem 11.4 is that we allow parameter dependence in the principal part of $A$. The proof is much simpler if this is not the case. More precisely, if we assume that

$$A(\zeta, t) = A_0(t) + B(\zeta, t) \tag{11.19}$$

where $B(\cdot\,,\cdot) \in C^{r,\rho}\big(\overline{\Lambda} \times [0,T], \mathcal{L}(X_\alpha, X_0)\big)$ for some $\alpha \in [0, 1)$ and $\big(A_0(t)\big)_{0\le t\le T}$ satisfies assumptions $(A1)$–$(A3)$ of section 2. Then we may consider $-B(\zeta, t)u$ as a nonlinearity and study the parameter depence of the solution of the equation

$$\begin{cases} \dot{u} + A_0(t)u = -B(\zeta, t)u & 0 \le s < t \le T \\ u(s) = x \end{cases} \tag{11.20}$$

For details we refer to Section 18. □

**E. Analytic dependence:** We assume in the sequel that $(\tilde{A}1)$, $(\tilde{A}2)$ and $(\tilde{A}3)_\omega$ are satisfied. By complexification we may assume that $X_0, X_1$ are complex Banach spaces.

Moreover, observe that by Theorem 1.4 one may extend the mapping $\zeta \mapsto A(\zeta, t)$ to a complex neigbourhood $\tilde{\Lambda}$ of $\Lambda$ such that $(\tilde{A}_1)$, $(\tilde{A}_2)$ and $(\tilde{A}_3)_\omega$ are satisfied with $\Lambda$ replaced by $\tilde{\Lambda}$.

Let $U_{n,\zeta}(t,s)$ be the evolution operator to the family $\big(A_n(\zeta,t)\big)_{0\le t\le T}$, where $A_n$ is the $n$-th Yosida approximation of $A$. Replacing $B$ by $A_n$ in (9.29) we see that $\zeta \to U_{\zeta,n}(t,s)$ is analytic. We used that the convergence in (9.30) is in $\mathcal{L}(X_0)$ and uniform with respect to $\zeta \in \overline{\Lambda}$ and that the uniform limit of analytic functions is again analytic (cf. [**47**], Chapter IX). For the same reason, Theorem 10.7 implies then that $\zeta \to U_\zeta(t,s)x$ is analytic for all $(t,s) \in \Delta_T$ and $x \in X_0$, where $U_\zeta(\cdot\,,\cdot)$ is the evolution operator for the family $\big(A(\zeta,t)\big)_{0\le t\le T}$. This can be expressed by saying that $\zeta \mapsto U_\zeta(t,s)$ is strongly analytic.

Similar arguments show, that $\zeta \mapsto A(t)U(t,s)A^{-1}(t,s)$ is also stronly analytic.

Using the fact that strongly analytic operator-valued functions are analytic with respect to the uniform operator topology (see Kato [**75**], Theorem III.3.12), we get the following theorem:

**11.8 Theorem**
*Let $(\tilde{A}1)$, $(\tilde{A}2)$ and $(\tilde{A}3)_\omega$ be satisfied and let $U_\zeta(\cdot\,,\cdot)$ be the evolution operator for the family $\big(A(\zeta,t)\big)_{0\le t\le T}$. Then for any $(t,s)\in\Delta_T$, the map*

$$\zeta \mapsto U_\zeta(t,s),\ \Lambda \to \mathcal{L}(X_i) \qquad (i=0,1)$$

*is analytic.*

Similar to the differentiable case we may prove the following corollary:

**11.9 Corollary**
*Under the assumptions of the above theorem, the map*

$$\zeta \mapsto U(t,s),\ \Lambda \to \mathcal{L}(X_\alpha)$$

*is analytic for any $\alpha \in [0,1]$ and $(t,s)\in\Delta_T$.*

**Proof**
By the above theorem, $\|U(t,s)\|_{i,i}$ is uniformly bounded for $\zeta$ in a small neighbourhood of $\zeta_0$, where $\zeta_0 \in \Lambda$ is chosen fixed. By interpolation, the same holds for $\|U(t,s)\|_{\alpha,\alpha}$ for fixed $\alpha \in (0,1)$. Using the interpolation inequality $(F2)$ from Definition 3.5 and the above theorem we see that the power series representing $U_\zeta(t,s)$ in a neighbourhood of $\zeta_0$ converges in $\mathcal{L}(X_\alpha)$ and represents $U\zeta(t,s)$. Hence the corollary is proved. □

**Notes and references:** We do not know of any reference where parameter dependence is established in this setting besides in Amann [**11**], where results on continuous, but not on differentiable, dependence are obtained. We took the basic techniques from Tanabe [**119**], where they are used to established regularity with respect to the time variable.

## 12. Ordered Banach spaces and positive operators

In many applications one is interested in positive solutions of an evolution equation. In fact, population densities or concentrations in chemical processes are always positive. In order to be able to formulate positivity properties of solutions to differential equations within the abstract framework we need the concepts of ordered vector spaces and positive operators.

**A. Ordered vector spaces:** A real vector space $E$ is called an *ordered vector space* if it is equipped with an order relation, i.e a transitive, reflexive and antisymmetric relation $\leq$, satisfying the following compatibility conditions:

$$\begin{aligned} x \leq y \quad &\Longrightarrow \quad x+z \leq y+z \qquad &&\text{for all } z \in E \\ x \leq y \quad &\Longrightarrow \quad \lambda x \leq \lambda y \qquad &&\text{for all } \lambda \geq 0. \end{aligned}$$

Clearly, $E_+ := \{x \in E;\ x \geq 0\}$ is a convex cone in $E$, that is $\lambda x + \mu y \in E_+$ whenever $x, y \in E_+$ and $\lambda, \mu \in \mathbb{R}$ are positive. $E_+$ is called the *positive cone* of E. Moreover, by antisymmetry, $E_+$ is a *proper cone*, that is $E_+ \cap (-E_+) = \{0\}$. On the other hand, if $K$ is any proper cone in $E$, then $E$ becomes an ordered vector space with $E_+ = K$ by setting $x \leq y$ whenever $y - x \in K$ holds.

When dealing with ordered vector spaces, we shall always use the following notation:

$$x < y \qquad \text{if} \quad x \leq y \quad \text{and} \quad x \neq y.$$

For $x \leq y$, we can define the *order interval with endpoints $x$ and $y$* in the space $E$ by setting

$$[x, y]_E := \{z \in E;\ x \leq z \leq y\}.$$

If no confusion seems possible we omit the index $E$ and simply write $[x, y]$.

For our purposes it will sufice to consider *ordered Banach spaces* and *Banach lattices.* We proceed to give the relevant definitions.

**B. Ordered Banach spaces:** Let $E$ be a Banach space. Then $E$ is called *ordered Banach space* if it is an ordered vector space such that the positive cone $E_+$ is norm closed.

On the dual space $E'$ we can define the *dual cone* $E'_+$ by setting

$$E'_+ := \{x' \in E';\ <x', x> \geq 0 \text{ for all } x \in E_+\}.$$

In general $E'_+$ is closed and convex but not proper. It follows from the Hahn-Banach theorem, that $E'_+$ is proper if and only if $E_+ - E_+$ is dense in $E$. We say

$$E_+ \text{ is } \textit{total} \text{ if } \overline{E_+ - E_+} = E \text{ and}$$
$$E_+ \text{ is } \textit{generating} \text{ if } E_+ - E_+ = E.$$

We have seen that $E'$ carries in a natural way the structure of an ordered Banach space if and only if $E_+$ is total. Since this is a major technical advantage and since most the spaces arising from applications have this property, we shall always assume that $E_+$ is total.

Sometimes it is important to know, that $E_+$ has nonempty interior. If $x$ is an interior point of $E_+$, then it is clear that the order intervall $[-x, x]$ has nonempty interior, and is thus a neigbourhood of zero. This implies that for any $x' \in E_+$ with $x' > 0$

$$<x', x> > 0 \quad \text{for all } x > 0 \tag{12.1}$$

holds. In some cases, it is sufficient to know that there exist points in $E_+$ for which (12.1) holds. Points with this property are called *quasi-interior points* of $E_+$. One can show that quasi-interior points and interior points coincide if $E_+$ has nonempty interior ([**31**], Proposition $A$.2.10). If $x$ is an interior point of $E_+$ then

$$\bigcup_{n \in \mathbb{N}} n[-x, x] = E. \tag{12.2}$$

This follows from the fact that $[-x, x]$ is a neighbourhood of zero and is as such absorbing. In fact, by Baires theorem, (12.2) characterizes interior points of $E_+$. For quasi-interior points we have a weaker version of (12.2)

$$\mathrm{cl}_E\Big(\bigcup_{n \in \mathbb{N}} n[-x, x]\Big) = E. \tag{12.3}$$

This is actually a characterization of quasi-interior points in the special case that $E$ is a Banach lattice ([**108**], Theorem II.6.3). For a definition of Banach lattices see the next subsection. We write

$$x \ll y$$

whenever $y - x$ is a quasi-interior (or even an interior) point of $E_+$. In this case we have for any $z \geq 0$

$$x \ll y + z.$$

In particular, it easily follows from this that the set of quasi-interior points of $E_+$ – if it is nonempty – is dense in $E_+$. Finally, observe that if $E_+$ has a quasi-interior point $x$, then $E_+$ is a total cone. Indeed, this follows from $[-x, x] \subset E_+ - E_+$ and (12.3). If $x$ is even an interior point of $E_+$, (12.2) implies that $E_+$ is generating.

**C. Banach lattices:** Let E be an ordered vector space and A a subset of E. A point $m \in E$ is called *supremum* of A, if $a \leq m$ for all $a \in A$, and for any $m' \in E$ satisfying $a \leq m'$ for all $a \in A$ we have $m \leq m'$. The *infimum* of $A$ is defined in an analogous way. Whenever they exist, we denote by $\sup A$ and $\inf A$ the supremum and the infimum respectively. Furthermore, for $x \in E$ we set

$$\begin{aligned} x^+ &:= \sup\{0, x\} && \text{the } \textit{positive part} \text{ of } x \\ x^- &:= \sup\{0, -x\} && \text{the } \textit{negative part} \text{ of } x \\ |x| &:= x^+ + x^- && \text{the } \textit{absolute value} \text{ of } x \end{aligned}$$

if they exist. A *vector lattice* is an ordered vector space in which $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for any $x, y \in E$. This implies in particular that $E_+$ is generating. If $E$ is an ordered Banach space which is a vector lattice satisfying

$$\| \, |x| \, \| = \|x\| \tag{12.4}$$

for all $x \in E$, then $E$ is called *Banach lattice.* If (12.4) is satisfied $E_+$ is said to be *normal.* So Banach lattices are very special ordered Banach spaces.

**D. Positive operators and spectral theory:** Suppose that $E$ and $F$ are ordered Banach spaces and that $T : E \to F$ is a given mapping. We introduce the following notations:

$$\begin{aligned} T &\geq 0 && \text{if } Tx \geq 0 \text{ for all } x \geq 0 \\ T &> 0 && \text{if } Tx > 0 \text{ for all } x > 0 \\ T &\gg 0 && \text{if } Tx \gg 0 \text{ for all } x > 0. \end{aligned}$$

In the first case $T$ is said to be *positive*, in the second *strictly positive* and in the last *strongly positive.* If $E_+$ is generating and $F$ is a Banach lattice, it can be shown that each positive linear operator from $E$ to $F$ is already bounded ([**31**] Prop. $A$.2.11). In particular, any positive linear functional on an ordered Banach space with generating positive cone is continuous. If $E$ is a Banach lattice and $T : E \to E$ is positive and linear, then by the above remark $T \in \mathcal{L}(E)$ and by (12.4):

$$\|T\| = \sup\{\|Tx\|;\ x \geq 0 \text{ and } \|x\| = 1\}. \tag{12.5}$$

We want now turn to the spectral theory for positive operators which is of particular beauty. As usual when dealing with linear operators on real Banach spaces, the spectrum is to be understood as the spectrum of the complexification (see Section 0.F). Due to the following result the spectral radius of a positive operator plays a prominent role (see [**109**], Appendix, Lemma 2.2).

**12.1 Lemma**
*Let $E$ be an ordered Banach space with generating positive cone and $T \in \mathcal{L}(E)$ a positive operator. Then the spectral radius* $\mathrm{r}(T)$ *is an element of the spectrum of $T$.*

The following Lemma gives a representation formula for the spectral radius of positive operators in Banach lattices. We include a proof since we were not able to find a reference.

**12.2 Lemma**
*Let $E$ be a Banach lattice and $T$ a positive operator. Then $T \in \mathcal{L}(E)$ and*

$$\mathrm{r}(T) = \lim_{n\to\infty} \Big( \sup_{\substack{x\in\partial\mathbb{B}_E\cap X_+ \\ x'\in\partial\mathbb{B}_{E'}\cap X'_+}} <x', T^n x> \Big)^{\frac{1}{n}} \tag{12.6}$$

*holds, where $\mathbb{B}_E$ and $\mathbb{B}_{E'}$ are the unit balls in $E$ and $E'$, respectively.*
**Proof**
That any positive operator on a Banach lattice is bounded was already mentioned at the beginning of this subsection. From the Hahn-Banach Theorem it follows that

$$\|T\| = \sup_{\substack{\|x\|=1 \\ \|x'\|=1}} |<x', Tx>|.$$

Since $E$ is a Banach lattice, its dual $E'$ is also a Banach lattice (see e.g. [**108**], Theorem II.5.5). Splitting $x$ and $x'$ in positive and negative part, we see that

$$\|T\| = \sup_{\substack{\|x\|=1 \\ \|x'\|=1}} |<x', Tx>| \le 4 \sup_{\substack{x\in\partial\mathbb{B}_E\cap X_+ \\ x'\in\partial\mathbb{B}_{E'}\cap X'_+}} <x', Tx>.$$

On the other hand it is clear that

$$\sup_{\substack{x\in\partial\mathbb{B}_E\cap X_+ \\ x'\in\partial\mathbb{B}_{E'}\cap X'_+}} <x', Tx> \le \|T\|$$

holds. The assertion of the lemma is now obtained using $\mathrm{r}(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ and $\lim_{n\to\infty} \sqrt[n]{4} = 1$. □

Suppose now that $E$ is an ordered Banach space. A positive operator $T \in \mathcal{L}(E)$ is said to be *irreducible*, if there exists a $\lambda > \mathrm{r}(T)$ such that

$$(\lambda - T)^{-1} \gg 0 \tag{12.7}$$

holds. In particular this implies that $E_+$ has quasi-interior points. For example a linear operator $T$ is irreducible if $T$ itself or a power of $T$ is strongly positive. To see this just note that for every $\lambda > \mathrm{r}(T)$ we have that

$$(\lambda - T)^{-1} = \sum_{k=0}^{\infty} \lambda^{-(k+1)} T^k \,. \tag{12.8}$$

We now state one of the most important theorems in the spectral theory for positive operators.In conjunction with the maximum principle it has established as a powerful tool in the theory of elliptic and parabolic partial differential equations.

**12.3 Theorem**
*Let $E$ be an ordered Banach space and $T \in \mathcal{L}(E)$ a positive irreducible operator. Suppose that in addition one of the following hypotheses is satisfied.*

- ($i$) *$T$ is compact and $E$ is a Banach lattice*
- ($ii$) *$T$ is compact and* $\mathrm{int}(E_+) \neq \emptyset$
- ($iii$) $\mathrm{r}(T)$ *is a pole of the resolvent of $T$ and $E$ is a Banach lattice*
- ($iv$) $\mathrm{r}(T)$ *is a pole of the resolvent and* $\mathrm{int}(E_+) \neq \emptyset$.

*Then, the following assertions hold:*

1. $\mathrm{r}(T) > 0$ *is a pole of the resolvent of $T$ of order* 1,
2. $\mathrm{r}(T)$ *is an algebraically simple eigenvalue of $T$ and $T'$. The eigenspace is spanned by a quasi-interior eigenvector and a strictly positive eigenfunctional respectively,*
3. $\mathrm{r}(T)$ *is the only eigenvalue of $T$ having a positive eigenfunction*

**Proof**
It is known that every compact irreducible operator on a Banach lattice has nonzero spectral radius (see [**96**], Theorem 4.2.2). By Lemma 12.1 it is an element of the spectrum of $T$. From the Riesz-Schauder theory of compact operators we conclude that $\mathrm{r}(T)$ is a pole of the resolvent of $T$. Therefore, ($i$) implies ($iii$).

If ($iii$) is satisfied, the assertions of the theorem follow from [**108**], Theorem $V$.5.2 and its corollary.

To prove $(ii)$ we apply the Krein-Rutman theorem (see e.g. [**44**], [**81**], [**86**], [**128**]) to $(\lambda - T)^{-1}$, where $\lambda > r(T)$ is chosen such that (12.7) holds. By the spectral mapping theorem, assertion (1)–(3) follow for $T$. Finally, if $(iv)$ is satisfied, the assertions follow from [**109**], Appendix, Theorem 3.2. The uniqueness of $\mathrm{r}(T)$ as an eigenvalue with positive eigenvector is a consequence of (2). Indeed, let and $x$ be a positive eigenvector corresponding to the (real) eigenvalue $\lambda_0$ and $x'$ the strictly positive eigenfunctional corresponding to $\mathrm{r}(T)$. Then, $\lambda_0 <x', y> = <x', Ty> = <T'x', y> = \mathrm{r}(T) <x', y>$, which implies that $\lambda_0 = \mathrm{r}(T)$. □

In many instances, the following corollary turns out to be useful.

**12.4 Corollary**
*Suppose that the assumptions of the above theorem are satisfied and consider the inhomogeneous equation*

$$\lambda x - Tx = y \tag{12.9}$$

*with* $y \geq 0$. *Then* (12.9) *has a unique positive solution if* $\lambda > \mathrm{r}(T)$, *no positive solution if* $\lambda < \mathrm{r}(T)$ *and no solution at all if* $\lambda = \mathrm{r}(T)$.

Next we consider some examples of ordered Banach spaces and Banach lattices

**E. Examples of ordered Banach spaces:** 1) Of course, $\mathbb{R}^n$ equipped with the order relation induced by the cone $\mathbb{R}^n_+$ (see Section 0.A) is a Banach lattice.

2) If $M$ is any set and $E$ an ordered vector space, then every vector space of functions from $M$ into $E$ carries in a natural way an order structure, which is given by

$$f \leq g \qquad \text{whenever} \quad f(x) \leq g(x) \quad \text{for all } x \in M.$$

When dealing with function spaces we shall always use this order relation without further comment.

3) We first consider some Banach lattices. Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. Then the spaces

$$BUC(\Omega), \qquad C_0(\Omega), \qquad L_p(\Omega) \quad (1 \leq p \leq \infty)$$

are Banach lattices. Condition (12.4) follows immediately from the definition of the norms in these spaces. A function $u \in BUC(\Omega)$ ($u \in L_\infty(\Omega)$) is interior to the positive cone, iff there exists a $\delta > 0$ such that $u(x) > 0$ for (almost) all $x \in \Omega$. The positive cones in $C_0(\Omega)$ and $L_p(\Omega)$ $(1 \leq p < \infty)$ have empty interior, but they do have quasi-interior points. In fact, the quasi-interior points are given exactly by those functions $u \in C_0(\Omega)$ ($L_p(\Omega)$) for which $u(x) > 0$ for (almost) all $x \in \Omega$.

4) Let now $\Omega$ be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Then, the spaces

$$W_p^\alpha(\Omega), \qquad H_p^\alpha(\Omega), \qquad C^\alpha(\overline{\Omega}) \quad (\alpha > 0,\ 1 < p < \infty)$$

are ordered Banach spaces. However, (12.4) is not satisfied, and, hence, they are not Banach lattices. Let $E$ and $F$ be ordered Banach spaces satisfying $F \hookrightarrow E$ and $F_+ = F \cap E_+$. If $E_+$ has nonempty interior, by the continuity of the inclusion, the same holds for $F_+$. Due to the imbeddings in Theorem A3.7, this implies that the positive cones in $W_p^\alpha(\Omega)$ and $H_p^\alpha(\Omega)$ have nonempty interior if $\alpha - \frac{n}{p} > 0$. The positive cone of $C^\alpha(\overline{\Omega})$ has always interior points.

5) Another example is the space

$$C_D^1(\overline{\Omega}) := C^1(\overline{\Omega}) \cap C_0(\Omega).$$

This space has a cone with nonempty interior. The interior points are given by those functions $u \in C_D^1(\overline{\Omega})$ satisfying $u(x) > 0$ for all $x \in \Omega$ and $\partial_\nu u(x) < 0$ for all $x \in \partial\Omega$, where $\nu$ is the outer unit normal on $\partial\Omega$.

**Notes and references:** Examples of books devoted to the abstract theory of ordered vector spaces are Meier-Nieberg [**96**] and Schaefer [**108**], [**109**]. A treatment from the perspective of applications to evolution equations may be found in the books of Clément et al. [**31**], Krasnosel'skiı̆ [**81**], [**83**], Nagel et al. [**98**]. The survey article of Amann [**5**] and the books of Deimling [**44**] and Zeidler [**128**] are concerned with interesting applications to nonlinear analysis.

## 13. The parabolic maximum principle and positivity

In the theory of elliptic and parabolic differential equations of second order, the strong positivity of a solution operator is usually shown applying the strong maximum principle. Since we are interested in evolution equations, we shall only consider the parabolic case. We formulate it here in an $L_p$-setting and in the form of a minimum principle which is more suitable to our purposes. Moreover, we shall not make any restriction on the sign of the zero-order coefficients of the differential operators neither in the domain nor on the boundary.

Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $T > 0$ and that $\mathcal{A} := \mathcal{A}(x,t,D)$ $(0 \le t \le T)$ is an elliptic differental operator of the form (2.8) with coefficients $a_{jk}$, $a_j$ and $a_0$ continuous on $\overline{\Omega} \times [0,T]$ satisfying (2.9).

**13.1 Proposition**

*Let $u \in W_p^{2,1}\big(\Omega \times (0,T)\big)$ for some $p \ge n+1$ and suppose that*

(13.1) $$\partial_t u + \mathcal{A}u \ge 0 \qquad \text{almost everywhere in } \Omega \times (0,T)$$

*holds. If $u$ attains its minimum $m$ at $(x_0, t_0) \in \Omega \times (0, T]$ and $m \leq 0$, then $u \equiv m$ in $\overline{\Omega} \times [0, t_0]$.*

**Proof**
First of all observe that by the imbedding theorem A3.14 $u$ lies in $C\big(\overline{\Omega} \times [0, T]\big)$ so that it makes sense to speak of a minimum of $u$. Replacing $u$ by $e^{-\lambda_0 t}u$ and $\mathcal{A}$ by $\mathcal{A} + \lambda_0$ with $\lambda_0 \geq \|a_0\|_\infty$ we may assume that the zero-order coefficient of $\mathcal{A}$ is nonnegative. It can be shown ([**48**], Theorem V.28) that

$$\lim_{\substack{x \to x_0 \\ t \nearrow t_0}} \text{ess-sup}\, \partial_t u + \mathcal{A}u \leq 0.$$

The proof of the proposition is now exactly the same as in the case of differentiable functions (see e.g. [**101**], [**58**]) □

For the next proposition concerning the derivative in an outward pointing direction we assume that $\Omega$ of class $C^2$.

**13.2 Proposition**
*Let $u \in W_p^{2,1}\big(\Omega \times (0, T)\big)$ for some $p > n + 2$ and suppose that (13.1) holds. Assume that $u$ attains its minimum $m$ at $(x_0, t_0) \in \partial\Omega \times (0, T]$ and that $b \neq 0$ is an outwards pointing nontangent vector on $\Omega$ at $x_0 \in \partial\Omega$. Then, if $u$ is nonconstant in $\overline{\Omega} \times (0, t_0]$, we have that*

$$\partial_b u(x_0, t_0) < 0.$$

**Proof**
By the imbedding theorem A3.14, $u$ lies in $C^{1,0}\big(\overline{\Omega} \times [0, T]\big)$. Then it is clear that $\partial_b u(x_0, t_0) \leq 0$. The assertion is now obtained in the same way as in the classical case (see e.g. [**101**], [**58**]). □

Let now $\mathcal{B} := \mathcal{B}(x, D)$ a boundary operator of the form

$$\mathcal{B}u = \begin{cases} 0 & \text{on } \Gamma_0 \\ \partial_b u + b_0 u & \text{on } \Gamma_1 \end{cases}$$

where $b_0 \in C^1(\Gamma_1)$ and $b \in C^1(\Gamma_1, \mathbb{R}^n)$ is a nowhere tangent and nowhere vanishing outwards pointing vector field. $\Gamma_0$ and $\Gamma_1$ are open and closed disjoint subsets of $\partial\Omega$ and $\Gamma_0 \cup \Gamma_1 = \partial\Omega$. We emphasize that we do not impose any restrictions on the sign of $b_0$.

We need the following two lemmas:

**13.3 Lemma**
*Let $1 < p < \infty$. Then there exists a map $\mathcal{R} \in \mathcal{L}\big(W_p^{2-1/p}(\Gamma_0) \times W_p^{1-1/p}(\Gamma_1), W_p^2(\Omega)\big)$ such that*

$$\mathcal{B}\mathcal{R} = \mathbb{1} \qquad \text{and} \qquad \gamma \circ \mathcal{R} = 0,$$

*where* $\gamma \in \mathcal{L}\big(W_p^1(\Omega), W_p^{1-1/p}(\partial\Omega)\big)$ *is the trace operator. Moreover,*

$$\mathcal{R}\big(C^2(\Gamma_0) \times C^1(\Gamma_1)\big) \subset C^2(\overline{\Omega}).$$

A proof of this result can be found in [**9**], Lemma 5.1. As a consequence we obtain the following lemma:

**13.4 Lemma**

*There exists a* $w \in C^2(\overline{\Omega})$*, such that* $w \gg 0$ *and* $\mathcal{B}w \geq 0$*.*

**Proof**

Let $\mathcal{R}$ be the map from Lemma 13.3. Put $w := 1+\varphi\mathcal{R}\|b_0\|_\infty$, where $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $0 \leq \varphi \leq 1$ and $\varphi = 1$ in a neighbourhood of $\Gamma_0$. Making the support of $\varphi$ small enough, we may assume that $w(x) > \frac{1}{2}$ holds for all $x \in \overline{\Omega}$. To see this recall that $\gamma \circ \mathcal{R} = 0$. Applying $\mathcal{B}$ to this function we obtain

$$\mathcal{B}w = b_0 + \|b_0\|_\infty \geq 0$$

and the assertion of the lemma follows. □

The following result allows to prove comparison theorems or positivity of the solution operator in many instances.

**13.5 Theorem**

*Let* $u \in W_p^{2,1}\big(\Omega \times (0,T)\big)$ *for some* $p > n+2$ *and suppose that*

$$\begin{cases} \partial_t u + \mathcal{A}(x,t,D)u \geq 0 & \text{in } \Omega \times (0,T] \\ \mathcal{B}(x,D)u \geq 0 & \text{on } \partial\Omega \times (0,T] \\ u(\cdot\,,0) \geq 0 & \text{in } \Omega \end{cases} \tag{13.2}$$

*holds. Then* $u \geq 0$ *holds in* $\overline{\Omega} \times [0,T]$*. If* $u(\cdot\,,0) > 0$*, then* $u(x,t) > 0$ *for all* $(x,t) \in \big(\Omega \cup \Gamma_1\big) \times (0,T]$ *and* $\partial_\nu u(x,t) < 0$ *for all* $(x,t) \in \Gamma_0 \times (0,T]$*, where* $\nu$ *denotes the outer unit normal on* $\Gamma_0$*.*

**Proof**

Let $v := \frac{u}{w}$, where $w$ is the function from Lemma 13.4. Then a simple calculation shows that

$$\begin{aligned} 0 &\leq \frac{1}{w}(\partial_t u + \mathcal{A}u) \\ &= \partial_t v - \sum_{j,k=1}^n a_{jk}\partial_j\partial_k v + \sum_{j=1}^n\Big(-\frac{2}{w}\sum_{k=1}^n a_{jk}\partial_k w + a_k\Big)\partial_j v + (\mathcal{A}w)v \end{aligned}$$

in $\Omega \times (0,T)$ and

$$0 \leq \frac{1}{w}\mathcal{B}u = \begin{cases} 0 & \text{on } \Gamma_0 \times (0,T] \\ \partial_b v + \left(\frac{1}{w}\mathcal{B}w\right)v & \text{on } \Gamma_1 \times (0,T]. \end{cases}$$

Clearly $v(\cdot\,,0) \geq 0$ and hence $v$ satisfies inequalities analogous to (13.2). Since $w \gg 0$ and $\mathcal{B}w \geq 0$, the zero-order coefficient $\left(\frac{1}{w}\mathcal{B}w\right)$ of the new boundary operator is nonnegative. The assertion of the theorem follows now from Proposition 13.1 and 13.2 and the definition of $v$. □

Let now $A(t)$ be the $L_p(\Omega)$-realization of $\big(\Omega, \mathcal{A}(x,t,D), \mathcal{B}(x,D)\big)$ for every $t \in [0,T]$ and for some $p \in (1,\infty)$. Assume that $U(\cdot\,,\cdot)$ is the evolution operator corresponding to the family $\big(A(t)\big)_{0\leq t\leq T}$ (see Example 2.9(d)). The domain of definition of $A(t)$ is given by $W^p_{p,\mathcal{B}}(\Omega)$ for every $t \in [0,T]$. We now set $X_0 := L_p(\Omega)$ and $X_1 := W^p_{p,\mathcal{B}}(\Omega)$. Put

$$C^1_{\mathcal{B}}(\overline{\Omega}) := \{u \in C^1(\overline{\Omega});\ \mathcal{B}u = 0\} \tag{13.3}$$

Then, if $p > n$, the imbedding Theorem A3.12 implies that

$$W^2_{p,\mathcal{B}}(\Omega) \hookrightarrow C^1_{\mathcal{B}}(\overline{\Omega}) \tag{13.4}$$

holds. For this reason, $U(t,s)$ maps $L_p(\Omega)$ into $C^1_{\mathcal{B}}(\overline{\Omega})$ whenever $p > n$ and $(t,s) \in \dot{\Delta}_T$. Observe that the positive cone in $C^1_{\mathcal{B}}(\overline{\Omega})$ has nonempty interior. The interior points are those functions $u$ for which $u(x) > 0$ for all $x \in \Omega \cup \Gamma_1$ and $\partial_\nu u(x) < 0$ for all $x \in \Gamma_0$, where $\nu$ is the outer unit normal on $\Gamma_0$. This implies in particular, that the positive cone in $X_1$ has nonempty interior.

As an easy consequence of Theorem 13.5 we prove the strong positivity of the evolution operator.

**13.6 Corollary**
*Let $p > n+2$ and $(t,s) \in \dot{\Delta}_T$. Then $U(t,s)\colon L_p(\Omega) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is strongly positive. In particular, $U(t,s) \in \mathcal{L}(X_0, X_1)$ is strongly positive and $U(t,s) \in \mathcal{L}(X_0)$ is irreducible. Moreover, if $\Gamma_0 = \emptyset$, $U(t,s) \gg 0$ as an operator from $L_p(\Omega)$ to $C(\overline{\Omega})$*

**Proof**
Let $u \in X_1$ with $u > 0$. Then we have that

$$U(\cdot\,,s)u \in C^1\big([0,T],X_0\big) \cap C\big([0,T],X_1\big).$$

Hence, we may identify $U(\cdot\,,s)u$ with an element of $W^{2,1}_p\big(\Omega \times (0,T)\big)$. Now, the assertion follows easily from Theorem 13.5: Since $X_1 \overset{d}{\hookrightarrow} X_0$ holds, $U(t,s)$ extends continuously to a positive operator on $X_0$. If now $u \in X_0$ with $u > 0$, we have $U(t,\tau)u > 0$ for $\tau$ near $s$

by continuity. Since $U(t,\tau)u \in X_1$, the assertion follows from the above considerations writing $U(t,s)u = U(t,\tau)U(\tau,s)u$ for suitable $\tau \in (s,t)$. $\square$

**13.7 Remark**

Theorem 13.5 contains as a special case the elliptic maximum principle. For completeness we would like to give here the precise statement.

Let $(\Omega, \mathcal{A}, \mathcal{B})$ be a second order elliptic boundary value problem of class $C^0$ and let $p > n$. Then, there exists a $\lambda_0 \in \mathbb{R}$ such that $u \geq 0$, whenever $u \in W_p^2(\Omega)$ satisfies the inequalities

$$\begin{cases} (\lambda + \mathcal{A})u \geq 0 & \text{in } \Omega \\ \mathcal{B}(x,D)u \geq 0 & \text{on } \partial\Omega. \end{cases}$$

for some $\lambda \geq \lambda_0$. Moreover, if $u \neq 0$, then $u(x) > 0$ for all $x \in \Omega \cup \Gamma_1$ and $\partial_\nu u(x) < 0$ for all $x \in \Gamma_0$, where $\nu$ denotes the outer unit normal on $\Gamma_0$.

The proof is completely analogous to the proof of Theorem 13.5 and can be found in [**9**], Theorem 6.1, from where we have actually taken the ideas. $\square$

**Notes and References:** For the classical maximum principles we refer to Friedman [**58**] or Protter and Weinberger [**101**]. In the elliptic case, a $W_p^2$-version was proved by Bony [**24**]. For a further disscussion of his result see Lions [**90**]. The result for parabolic operators analogous to the elliptic case was taken from Dong [**48**]. The positivity result in Remark 13.7 is due to Amann [**9**] and gave us the idea for the proof of Theorem 13.5. Note that a similar proof works in the case of time dependent boundary conditions. The essential ingredient is the existence of the operator $\mathcal{R}$ in Lemma 13.3. If the boundary operator $\mathcal{B}$ is time dependent, the existence of $\mathcal{R}$, which is in general time dependent, is ensured by Amann [**15**], Theorem B.3.

## 14. Superconvexity and periodic-parabolic eigenvalue problems

In this section we consider a generalization of the log-convexity of a real valued function to vector valued functions.

**A. Motivation:** As a motivation we may consider a periodic-parabolic eigenvalue problem with indefinite weight function on a bounded smooth domain $\Omega$ in $\mathbb{R}^n$:

$$(14.1)\qquad \begin{cases} \partial_t\varphi + \mathcal{A}(x,t,D)\varphi - \lambda m(x,t)\varphi = \mu(\lambda)\varphi & \text{in } \Omega \times \mathbb{R} \\ \mathcal{B}(x,D)\varphi = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ \varphi(\cdot\,,0) = \varphi(\cdot\,,T) & \text{in } \Omega, \end{cases}$$

where $T > 0$ and $\big(\Omega, \mathcal{A}(x,t,D), \mathcal{B}(x,D)\big)$ is an elliptic boundary value problem of class $C^0$ as described in Example 2.9(d), and $m \in C(\overline{\Omega} \times \mathbb{R})$. Moreover, we assume that the coefficients of $\mathcal{A}(x,t,D)$ and $m$ depend $T$-periodically on $t \in \mathbb{R}$.

We are then interested in the existence of a positive solution $\varphi$ of (14.1) and the behaviour of $\mu(\lambda)$ in dependence of the real parameter $\lambda$. We call $\mu(\lambda)$ the *principal eigenvalue* and $\varphi$ the *principal eigenfunction* corresponding to (14.1). By a principal eigenvalue one means an eigenvalue with positive eigenfunction.

These kind of eigenvalue problems were considered by Lazer [**89**], Beltramo and Hess [**21**], Beltramo [**20**] and Hess [**67**]. They are intimately related to stability properties of nonlinear reaction-diffusion equations as illustrated in [**67**] or [**89**] (see Section 22.C and 24.C).

As described in Example 2.9(d) we can take the $L_p(\Omega)$-realization of the problem. Then (14.1) takes the form

$$\begin{cases} \dot{\varphi} + A(t)\varphi - \lambda M(t)\varphi = \mu(\lambda)\varphi & \text{for } t \in \mathbb{R} \\ \varphi(0) = \varphi(T) \end{cases} \tag{14.2}$$

in $X_0 := L_p(\Omega)$, where $M(t)$ is the multiplication operator induced by $m(\cdot\,,t)$ on $L_p(\Omega)$. The problem is equivalent to the existence of an eigenvalue $\gamma(\lambda)$ of $U_\lambda(T,0)$ with positive eigenvector, where $U_\lambda(\cdot\,,\cdot)$ is the evolution operator to the family $\big(A(t) - \lambda M(t)\big)_{t\in\mathbb{R}}$ (see Section 6). As it is easily seen, the relationship between $\mu(\lambda)$ and $\gamma(\lambda)$ and the corresponding eigenvectors $\varphi(\cdot)$ and $\varphi_0$, respectively is given by

$$\mu(\lambda) = -T^{-1}\log\gamma(\lambda) \qquad \text{and} \qquad \varphi(t) = e^{\mu(\lambda)t}U(t,0)\varphi_0 \tag{14.3}$$

respectively.

As shown in Corollary 13.6, $U_\lambda(T,0)$ is irreducible. Since it is also compact, by Theorem 12.3 there exists a unique eigenvalue with positive eigenfunction and thus (14.2) has a unique solution. Moreover, this eigenvalue is the spectral radius of $U_\lambda(T,0)$.

It turns out, that $\mathrm{r}\big(U_\lambda(T,0)\big)$ is a log-convex function of $\lambda$ and by (14.3) $\mu(\cdot)$ a concave function. As already seen in Section 6 the stability of the zero solution of (14.2) is determined by whether $\mathrm{r}\big(U_\lambda(T,0)\big)$ is smaller or greater than one. Consequently, it also determined by the sign of $\mu(\lambda)$. By the concavity of $\mu(\cdot)$ (or the convexity of $\lambda \mapsto \mathrm{r}\big(U_\lambda(T,0)\big)$) there exist at most two values of $\lambda$, where the zero solution changes its stability properties as $\lambda$ crosses them.

We introduce here a general method to prove log-convexity of $\mathrm{r}(U_\lambda(T,0)$ as a function of $\lambda$. In particular, we do not require that $\mathrm{r}\big(U_\lambda(T,0)\big)$ is an eigenvalue.

In a first subsection we introduce the notion of log-convexity for vector- and operator-valued functions, called superconvexity, and prove a few properties of such functions. Of particular interest, is the fact that superconvexity of an operator-valued function implies that the spectral radius is a log-convex function of the parameter. This result was obtained by Kingman [**78**] for matrices and extended by Kato [**77**] to operators on Banach spaces. In the next two subsections we apply these results to semigroups and evolution operators. As an illustration the final subsection is devoted to a complete proof of the log-convexity of $\gamma(\lambda)$ defined above.

**B. Superconvex vector- and operator-valued functions:** In the sequel we assume that $E$ is a Banach lattice (see Section 11) and that $\Lambda$ is a nontrivial interval in the real axis.

By a real-valued log-convex function on $\Lambda$ we mean a function $\varphi\colon \Lambda \to \mathbb{R}$ such that either $\varphi \equiv 0$ or $\varphi > 0$ and $\log \varphi$ is convex. The set of *log*-convex functions has many nice properties: Linear combinations with nonnegative scalars, products and pointwise limits of log-convex functions are again log-convex. Moreover, any positive power of a log-convex function is log-convex.

After giving the definition of vector-valued superconvex functions, we show, that this class of functions is also closed under various operations.

**14.1 Definition**

(*i*) A function $u\colon \Lambda \to E$ is said to be *superconvex*, if for any $\varepsilon > 0$ and any triple $\zeta_1 < \zeta_0 < \zeta_2$ in $\Lambda$ there exists a finite number of $x_1, \dots, x_m \in E_+$ and log-convex real-valued functions $\varphi_1, \dots, \varphi_m$ defined on $\Lambda$, such that

$$\Big\| u(\zeta_k) - \sum_{j=1}^{m} \varphi_j(\zeta_k) x_j \Big\| \leq \varepsilon \qquad (k = 0, 1, 2) \tag{14.4}$$

holds.

(*ii*) An operator valued function $T(\cdot)\colon \Lambda \to \mathcal{L}(E)$ is called *superconvex* if $T(\cdot)x\colon \Lambda \to E$ is superconvex for all $x \in E_+$. □

In the next remark we list some simple properties of superconvex functions which follow immediately from the above definition.

**14.2 Remarks**

(a) It is not hard to see that a function $u\colon \Lambda \to \mathbb{R}^n$ is superconvex if and only if every component function is log-convex. Similarly, it is easy to see that an operator valued mapping $T(\cdot)\colon \Lambda \to \mathcal{L}(\mathbb{R}^n)$ is superconvex if and only if every matrix element of the representation matrix of $T(\lambda)$ is a log-convex function of $\lambda$.

(b) From the definition it follows that every superconvex function is positive, since by (14.4) $u(\zeta)$ is for every $\zeta \in \Lambda$ approximated by elements of $E_+$. Also for every superconvex operator-valued function $T(\cdot)$ we have by definition that $T(\zeta)$ is positive for all $\zeta \in \Lambda$. □

In the next lemma we prove further properties of superconvex functions:

**14.3 Lemma**

(*a*) *Let* $u\colon\Lambda\to E$ *be superconvex. Then for any* $x'\in E'_+$ *the function* $<x',u(\cdot)>\colon\Lambda\to\mathbb{R}$ *is* log-*convex.*

(*b*) *If* $u$ *and* $v\colon\Lambda\to E$ *are superconvex, so is* $\lambda u+\mu v$ *for any* $\lambda,\mu\geq 0$.

(*c*) *If* $u_n\colon\Lambda\to E$ *is superconvex for all* $n\in\mathbb{N}$ *and* $u(\zeta):=\lim_{n\to\infty}u_n(\zeta)$ *exists for every* $\zeta$ *in* $\Lambda$, *then* $u(\cdot)$ *is superconvex.*

**Proof**

(*a*) Fix $\zeta_1<\zeta_2$ in $\Lambda$ and set $\zeta_0:=\vartheta\zeta_1+(1-\vartheta)\zeta_2$ for any $\vartheta\in(0,1)$. To prove (*a*) we have to show that for any $x'\in E'_+$

$$<x',u(\zeta_0)>\;\leq\;<x',u(\zeta_1)>^{\vartheta}<x',u(\zeta_2)>^{1-\vartheta}. \tag{14.5}$$

Let $\varepsilon>0$ be arbitrary. By definition, we find $x_1,\dots,x_m$ in $E_+$ and log-convex functions $\varphi_1,\dots,\varphi_m$ such that (14.4) holds. Consequently, we get

$$\Big|<x',u(\zeta_k)>-\sum_{j=1}^{m}\varphi_j(\zeta_k)<x',x_j>\Big|\leq\varepsilon\|x'\|\qquad(k=0,1,2). \tag{14.6}$$

As a linear combination with positive scalars, $\sum_{j=1}^{m}\varphi_j(\cdot)<x',x_j>$ is log-convex. Hence, we see, that (14.5) holds up to an error of order $\varepsilon$. Letting $\varepsilon$ to zero, we obtain the assertion.

(*b*) is obvious from the definition.

(*c*) Fix $\zeta_1<\zeta_0<\zeta_2$ in $\Lambda$ and $\varepsilon>0$. Then choose $n$ so large that $\|u_n(\zeta_k)-u(\zeta_k)\|\leq\varepsilon$ holds for $k=0,1,2$. Then, by definition, there exist $x_1,\dots,x_m$ in $E_+$ and log-convex functions $\varphi_1,\dots,\varphi_m$ such that (14.4) holds. If we replace in (14.4) $u$ by $u_n$, the same estimate holds with $\varepsilon$ replaced by $2\varepsilon$. This proves assertion (*c*). □

Clearly, the product of log-convex functions is again log-convex. A similar result holds in the vector valued case:

**14.4 Lemma**

*Let* $u\colon\Lambda\to E$ *and* $T(\cdot)\colon\Lambda\to\mathcal{L}(E)$ *be superconvex. Then* $\zeta\mapsto T(\zeta)u(\zeta)$ *is superconvex.*

**Proof**

Let $\zeta_1<\zeta_0<\zeta_2$ in $\Lambda$ and $\varepsilon>0$ be arbitrary. By definition there exist $x_1,\dots,x_m$ in $E_+$ and log-convex functions $\varphi_1,\dots,\varphi_m$, such that (14.4) holds with $\varepsilon$ replaced by $\varepsilon/2M$, where $M:=\sup_{k=0,1,2}\|T(\zeta_k)\|$. Then, applying $T(\zeta_k)$ we obtain

$$\Big\|T(\zeta_k)u(\zeta_k)-\sum_{j=1}^{m}\varphi_j(\zeta_k)T(\zeta_k)x_j\Big\|\leq\frac{\varepsilon}{2}\qquad(k=0,1,2). \tag{14.7}$$

Since by definition $T(\cdot)x_j$ is superconvex for all $j = 1, \ldots, m$, we can choose $x_{j1}, \ldots, x_{jm_j}$ in $E_+$ and log-convex functions $\varphi_{j1}, \ldots, \varphi_{jm_j}$ such that

$$\Big\|T(\zeta_k)x_j - \sum_{i=1}^{m_j} \varphi_{ji}(\zeta_k)x_{ji}\Big\| \leq \frac{\varepsilon}{2mN} \qquad (k = 0, 1, 2), \tag{14.8}$$

where $N := \max_{\{j,k\}} \varphi_j(\zeta_k)$. Putting (14.7) and (14.8) together, we get

$$\Big\|T(\zeta_k)u(\zeta_k) - \sum_{j=1}^{m}\sum_{i=1}^{m_j} \varphi_j(\zeta_k)\varphi_{ji}(\zeta_k)x_{ji}\Big\| \leq \varepsilon \qquad (k = 0, 1, 2). \tag{14.9}$$

As noted at the beginning of this subsection, the product of log-convex functions is log-convex and thus (14.9) proves the lemma. □

As a consequence of the two lemmas, we obtain the following corollary on operator-valued superconvex functions.

**14.5 Corollary**

*Let $u_n : \Lambda \to E$ and $T_n : T_n(\cdot) \to \mathcal{L}(E)$ be superconvex for all $n \in \mathbb{N}$. Then*

(*a*) *$\zeta \mapsto \langle x', T_1(\zeta)x\rangle$ is* log-*convex for all $x$ in $E_+$ and $x'$ in $E'_+$.*

(*b*) *If $\mu_1$, $\mu_2 \geq 0$, then $\mu_1 T_1(\cdot) + \mu_2 T(\cdot)$ and $\zeta \mapsto T_1(\zeta)T_2(\zeta)$ are superconvex.*

(*c*) *If $T(\zeta) = \text{s-lim}_{n\to\infty} T_n(\zeta)$ exists for every $\zeta \in \Lambda$, then $T(\cdot)$ is superconvex.*

(*d*) *Suppose that $t > 0$ and $T : \Lambda \times [0, T] \to \mathcal{L}(E)$ such that $T(\cdot\,, \tau)$ is superconvex for every $\tau \in [0, t]$ and that $T(\zeta, \cdot)$ is a continuous functions from $[0, t]$ into $\mathcal{L}_s(E)$ for all $\zeta \in \Lambda$. Then the mapping*

$$\Lambda \to \mathcal{L}(E), \zeta \mapsto \int_0^t T(\zeta, \tau)\, d\tau$$

*is superconvex.*

**Proof**

(*a*)–(*c*) follow directly from Lemma 14.3 and 14.4. Assertion (*d*) is a consequence of these facts and the fact that we may approximate the integral by Riemann sums. □

After these preparations it is easy to prove the following theorem due to Kingman and Kato.

**14.6 Theorem**

*If $T(\cdot) : \Lambda \to \mathcal{L}(E)$ is superconvex, so is $\mathrm{r}\big(T(\cdot)\big)$.*

**Proof**
By Corollary 14.5($b$) the function $T^n(\cdot)$ is superconvex for any $n \in \mathbb{N}$ and by ($a$) of the same corollary, $<x', T^n(\cdot)x>$ is log-convex. Since the supremum and any positive power of log-convex functions is again log-convex, the assertion follows from Lemma 12.2. $\square$

**C. Superconvexity and semigroups:** Let $\Lambda$ and $E$ be as in the above subsection. Suppose that $\big(A(\zeta)\big)_{\zeta\in\Lambda}$ is a family of linear operators such that $-A(\zeta)$ is the generator of a $C_0$-semigroup on $E$ for each $\zeta \in \Lambda$. Moreover, assume that there exists a $\omega_0 \in \mathbb{R}$ such that

$$\varrho\big(-A(\zeta)\big) \subset [\operatorname{Re}\mu \geq \omega_0] \tag{14.10}$$

for all $\zeta \in \Lambda$. If in addition, there exists a $\lambda_0 \geq \omega_0$, such that the map

$$\zeta \mapsto \big(\lambda + A(\zeta)\big)^{-1}$$

is superconvex for all fixed $\lambda \geq \lambda_0$, the family $\big(-A(\zeta)\big)_{\zeta\in\Lambda}$ is called *resolvent superconvex.* Such a family is called *semigroup superconvex*, if the map

$$\zeta \mapsto e^{-tA(\zeta)}$$

is superconvex for all $t \geq 0$.

From (1.5), (1.6) and Corollary 14.5, it follows immediately that

$$\begin{aligned} &\big(-A(\zeta)\big)_{\zeta\in\Lambda} \text{ is resolvent superconvex} \\ &\qquad \iff \big(-A(\zeta)\big)_{\zeta\in\Lambda} \text{ is semigroup superconvex.} \end{aligned} \tag{14.11}$$

We proceed now by proving a perturbation theorem for resolvent superconvex families. This theorem allows us to construct nontrivial resolvent superconvex families and is the key to solving the problem posed in Subsection A.

**14.7 Theorem**
*Let $\big(-A(\zeta)\big)_{\zeta\in\Lambda}$ and $\big(-B(\zeta)\big)_{\zeta\in\Lambda}$ be resolvent superconvex families with $\|B(\zeta)\|$ bounded for all $\zeta \in \Lambda$. Then the family $\big(-A(\zeta) - B(\zeta)\big)_{\zeta\in\Lambda}$ is resolvent superconvex.*
**Proof**
Let $\zeta \in \Lambda$ be fixed. Then by a perturbation theorem for $C_0$-semigroups (see e.g. [**100**], Theorem 3.1.1), we have that $-\big(A(\zeta) + B(\zeta)\big)$ is the generator of a $C_0$-semigroup. The corresponding semigroup can be represented by the following product formula (see e.g. [**100**], Corollary 3.5.5):

$$e^{-t(A(\zeta)+B(\zeta))} = \operatorname*{s-lim}_{n\to\infty}\big(e^{-\frac{t}{n}A(\zeta)}e^{-\frac{t}{n}B(\zeta)}\big)^n. \tag{14.12}$$

(The conditions assuring that (14.12) holds are obtained by using a suitable equivalent norm on $E$.) From (14.11), Corollary 14.5 as well as (14.12) we obtain the assertion of the theorem. $\square$

**D. Superconvexity and the evolution operator:** Assume that $X_0$ and $X_1$ are Banach spaces satisfying $X_1 \overset{d}{\hookrightarrow} X_0$. Moreover, assume that $X_0$ is a Banach lattice.

Let $\big(A(\zeta,t)\big)$ be a family of closed linear operators on $X_0$ satisfying $(\tilde{A}1)$–$(\tilde{A}3)$ of Section 10. Suppose in addition, that the family $\big(A(\zeta,t)\big)_{\zeta\in\Lambda}$ is resolvent superconvex for each fixed $t \in [0,T]$. Finally, let $U_\zeta(\cdot\,,\cdot)$ be the evolution operator corresponding to the family $\big(A(\zeta,t)\big)_{0\le t\le T}$. Then, the following holds:

### 14.8 Theorem

*Let the assumptions given above be satisfied. Then, for each $(t,s) \in \Delta_T$, the map*

$$\zeta \mapsto U_\zeta(t,s),\ \Lambda \to \mathcal{L}(X_0)$$

*is superconvex.*

**Proof**

In view of Theorem 10.7 and Corollary 14.5 it is sufficient to prove the assertion for the $n$-th Yosida approximations $U_{\zeta,n}(\cdot\,,\cdot)$ for large $n \in \mathbb{N}$.

Note that (14.11), Corollary 14.5, the assumption as well as the representation formula (1.5) and (1.14) imply that $n^2\big(n + A(\cdot\,,t)\big)^{-1}$ is superconvex for $n \ge 1$.

Solving equation (10.6) by means of (9.29) and (9.30), it is readily seen from Corollary 14.5, that $\zeta \mapsto U_{\zeta,n}(t,s)$ is superconvex for any $n \in \mathbb{N}^*$ and $(t,s) \in \Delta_T$ and the assertion of the Theorem follows. $\square$

The following corollary is an easy consequence of the above theorem and Theorem 14.6.

### 14.9 Corollary

*Let the same hypotheses of the preceding theorem be satisfied. Then the function*

$$\zeta \mapsto \mathrm{r}\big(U_\zeta(t,s)\big)$$

*is* log-*convex for each $(t,s) \in \Delta_T$.*

We resume now the study of the periodic-parabolic eigenvalue problem started in Subsection A to illustrate how the abstract results might be used in a concrete situation.

**E. Application: Periodic-parabolic eigenvalue problems:** In this subsection, we use the same notations and hypotheses as in Subsection A.

**14.10 Proposition**

*Let $m \in L_\infty(\Omega)$ and $M$ the corresponding multiplication operator $u \mapsto mu$ on $L_p(\Omega)$. Then the function*

$$\mathbb{R} \to \mathcal{L}\big(L_p(\Omega)\big),\ \lambda \mapsto e^{\lambda M}$$

*is superconvex.*

**Proof**

Fix $u \in L_p(\Omega)$ and let $\lambda_1 < \lambda_0 < \lambda_2$ and $\varepsilon > 0$ be arbitrary. Approximate $m$ by a simple function

$$\tilde{m} := \sum_{j=0}^{l} m_j \chi_{\Omega_j} \tag{14.13}$$

such that

$$\|e^{\lambda_k m} - e^{\lambda_k \tilde{m}}\|_\infty \leq \frac{\varepsilon}{\|u\|_p} \qquad (k = 0, 1, 2). \tag{14.14}$$

Here, the $\Omega_j$ are disjoint measurable subsets of $\Omega$ such that $\Omega = \bigcup_{j=1}^{l} \Omega_j$ and $\chi_{\Omega_j}$ the function on $\Omega$ which has the value 1 on $\Omega_j$ and zero otherwise, and $m_j$ are just real numbers.

Therefore, we get that

(14.15)
$$\|e^{\lambda_k m} u - \sum_{j=1}^{l} e^{\lambda_k m_j} \chi_{\Omega_j} u\|_p^p = \sum_{j=1}^{l} \int_{\Omega_j} |e^{\lambda_k m(x)} - e^{\lambda_k m_j}|^p |u(x)|^p \, dx \leq \varepsilon^p \qquad (k = 0, 1, 2).$$

holds, which completes the proof of the proposition. □

**14.11 Remark**

Proposition 14.10 holds also if we consider the multiplication operator induced on $C(\overline{\Omega})$ or $C_0(\mathbb{R}^n)$ by a bounded continuous function $m$. The approximation (14.13) is then replaced using an appropriate smooth partition of unity on $\Omega$ and $\mathbb{R}^n$ respectively. □

**14.12 Theorem**

*The evolution operator $U_\lambda(t, s)$ corresponding to the family $\big(A(t) - \lambda M(t)\big)_{0 \leq t \leq T}$ depends superconvex on $\lambda \in \mathbb{R}$.*

**Proof**
By the above proposition and (14.11) it is clear that the family of bounded operators $\big(\lambda M(t)\big)_{\lambda\in\mathbb{R}}$ is resolvent superconvex for each $t \in \mathbb{R}$. By our perturbation Theorem 14.7 if follows readily that this is also true for $\big(A(t) - \lambda M(t)\big)_{\lambda\in\mathbb{R}}$. The assertion of the theorem is now a consequence of Theorem 14.8. □

The result on the principal eigenvalue in which we were originally interested now follows from Theorem 14.6 and (14.3):

**14.13 Corollary**
*The function* $\gamma(\lambda) = \mathrm{r}\big(U_\lambda(T,0)\big)$ *is a* log-*convex function of* $\lambda \in \mathbb{R}$. *Equivalently,* $\mu(\cdot)$ *is a concave function.*

Finally, we remark that it is by no means necessary to consider a periodic problem to prove Theorem 14.12 and the first part of Corollary 14.13.

**Notes and references:** The material in Subsections B and C is essentially taken from Kato [**77**]. Theorem 14.6 was proved by Kingman [**78**] for operators acting on finite dimensional spaces. The results in Subsection D are taken from Daners and Koch Medina [**38**].

The notion of periodic-parabolic eigenvalues was introduced by Lazer in [**89**] and extensively studied by Beltramo and Hess [**21**], Beltramo [**20**], and Hess [**67**] for periodic-parabolic equations in bounded domains. For problems on $\mathbb{R}^n$ see Daners and Koch Medina [**38**] and Koch Medina and Schätti [**80**].

Principal eigenvalues in the elliptic case are studied for example in Amann [**5**], Hess and Kato [**68**] or Nussbaum [**99**].

# IV. Semilinear Evolution Equations of Parabolic Type

In this chapter we prove local and global existence of solutions of semilinear evolution equations, as well as continuous dependence on the initial data by exploiting the estimates for the evolution operator which we obtained in the first chapter. We shall also establish a theorem on parameter dependent problems.

## 15. Mild Solutions of Semilinear Equations

Let $X_0$ and $X_1$ be Banach spaces with $X_1 \overset{d}{\hookrightarrow} X_0$, and let $\{A(t); t \in [0,T]\}$, $T > 0$, be a family of closed linear operators in $X_0$ satisfying conditions $(A1)$–$(A3)$ of Section 2. In particular we have: $X_1 \doteq D(A(t))$, for all $t \in [0,T]$. The evolution operator associated to this family will be denoted by $U(\cdot\,,\cdot)$. Throughout this chapter the notation of Chapter I will be freely used.

**A. Basic definitions:** We fix a number $\alpha \in [0,1]$ and consider the following *semilinear initial value problem*

$$(15.1) \qquad \begin{cases} \partial_t u + A(t)u = g(t,u) \text{ for } t \in (s,T] \\ u(s) = x, \end{cases}$$

where $(s,x) \in [0,T) \times X_\alpha$, and the *nonlinearity* $g$ satisfies one of the following two sets of assumptions:

$(G0)$ $g \in C^{0,1-}([0,T] \times X_\alpha, X_\gamma)$ uniformly in $[0,T]$ for some $\gamma \in (0,1]$, i.e. to every $\rho > 0$, there exists a $\kappa(\rho) > 0$, such that

$$\|g(t,x) - g(t,y)\|_\gamma \le \kappa(\rho)\|x - y\|_\alpha,$$

for all $t, s \in [0,T]$, and all $x, y \in X_\alpha$ with $\|x\|_\alpha, \|y\|_\alpha \le \rho$.

$(G0')$ $\alpha < 1$ and $g \in C^{0,1-}([0,T] \times X_\alpha, X_0)$ uniformly in $[0,T]$, i.e. to every $\rho > 0$, there exists a $\kappa(\rho) > 0$, such that

$$\|g(t,x) - g(t,y)\| \le \kappa(\rho)\|x - y\|_\alpha,$$

for all $t, s \in [0,T]$, and all $x, y \in X_\alpha$ with $\|x\|_\alpha, \|y\|_\alpha \le \rho$.

These will be our minimal requirements on our nonlinearities. Note that we allow $g$ to be defined on $X_1$ only at the cost of imposing more regularity on its range. Assumption $(G0)$ will prove to be strong enough to obtain classical solutions to the above initial value problem. Assumption $(G0')$, however, will only yield what are usually called mild solutions. Thus, if we are interested in classical solutions when the range of $g$ is not regular enough, we are forced to consider nonlinearities satisfying the following stronger condition:

$(G1)$ $\alpha < 1$ and $g \in C([0,T] \times X_\alpha, X_0)$, and there exist $\nu \in (0,1)$ and an increasing function $\iota: \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $\rho > 0$,

$$\|g(t,x) - g(s,y)\| \leq \iota(\rho)(|t-s|^\nu + \|x-y\|_\alpha),$$

holds for all $t, s \in [0,T]$ and all $x, y \in X_\alpha$ with $\|x\|_\alpha, \|y\|_\alpha \leq \rho$.

A few definitions are in order:

**15.1 Definition**
A (*classical local*) *solution* of (15.1) (*on* $I$), is a function

$$u \in C(I, X_\alpha) \cap C^1(\dot{I}, X_0),$$

where $I$ is any non-trivial subinterval of $[s,T]$ containing $s$, such that $u(t) \in X_1$ for all $t \in \dot{I}$, satisfying $u(s) = x$ and $\partial_t u(t) + A(t)u(t) = g(t, u(t))$ for all $t \in \dot{I}$. Here we have used the notation $\dot{I} := I \setminus \{s\}$. The solution is called *maximal* if it cannot be extended to a solution on a strictly larger subinterval of $[s,T]$, and *global* if $I = [s,T]$. □

Let $u: I \to X_0$ be a solution of (15.1), and set $f(t) := g(t, u(t))$ for $t \in I$. Now, $f$ obviously lies in $C(I, X_0)$ and $u$ solves the linear Cauchy-problem

$$\begin{cases} \partial_t u + A(t)u = f(t) & \text{for } t \in \dot{I} \\ u(s) = x, & \end{cases}$$

The variation-of-constants formula then implies that u satisfies the following integral equation on $I$:

$$u(t) = U(t,s)x + \int_s^t U(t,\tau) g(\tau, u(\tau))\, d\tau. \tag{15.2}$$

**15.2 Definition**
A *mild solution* (on $I$) is a function

$$u \in C(I, X_\alpha),$$

where I is any non-trivial subinterval of $[s, T]$ containing $s$, such that (15.2) is satisfied for all $t \in I$. The solution is called *maximal* if it cannot be extended to a solution on a strictly larger subinterval of $[s, T]$, and *global* if $I = [s, T]$. □

**B. When are mild solutions classic?** The discussion preceding Definition 15.2 shows that any classic solution of (15.1) is a mild solution. The next proposition shows that the converse is also true if we either require $(G0)$ or we are willing to impose some additional regularity on the nonlinearity $g$. This fact is crucial for the results of the next section.

**15.3 Proposition**

*Assume that either $(G0)$ or $(G1)$ holds, and let $(s, x) \in [0, T) \times X_\alpha$. Suppose that $I$ is a non-trivial subinterval of $[s, T]$ containing $s$. Furthermore, let $u$ be a mild solution of (15.1) on I. Then $u$ is a classical solution of (15.1) on $I$.*

**Proof**

Assume that $(G1)$ holds and let $s < r < \sup(I)$ and set $I_r := I \cap [r, T]$, as well as $x_r = u(r)$. By Lemma 5.3 and 5.5, $x_r \in X_\beta$ for every $\beta \in [0, 1)$. Furthermore we have for any $t \in I_r$ :

$$\begin{aligned} u(t) &= U(t,r)U(r,s)x + U(t,r)\int_s^r U(r,\tau)g(\tau, u(\tau))d\tau + \int_r^t U(t,\tau)g(\tau, u(\tau))d\tau \\ &= U(t,r)x_r + \int_r^t U(t,\tau)g(\tau, u(\tau))d\tau. \end{aligned}$$

By Corollary 5.6 it follows that $u \in C^{\beta-\alpha}(J, X_\alpha)$ for any closed subinterval $J$ of $I_r$ containing $r$. This implies that $[t \mapsto g(t, u(t))]$ is a Hölder-continuous function on $J$. Now we infer from Theorem 5.9 in case (1), that $u$ lies in $C^1(\dot{J}, X_0)$ and satisfies $\partial_t u(t) + A(t)u(t) = g(t, u(t))$ for $t \in \dot{J}$, for any closed subinterval $J$ of $I$ containing $r$. As $r \in (s, \sup(I))$ was arbitrarily chosen, it follows that $u \in C^1(\dot{I}, X_0)$, and $u$ is a solution of (15.1) on $I$, proving the theorem in case that $(G1)$ holds.

If $(G0)$ holds the proof is very similar but instead of invoking case (1) of Theorem 5.9 we have to invoke case (2). □

**15.4 Remarks**

(a) It is also possible to treat nonlinearities which are not defined on the whole of $[0, T] \times X_\alpha$ but only on $[0, T] \times D$, where $D$ is an open subset of $X_\alpha$. This does not present any new conceptual difficulties but only makes things technically slightly more involved, obscuring the essence of the arguments. For this reason we have refrained from treating this more general case. However, the reader should have no difficulties supplying the details in order to deal with this problem.

(b) The preceding result shows that, if $(G0)$ or $(G1)$ holds, any mild solution is also a classical solution. Note that this result is not available in the context of fractional power spaces. What one can show in these spaces is that if $x \in X_\beta$ for $\beta \in (\alpha, 1]$, then a mild solution of the semilinear initial value problem is also a classic solution (see Lemma 3.2 in [**7**]). This difference is one of the major advantages of the approach via interpolation spaces over the one via fractional powers since it seems only natural that one should be able to take initial values in the whole 'phase space' $X_\alpha$ and not only in a smaller subspace $X_\beta$, $\alpha < \beta \leq 1$ . □

**C. Continuity and differentiability of Nemitskii-operators on Hölder spaces:** Before being able to formulate semilinear parabolic equations on subdomains of $\mathbb{R}^n$ as abstract evolution equations on a suitable function space, we are forced to spend some time investigating maps between function spaces which are induced by composition with a fixed function. In order to justify the sacrifice of working his way through this rather technical material the reader might want to throw a glance at Subsections D and E.

We assume that $n$ , $m$ and $k$ are positive natural numbers and that $\Omega$ is an open subset of $\mathbb{R}^n$. Observe that we do not require that $\Omega$ be bounded. Consider a fixed function

$$f\colon \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^k,$$

and define for any $u\colon \overline{\Omega} \to \mathbb{R}^m$ a new function $g_f(u)\colon \overline{\Omega} \to \mathbb{R}^k$ by setting

$$g_f(u)(x) := f(x, u(x))$$

for $x \in \overline{\Omega}$. The mappping $u \mapsto g_f(u)$ is called the *Nemitskii-operator* induced by $f$. Sometimes we shall refer to it as the *superposition-* or the *substitution-operator* induced by $f$.

Given two Banach spaces, $\mathcal{F}_1(\overline{\Omega}, \mathbb{R}^m)$ and $\mathcal{F}_2(\overline{\Omega}, \mathbb{R}^k)$, of functions from $\overline{\Omega}$ into $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively, the following question arises: Under what conditions on $f$ do we have $g_f\colon \mathcal{F}_1(\overline{\Omega}, \mathbb{R}^m) \to \mathcal{F}_2(\overline{\Omega}, \mathbb{R}^k)$ and what are the continuity or differentiability properties of this mapping. In this subsection we shall be concerned solely with continuity and differentiability of $g_f$ between spaces of Hölder type.

For notational convenience we set

$$t \cdot (1{-}) := t \qquad \text{and} \qquad s^{1-} := s$$

whenever $t \geq 0$ and $s > 0$. Moreover, $s \leq 1{-}$ is to be interpreted as $s \leq 1$.

**15.5 Remarks**

At this point we mention some facts which shall be used repeatedly below. In the sequel $\alpha$, $\beta$, $\mu$ and $\nu$ denote elements in the set $(0, 1) \cup \{1{-}\}$.

(a) Suppose that $u \in BUC^{\alpha}(\Omega, \mathbb{R}^k)$ is given. Then,

$$|u(x) - u(y)| \leq \|u\|_{\alpha}|x - y|^{\alpha}$$

holds for all $x, y \in \overline{\Omega}$. Moreover, by Proposition A1.1, we have that $\|u\|_{\beta} \leq \|u\|_{\alpha}$ whenever $\beta \leq \alpha$. In particular, we have that

$$|u(x) - u(y)| \leq \|u\|_{\alpha}|x - y|^{\beta}$$

holds for all $x, y \in \overline{\Omega}$.

(b) Suppose that $f \in C^{\mu,\nu}\big(\overline{\Omega} \times \mathbb{R}^m, \mathbb{R}^k\big)$, uniformly on subsets of the form $\overline{\Omega} \times \mathbb{B}(0, R)$, $R > 0$. This means that $f \in BUC^{\mu,\nu}\big(\Omega \times \mathbb{B}(0, R), \mathbb{R}^k\big)$ for each $R > 0$, which implies that the expressions

$$p(f, R) := \sup_{(x,\xi) \in \overline{\Omega} \times \overline{\mathbb{B}}(0,R)} |f(x, \xi)|,$$

and

$$q(f, \mu, \nu, R) := [f]_{\overline{\Omega}, \mathbb{B}(0,R), \mu, \nu}$$

are finite. We set

$$\kappa_0(f, \mu, \nu, R) := p(f, R) + q(f, \mu, \nu, R).$$

By definition of $p$ and $q$ we have

$$|f(x, \xi)| \leq \kappa_0(f, \mu, \nu, R) \tag{15.3}$$

for all $(x, \xi) \in \overline{\Omega} \times \overline{\mathbb{B}}(0, R)$, as well as

$$|f(x, \xi) - f(y, \zeta)| \leq \kappa_0(f, \mu, \nu, R)\big(|x - y|^{\nu} + |\xi - \zeta|^{\mu}\big) \tag{15.4}$$

for all $x, y \in \overline{\Omega}$ and $\xi, \zeta \in \overline{\mathbb{B}}(0, R)$. Using Remark (a) it is very easy to see that whenever $\alpha \leq \mu$ and $\beta \leq \nu$, we have that $\kappa_0(f, \alpha, \beta, R) \leq \kappa_0(f, \mu, \nu, R)$. In particular,

$$|f(x, \xi) - f(y, \zeta)| \leq \kappa_0(f, \mu, \nu, R)\big(|x - y|^{\alpha} + |\xi - \zeta|^{\beta}\big) \tag{15.5}$$

holds for all $x, y \in \overline{\Omega}$ and $\xi, \zeta \in (0, R)$.

Assume now that $f \in C^{\mu,1+\nu}(\overline{\Omega} \times \mathbb{R}^m, \mathbb{R}^k)$, uniformly on subsets of the form $\overline{\Omega} \times \mathbb{B}(0, R)$, $R > 0$. Hence, $f \in BUC^{\mu,1+\nu}\big(\Omega \times \mathbb{B}(0, R)\big)$ for each $R > 0$, which implies that

$$\kappa_1(f, \mu, \nu, R) := \kappa_0(f, \mu, \nu, R) + \kappa_0(\partial_2 f, \mu, \nu, R)$$

is finite. Since by definition $\kappa_0(f, \mu, \nu, R) \leq \kappa_1(f, \mu, \nu, R)$ holds, we obtain the estimates (15.3) and (15.4) also with $\kappa_1(f, \mu, \nu, R)$ instead of $\kappa_0(f, \mu, \nu, R)$. Moreover, we additionally have that

$$|\partial_2 f(x, \xi) - \partial_2 f(y, \zeta)| \leq \kappa_1(f, \mu, \nu, R)\big(|x - y|^{\nu} + |\xi - \zeta|^{\mu}\big) \tag{15.6}$$

holds for all $x, y \in \overline{\Omega}$ and $\xi, \zeta \in \overline{\mathbb{B}}(0, R)$. Again we have

$$\kappa_1(f, \alpha, \beta, R) \le \kappa_1(f, \mu, \nu, R),$$

whenever $\alpha \le \mu$ and $\beta \le \nu$ holds.

(c) Suppose that $f \in C^{\mu,\nu}(\overline{\Omega} \times \mathbb{R}^m, \mathbb{R}^k)$, uniformly on subsets of the form $\overline{\Omega} \times \mathbb{B}(0, R)$, $R > 0$. Then, it is easy to see that

$$g_f \in C^\nu\big(BUC(\Omega, \mathbb{R}^m), BUC(\Omega, \mathbb{R}^k)\big)$$

uniformly on bounded sets. More precisely, we shall need that for each $R > 0$ we have that

$$\|g_f(u) - g_f(v)\|_\infty \le \kappa_0(f, \mu, \nu, R)\|u - v\|_\infty^\nu \tag{15.7}$$

holds for all $u, v \in BUC(\Omega, \mathbb{R}^k)$ with $\|u\|_\infty, \|v\|_\infty \le R$.

(d) Suppose that $f \in C^{\mu,1}(\overline{\Omega} \times \mathbb{R}^m, \mathbb{R}^k)$, uniformly on subsets of the form $\overline{\Omega} \times \mathbb{B}(0, R)$, $R > 0$. Then, an easy application of the Mean Value Theorem immediately yields that $f \in C^{\mu,1-}(\overline{\Omega} \times \mathbb{R}^m, \mathbb{R}^k)$ uniformly on subsets of the above form. □

We start with the so called *acting conditions* on Hölder spaces.

**15.6 Lemma**
*Let $\mu, \nu \in (0, 1) \cup \{1-\}$ and assume that $f \in C^{\mu,\nu}(\overline{\Omega} \times \mathbb{R}^m, \mathbb{R}^k)$, uniformly on subsets of the form $\overline{\Omega} \times \mathbb{B}(0, R)$, $R > 0$. Then, if $\alpha \in (0, 1) \cup \{1-\}$ is arbitrary,*

$$g_f\big(BUC^\alpha(\overline{\Omega}, \mathbb{R}^m)\big) \subset BUC^\beta(\Omega, \mathbb{R}^k)$$

*whenever $0 \le \beta \le \min\{\alpha\nu, \mu\}$. Moreover, $g_f$ maps bounded subsets of $BUC^\alpha(\Omega, \mathbb{R}^m)$ into bounded subsets of $BUC^\beta(\Omega, \mathbb{R}^k)$ and if $\|u\|_\alpha \le R$ for some $R > 0$, the image of $u$ under $g_f$ is bounded by a constant depending only on $R$ and $\kappa_1(f, \mu, \nu, R)$.*

**Proof**
Take any $u \in BUC^\alpha(\Omega, \mathbb{R}^m)$ and set $R := \|u\|_\infty$. If $0 \le \beta \le \min\{\alpha\nu, \mu\}$ then, in particular, $\beta \le \mu$ and $\beta/\alpha \le \nu$. Hence, using (15.5) and Remark 15.5(c), we obtain that

$$\begin{aligned} |f\big(x, u(x)\big) - f\big(y, u(y)\big)| &\le \kappa_0(f, \mu, \nu, R)\big(|x - y|^\beta + |u(x) - u(y)|^{\beta/\alpha}\big) \\ &\le \kappa_0(f, \mu, \nu, R)\big(|x - y|^\beta + \|u\|_\alpha^{\beta/\alpha}|x - y|^\beta\big) \\ &= \kappa_0(f, \mu, \nu, R)\big(1 + \|u\|_\alpha^{\beta/\alpha}\big)|x - y|^\beta, \end{aligned}$$

holds for all $x, y \in \overline{\Omega}$. From this the assertion follows. □

The above lemma gives only a sufficient condition for the image of $BUC^{\alpha}(\Omega,\mathbb{R}^m)$ to be contained in $BUC^{\beta}(\Omega,\mathbb{R}^k)$, but says nothing whatsoever about the continuity properties of $g_f$ as a mapping between these two spaces. Indeed, this turns out to be a rather delicate question for the critical value $\beta = \min\{\alpha\nu,\mu\}$. For counterexamples under the assumptions of our lemma consult Section 7.3 in [**19**] or Section III.26 in [**67**]. In regard to continuity we shall be content with the following result.

**15.7 Lemma**
*Let $\mu,\nu \in (0,1)\cup\{1-\}$ and assume that $f \in C^{\mu,\nu}(\overline{\Omega}\times\mathbb{R}^m,\mathbb{R}^k)$, uniformly on subsets of the form $\overline{\Omega}\times\mathbb{B}(0,R)$, $R>0$. Then, for each $\alpha \in (0,1)\cup\{1-\}$, there exists a $\sigma \in (0,1)$ such that*

$$g_f \in C^{\sigma}\big(BUC^{\alpha}(\Omega,\mathbb{R}^m), BUC^{\beta}(\Omega,\mathbb{R}^k)\big),$$

*holds, uniformly on bounded sets, whenever $0 \le \beta < \min\{\alpha\nu,\mu\}$.*

**Proof**

(*i*) From Remark 15.5(c) and the imbedding $BUC^{\alpha}(\Omega,\mathbb{R}^m) \hookrightarrow BUC(\Omega,\mathbb{R}^m)$ we conclude that

$$g_f \in C^{\nu}\big(BUC^{\alpha}(\Omega,\mathbb{R}^m), BUC(\Omega,\mathbb{R}^k)\big),$$

holds, uniformly on bounded sets.

(*ii*) Now, let $0<\beta<\gamma<\min\{\alpha\nu,\mu\}$ and $R>0$. For any $u,v \in BUC^{\alpha}(\Omega,\mathbb{R}^m)$ satisfying $\|u\|_{\alpha},\|v\|_{\alpha}<R$ we obtain that

$$\begin{aligned}\|g_f(u)-g_f(v)\|_{\beta} &\le \|g_f(u)-g_f(v)\|_{\gamma}^{1-\beta/\gamma}\|g_f(u)-g_f(v)\|_{\infty}^{\beta/\gamma}\\ &\le c(R)\|u-v\|_{\alpha}^{\nu\beta/\gamma},\end{aligned}$$

Here we used the interpolation inequality $(A1.1)$, part $(i)$ as well as Lemma 15.6. From Lemma 15.6 we also see that $c(R)$ depends only on $R$ and $\kappa_0(f,\mu,\nu,R)$. Setting $\sigma := \nu\beta/\gamma$, the assertion of the lemma follows. □

We are now ready to prove a differentiability result. In order that things do not loose transparency due to cumbersome notation we deal only with the case $k=1$. Of course, for other $k$'s we do it component wise.

**15.8 Proposition**
*Let $\mu,\nu \in (0,1)\cup\{1-\}$ and assume that $f \in C^{\mu,1+\nu}(\overline{\Omega}\times\mathbb{R}^m)$, uniformly on subsets of the form $\overline{\Omega}\times\mathbb{B}(0,R)$, $R>0$. Then, for any $\alpha \in (0,1)\cup\{1-\}$, we have that*

$$g_f \in C^1\big(BUC^{\alpha}(\Omega,\mathbb{R}^m), BUC^{\beta}(\Omega)\big)$$

*uniformly on bounded subsets of $BUC^{\alpha}(\Omega,\mathbb{R}^m)$, whenever $0\le\beta<\min\{\alpha\nu,\mu\}$. The derivative*

$$Dg_f(u_0) \in \mathcal{L}\big(BUC^{\alpha}(\Omega), BUC^{\beta}(\Omega)\big)$$

*is given by*

$$g_{\partial_2 f}(u_0)v := \sum_{i=1}^{m} \partial_{\xi_i} f(\cdot\,, u_0(\cdot))v(\cdot). \tag{15.8}$$

*for all* $v \in BUC^\alpha(\Omega)$*, where* $u_0 \in BUC^\alpha(\Omega)$ *is fixed. Moreover, to each* $R > 0$ *there exists a constant* $c(f,\mu,\nu,R) > 0$ *such that*

$$\|g_f(u) - g_f(v)\|_\beta \le c(f,\mu,\nu,R)\|u - v\|_\alpha$$

*holds for all* $u, v \in BUC^\alpha(\Omega,\mathbb{R}^m)$ *with* $\|u\|_\alpha, \|v\|_\alpha \le R$*. The constant* $c(f,\mu,\nu,R)$ *depends only on* $R$ *and* $\kappa_1(f,\mu,\nu,3R)$*.*

**Proof**

Let $0 \le \beta \le \min\{\alpha\nu,\mu\}$. By the previous lemma we have that $g_f$ maps $BUC^\alpha(\Omega,\mathbb{R}^m)$ into $BUC^\beta(\Omega)$.

($i$) First we take a closer look at the obvious candidate for the derivative of $g_f$. Observe that $\partial_2 f \in C^{\mu,\nu}(\overline{\Omega} \times \mathbb{R}^m, \mathbb{R}^m)$, uniformly on subsets of the form $\overline{\Omega} \times \mathbb{B}(0,R)$, $R > 0$. Applying the pevious lemma we see that

$$g_{\partial_2 f} \in C^\sigma\big(BUC^\alpha(\Omega,\mathbb{R}^m), BUC^\beta(\Omega,\mathbb{R}^m)\big),$$

uniformly on bounded sets for some suitable $\sigma \in (0,1)$. But since $BUC^\beta(\Omega)$ is a Banach algebra we immediately get that

$$g_{\partial_2 f} \in C^\sigma\big(BUC^\alpha(\Omega,\mathbb{R}^m), \mathcal{L}(BUC^\beta(\Omega,\mathbb{R}^m))\big), \tag{15.9}$$

uniformly on bounded sets, where for any $u_0 \in BUC^\alpha(\Omega,\mathbb{R}^m)$ the map $g_{\partial_2 f}(u_0)$ acts as a bounded operator on $BUC^\beta(\Omega,\mathbb{R}^m)$ as described by (15.8).

($ii$) Applying Remark 15.5(c) to $\partial_2 f$ and the Mean Value Theorem, we readily see that

$$g_f \in C^1\big(BUC(\Omega,\mathbb{R}^m), BUC(\Omega)\big)$$

uniformly on bounded sets, with $g_{\partial_2 f}$ as derivative acting as a linear operator as described in part ($i$). Hence, it follows from the imbedding $BUC^\alpha(\Omega,\mathbb{R}^m) \hookrightarrow BUC(\Omega,\mathbb{R}^m)$, that

$$g_f \in C^1\big(BUC^\alpha(\Omega,\mathbb{R}^m), BUC(\Omega)\big)$$

uniformly on bounded sets.

($iii$) Let $0 < \gamma < \min\{\alpha\nu,\mu\}$ and $R > 0$ be fixed. Suppose that $u, h \in BUC^\alpha(\Omega)$ such that $\|u\|_\alpha$, $\|h\|_\alpha \le R$. We show now that the difference quotient

$$\frac{\|g_f(u+h) - g_f(u) - g_{\partial_2 f}(u)h\|_\gamma}{\|h\|_\alpha} \tag{15.10}$$

remains bounded in $BUC^\gamma(\Omega)$ as $\|h\|_\alpha$ tends to zero. To do this observe that for any $x \in \Omega$

$$f\big(x, u(x) + h(x)\big) - f\big(x, u(x)\big) - \partial_2 f\big(x, u(x)\big)h(x) = F\big(x, u(x), h(x)\big)h(x),$$

where

$$F\big(x, u(x), h(x)\big) := \int_0^1 \partial_2 f\big(x, u(x) + \tau h(x)\big) - \partial_2 f\big(x, u(x)\big)\, d\tau.$$

Note that by Lemma 15.6 applied to $\partial_2 f$ the function $F\big(\cdot\,, u(\cdot), h(\cdot)\big)$ is bounded in $BUC^\gamma(\Omega)$ by a constant depending only on $R$ and $\kappa_1(f, \mu, \nu, 2R)$. Using that $BUC^\gamma$ is a Banach algebra, (15.8) as well as (15.5), it follows that

$$\|F\big(\cdot\,, u(\cdot), h(\cdot)\big)h\|_\gamma \le \|F\big(\cdot\,, u(\cdot), h(\cdot)\big)\|_\gamma\, \|h\|_\gamma \le \|F\big(\cdot\,, u(\cdot), h(\cdot)\big)\|_\gamma\, \|h\|_\alpha\,.$$

Hence the difference quotient (15.10) is uniformly bounded by a constant $C > 0$ depending only on $R$ and $\kappa_1(f, \mu, \nu, 2R)$

(*iv*) Suppose that $\beta < \gamma < \min\{\alpha\nu, \mu\}$. The interpolation inequality ($A1.1$) now yields

$$\begin{aligned} &\frac{\|g_f(u_0 + h) - g_f(u_0) - g_{\partial_2 f} h\|_\beta}{\|h\|_\alpha} \\ &\qquad \le \frac{\|g_f(u_0 + h) - g_f(u_0) - g_{\partial_2 f} h\|_\gamma^{1-\beta/\gamma}}{\|h\|_\alpha^{1-\beta/\gamma}}\; \frac{\|g_f(u_0 + h) - g_f(u_0) - g_{\partial_2 f} h\|_\infty^{\beta/\gamma}}{\|h\|_\alpha^{\beta/\gamma}}. \end{aligned}$$

By part (*iii*) the first factor is bounded, uniformly in $\|h\|_\alpha < R$ and by part (*i*) the second factor converges to zero as $h \to 0$. This gives the differentiability of $g_f$. The continuity of $\partial g_f$ follows from part (*ii*).

(*v*) Finally we have to show the Lipschitz continuity on bounded sets of $BUC^\alpha(\Omega)$. But this follows from the considerations in part (*ii*) and the Mean Value Theorem. □

We shall also have occasion to consider Nemitskii-operators which act not only on a function $u$ but also on its gradient $\nabla u$. Suppose now that

$$f\colon \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

is a given mapping. For any $u \in C^1(\overline{\Omega})$ we define

$$g_f(u)(x) := f\big(x, u(x), \nabla u(x)\big)$$

for each $x \in \overline{\Omega}$. The mapping $[u \mapsto g_f(u)]$ is also called the *Nemitskii-operator* induced by $f$. Since for any $\alpha \in [0, 1) \cup \{1-\}$ the linear operator

$$BUC^{1+\alpha}(\Omega) \to BUC^\alpha(\Omega, \mathbb{R} \times \mathbb{R}^n), \quad u \mapsto (u, \nabla u)$$

is bounded, we immediately obtain the following result from Proposition 15.8.

**15.9 Corollary**
*Let $\mu, \nu \in (0,1) \cup \{1-\}$ and assume that $f \in C^{\mu,1+\nu}(\overline{\Omega} \times \mathbb{R}^{1+n})$, uniformly on subsets of the form $\overline{\Omega} \times [-R, R] \times \mathbb{B}(0,R)$, $R > 0$. Then,*

$$g_f \in C^1\big(BUC^{1+\alpha}(\Omega), BUC^{\beta}(\Omega)\big)$$

*uniformly on bounded subsets of $BUC^{1+\alpha}(\Omega)$, whenever $0 \le \beta \le \min\{\alpha\nu, \mu\}$. More precisely, To each $R > 0$ there exists a constant $c(f, \mu, \nu, R) > 0$ such that*

$$\|g_f(u) - g_f(v)\|_{\beta} \le c(f, \mu, \nu, R)\|u - v\|_{1+\alpha}$$

*holds for all $u, v \in BUC^{1+\alpha}(\Omega, \mathbb{R}^m)$ with $\|u\|_{1+\alpha}, \|v\|_{1+\alpha} \le R$. Moreover, $c(f, \mu, \nu, R)$ depends only on $R$ and $\kappa_1(f, \mu, \nu, 3R)$.*

**D. Semilinear parabolic initial-boundary value problems:** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^\infty$, for some $n \ge 1$ and $\eta \in (0,1)$. Suppose that $\mathcal{A}(x,t,D)$ and $\mathcal{B}(x,D)$ are defined as in Example 2.9(d). Consider the following *semilinear parabolic initial-boundary value problem*:

$$(15.11)\qquad \begin{cases} \partial_t u(x,t) + \mathcal{A}(x,t,D)u(x,t) = f\big(x,t,u(t,x),\nabla u(x,t)\big) & (x,t) \in \Omega \times (0,T] \\ \mathcal{B}(x,D)u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where $u_0 \colon \Omega \to \mathbb{R}$ is the *initial value* and the *nonlinearity* $f : \overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given function satisfying

$$(15.12)\qquad \begin{cases} f \in C\big(\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^n\big) \text{ and } f(\cdot\,,\cdot\,,\xi,\zeta) \in C^{\eta,\frac{\eta}{2}}\big(\overline{\Omega} \times [0,T]\big) \text{ uniformly} \\ \text{for } (\xi,\zeta) \text{ on bounded subsets of } \mathbb{R} \times \mathbb{R}^n. \text{ Furthermore, we assume that} \\ \partial_\xi f,\ \partial_{\zeta_1} f, \ldots, \partial_{\zeta_n} f \text{ exist and are continuous on } \overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^n. \end{cases}$$

**15.10 Remark**
Let $R > 0$ be given. Applying the Mean Value Theorem as in Remark 15.5(d) we obtain that there exists a constant $\kappa(f,R) > 0$ such that

$$(15.13)\qquad |f(x,t,\xi,\zeta) - f(y,s,\bar{\xi},\bar{\zeta})| \le \kappa(f,R)\big(|x-y|^\eta + |t-s|^{\frac{\eta}{2}} + |\xi - \bar{\xi}| + |\zeta - \bar{\zeta}|\big)$$

holds for all $x, y \in \overline{\Omega}$, $t, s \in [0,T]$, $\xi, \bar{\xi} \in [-R, R]$ and $\zeta, \bar{\zeta} \in \overline{\mathbb{B}}(0,R)$. □

Now, for each $(t,u) \in [0,T] \times X_\alpha$ and $x \in \overline{\Omega}$ we may define the expression

$$(15.14)\qquad g_f(t,u)(x) := f\big(t,x,u(x),\nabla u(x)\big).$$

It is an easy exercise to show that (15.13) implies that

$$g_f \in C^{\frac{\eta}{2},1-}\big([0,T] \times C^1(\overline{\Omega}), C(\overline{\Omega})\big), \tag{15.15}$$

uniformly on bounded subsets of $[0,T] \times C^1(\overline{\Omega})$.

Let now $1 < p < \infty$ and set

$$X_0 := L_p(\Omega) \qquad \text{and} \qquad X_1 := W^2_{p,\mathcal{B}}(\Omega).$$

For any $\alpha \in (0,1)$ we define

$$X_\alpha := [X_0, X_1]_\alpha \qquad \text{or} \qquad (X_0, X_1)_{\alpha,p}\,.$$

Fix now $\frac{1}{2} + \frac{n}{2p} < \alpha \leq 1$. By Corollary 4.17 we have

$$X_\alpha \overset{d}{\hookrightarrow} C^{1+\nu}(\overline{\Omega}), \tag{15.16}$$

whenever $\nu > 0$ is sufficiently small. Therefore, (15.15) and (15.16) together with the imbedding $C(\overline{\Omega}) \hookrightarrow L_p(\Omega)$ yields the following Proposition.

**15.11 Proposition**
*If* (15.12) *is satisfied and* $\frac{1}{2} + \frac{n}{2p} < \alpha \leq 1$, *we have*

$$g_f \in C^{\frac{\eta}{2},1-}\big([0,T] \times X_\alpha, X_0\big)$$

*uniformly on bounded subsets, so that* $g_f$ *satisfies* $(G1)$.

If we require more regularity from $f$ we can show the continuous differentiability of $g_f$ in the second argument. This implies in particular that $(G0)$ holds. To this end we need the following assumption.

$$\left\{ \begin{array}{l} f \in C\big(\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^n\big) \text{ and } f(\cdot\,,\cdot\,,\xi,\zeta) \in C^{\eta,\frac{\eta}{2}}\big(\overline{\Omega} \times [0,T]\big) \text{ uniformly} \\ \text{for } (\xi,\zeta) \text{ on bounded subsets of } \mathbb{R} \times \mathbb{R}^n. \text{ Furthermore, we assume that} \\ \partial_\xi f, \partial_{\zeta_1} f, \ldots, \partial_{\zeta_n} f \text{ exist and are Hölder-continuous on } \overline{\Omega} \times [0,T] \times \mathbb{R} \times \\ \mathbb{R}^n. \end{array} \right. \tag{5.17}$$

**15.12 Remark**
For each $t \in [0,T]$ set

$$f_t(x,\xi,\zeta) := f(x,t,\xi,\zeta)$$

whenever $(x,\xi,\zeta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. By assumption (15.7) there exists a $\sigma \in (0,1)$ such that

$$f_t \in C^{\eta,1+\sigma}\big(\overline{\Omega} \times (\mathbb{R} \times \mathbb{R}^n)\big)$$

holds. Hence, for each $R > 0$ the expression $\kappa_1(f_t, \eta, \sigma, R)$ from Remark 15.5(b) is well defined. Moreover, (15.13) yields that for each $R > 0$

$$\kappa_1(f_t, \eta, \sigma, R) < M \tag{15.18}$$

holds for a constant $M > 0$ independent of $t \in [0, T]$.

Suppose that $0 < \nu < 1$ is such that (15.16) holds. Then Corollary 15.9 gives that for some small $0 < \gamma < \nu$

$$g(t, \cdot) \in C^1\big(BUC^{1+\nu}(\Omega), BUC^{\gamma}(\Omega)\big)$$

uniformly on bounded sets.

Choose now $0 < \mu < \gamma < \nu$ and $R > 0$. Then for any $u, v \in BUC^{1+\nu}(\Omega)$ satisfying $\|u\|_{1+\nu}, \|v\|_{1+\nu} < R$, and every $t \in [0, T]$ we get from the interpolation inequality (A1.1)

$$\|g(t,u) - g(s,v)\|_{\mu} \leq \|g(t,u) - g(s,v)\|_{\gamma}^{1-\mu/\gamma} \|g(t,u) - g(s,v)\|_{\infty}^{\mu/\gamma} \tag{15.19}$$

Since by Lemma 15.6 and by (15.18) the first factor is bounded independently of $t \in [0, T]$ and $\|u\|_{1+\nu}, \|v\|_{1+\nu} < R$, and since by Remark 15.10 the second factor tends to zero as $(t, u) \to (s, v)$, we obtain that

$$g_f \in C\big([0, T] \times BUC^{1+\nu}(\Omega), BUC^{\gamma}(\Omega)\big)$$

A similar argument gives

$$g_f \in C^{0,1}\big([0, T] \times BUC^{1+\nu}(\Omega), BUC^{\mu}(\Omega)\big) \tag{15.20}$$

holds, uniformly on bounded sets, whenever $0 < \mu < \nu < 1$ are small enough. □

We may now prove that under the stronger conditions (15.17), $g_f$ satisfies $(G0)$, and even more.

**15.13 Proposition**

*Let* (15.17) *hold and fix* $\frac{1}{2} + \frac{n}{2p} < \alpha \leq 1$. *Then,*

$$g_f \in C^{0,1}\big([0, T] \times X_{\alpha}, X_{\gamma}\big),$$

*uniformly on bounded subsets, whenever* $\gamma > 0$ *sufficiently small. Hence,* $g_f$ *satisfies* $(G0)$.

**Proof**
Let $0 < \mu < \nu < 1$ be so small that (15.16) and (15.20) hold. Since by Corollary 4.17 we have that

$$C^\mu(\overline{\Omega}) \hookrightarrow X_\gamma$$

is true for $\gamma > 0$ small enough, we obtain from (15.16) and (15.20) the assertion. □

**15.14 Remarks**

(a) Observe that any $f \in C^2\big([0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n\big)$ satisfies both (15.12) and (15.17).

(b) In case that $f$ does not depend explicitly on $\nabla u$, Proposition 15.11 and 15.13 remain valid whenever $\frac{n}{2p} < \alpha \leq 1$, as is easily seen.

(c) The assumptions made on $f$ are not optimal. Propositions 15.11 and 15.13 hold also under weaker regularity conditions on $f$. But since at a later stage we shall need to resort to the Schauder theory for linear parabolic equations (cf. [**87**]), we shall adhere to the present assumptions. □

Consider now for each $t \in [0,T]$ the $L_p$-realization of $\big(\Omega, \mathcal{A}(x,t,D), \mathcal{B}(x,D)\big)$, i.e. the operator

$$A(t)\colon D(A(t)) \subset X_0 \to X_0,$$

with $D(A(t)) = X_1$, as described in Section 1.D. With this notation (15.8) may be reformulated as an abstract semilinear initial value problem

$$\begin{cases} \partial_t u + A(t)u = g_f(t,u) \text{ for } t \in (s,T] \\ \qquad u(0) = u_0, \end{cases} \tag{15.21}$$

on $X_\alpha$, whenever $\frac{1}{2} + \frac{n}{2p} < \alpha \leq 1$ (or $\frac{n}{2p} < \alpha \leq 1$ if $f$ does not depend on $\nabla u$). We shall call (15.21) the *$L_p$-formulation* of (15.11).

**E. Semilinear parabolic initial value problems on $\mathbb{R}^n$:** Let $n \geq 1$ be fixed and consider the following semilinear initial value problem

$$\begin{cases} \partial_t u(x,t) + k(t)\Delta u(x,t) = f\big(x,t,u(t,x),\nabla u(x,t)\big) & (x,t) \in \mathbb{R}^n \times (0,T] \\ \qquad u(x,0) = u_0(x) & x \in \mathbb{R}^n, \end{cases} \tag{15.22}$$

where $u_0\colon \mathbb{R}^n \to \mathbb{R}$ is the *initial value*, the *diffusion coefficient* $k \in C^{\frac{\eta}{2}}([0,T])$ is strictly positive and the *nonlinearity* $f\colon \mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\begin{cases} f \in C\big(\mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R}^n\big) \text{ and } f(\cdot\,,\cdot\,,\xi,\zeta) \in BUC^{\eta,\frac{\eta}{2}}\big(\mathbb{R}^n \times [0,T]\big) \\ \text{uniformly for } (\xi,\zeta) \text{ on bounded subsets of } \mathbb{R} \times \mathbb{R}^n. \text{ Furthermore, we} \\ \text{assume that } \partial_\xi f, \partial_{\zeta_1} f, \ldots, \partial_{\zeta_n} f \text{ exist and are uniformly continuous on} \\ \text{sets of the form } \mathbb{R}^n \times [0,T] \times [-R,R] \times \mathbb{B}(0,R),\ R > 0. \end{cases} \tag{15.23}$$

Analogously to condition (15.17) in the previous subsection we shall also consider the situation where the nonlinearity satisfies the following requirement.

$$
(15.24)\qquad \begin{cases} f \in C\big(\mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R}^n\big) \text{ and } f(\cdot\,,\cdot\,,\xi,\zeta) \in C^{\eta,\frac{\eta}{2}}\big(\mathbb{R}^n \times [0,T]\big) \text{ uniformly for } (\xi,\zeta) \text{ on bounded subsets of } \mathbb{R}\times\mathbb{R}^n. \text{ Furthermore, we assume that } \partial_\xi f,\ \partial_{\zeta_1} f, \dots, \partial_{\zeta_n} f \text{ exist and are uniformly Hölder-continuous on sets of the form } \mathbb{R}^n \times [0,T] \times [-R,R] \times \mathbb{B}(0,R),\ R>0. \text{ For any } (t,u) \in [0,T] \times BUC^1(\mathbb{R}^n) \text{ we define} \\ \qquad g_f(t,u)(x) := f\big(x,t,u(x),\nabla u(x)\big) \\ \text{for every } x \in \mathbb{R}^n. \end{cases}
$$

**15.15 Remarks**

(a) As in (15.15) it is easy to show that (15.23) implies that

$$
(15.25)\qquad g_f \in C^{\frac{\eta}{2},1-}\big([0,T] \times BUC^1(\mathbb{R}^n), BUC(\mathbb{R}^n)\big),
$$

uniformly on bounded sets.

(b) The same arguments as in Remark 15.12 imply that whenever (15.24) holds and $0 < \mu < \nu < 1$ are small enough, we have

$$
(15.26)\qquad g_f \in C^{0,1-}\big([0,T] \times BUC^{1+\nu}(\mathbb{R}^n), BUC^{\mu}(\mathbb{R}^n)\big)
$$

uniformly on bounded sets.

(c) It is immediately clear that if we additionally require that

$$
(15.27)\qquad f(x,t,0,\zeta) = 0
$$

holds for every $(x,t,\zeta) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$, we may replace $BUC$ by $C_0$ in (15.25) and (15.26). □

Set now

$$
X_0 := BUC(\mathbb{R}^n) \qquad \text{or} \qquad C_0(\mathbb{R}^n),
$$

and define

$$
X_1 := \{u \in X_0;\ \Delta u \in X_0\} \qquad \text{as well as} \qquad X_\alpha := (X_0, X_1)^0_{\alpha,\infty}
$$

for any $\alpha \in (\frac{1}{2}, 1)$. Recall now from Corollary 4.19 and the definition of the little Hölder spaces that

$$
(15.28)\qquad X_\alpha \doteq \begin{cases} buc^{1+\nu}(\mathbb{R}^n) \hookrightarrow BUC^{1+\nu}(\mathbb{R}^n) & \text{if } X_0 = BUC(\mathbb{R}^n) \\ c_0^{1+\nu}(\mathbb{R}^n) \hookrightarrow C_0^{1+\nu}(\mathbb{R}^n) & \text{if } X_0 = C_0(\mathbb{R}^n), \end{cases}
$$

where we $\nu$ is the uniquely determined number such that $2\alpha = 1 + \nu$.

Now (15.28) and Remarks 15.15(a) and (c) yield the following analogon to Proposition 15.11.

**15.16 Proposition**
*If* (15.23) *is satisfied and* $\frac{1}{2} < \alpha \leq 1$, *we have*

$$g_f \in C^{\frac{\eta}{2},1-}\big([0,T] \times X_\alpha, X_0\big)$$

*uniformly on bounded subsets, so that* $g_f$ *satisfies* $(G1)$. *Of course, if* $X_0 = C_0(\mathbb{R}^n)$ *we have to assume that* (15.27) *holds.*

Moreover, (15.28) in conjunction with Remarks 15.15(b) and (c) give the corresponding analogon to Proposition 15.13.

**15.17 Proposition**
*Let* (15.24) *hold and fix* $\frac{1}{2} < \alpha \leq 1$. *Then,*

$$g_f \in C^{0,1}\big([0,T] \times X_\alpha, X_\gamma\big),$$

*uniformly on bounded subsets, whenever* $\gamma > 0$ *sufficiently small. Hence,* $g_f$ *satisfies* $(G0)$. *If* $X_0 = C_0(\mathbb{R}^n)$, *we have to assume that* (15.27) *holds.*

**Proof**
Observe that (15.20) holds also in the case of $\Omega = \mathbb{R}^n$. Hence, the assertion is an easy consequence of (15.28) and Proposition A1.4. □

More or less the same observations as in Remarks 15.14 apply to this case.

**15.18 Remarks**
(a) Observe that any $f \in C^2\big([0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n\big)$, uniformly on sets of the form $\mathbb{R}^n \times [0,T] \times [-R,R] \times \mathbb{B}(0,R)$, $R > 0$. satisfies both (15.23) and (15.24).

(b) In case that $f$ does not depend explicitly on $\nabla u$, Propositions 15.16 and 15.17 remain valid whenever $0 < \alpha \leq 1$, as is easily seen. One may also choose $\alpha = 0$ for the validity of Proposition 15.16 in this case.

(c) The assumptions made on $f$ are not optimal. Propositions 15.16 and 15.17 hold also under weaker regularity conditions on $f$. But also here at a later stage we shall need to resort to the Schauder theory for linear parabolic equations so we shall keep the present assumptions. □

Consider now for each $t \in [0,T]$ the $X_0$-realization of $k(t)\Delta$, i.e. the operator

$$A(t)\colon D(A(t)) \subset X_0 \to X_0,$$

with $D(A(t)) = X_1$, as described in Section 1.D. With this notation (15.22) may be reformulated as an abstract semilinear initial value problem

$$\begin{cases} \partial_t u + A(t)u = g_f(t,u) \text{ for } t \in (s,T] \\ \qquad u(0) = u_0, \end{cases} \tag{15.28}$$

on $X_\alpha$, whenever $\frac{1}{2} < \alpha \leq 1$ (or $0 < \alpha \leq 1$ if $f$ does not depend on $\nabla u$). We shall call (15.28) the $X_0$*-formulation* of (15.22).

**Notes and references:** Most of the material in Subsections A and B is well known to the specialist. However, a precise reference for the results as we have formulated them does not seem to exist. We have included the account of Nemitskii-operators on Hölder spaces for the same reason. While there exists a general reference on Nemitskii-operators (Appell and Zabrejko [**19**]), it does not contain the statements in the form we require. The concept underlying this, and in fact most of the subsequent sections, is taken from Amann [**7**], [**10**], Henry [**66**] and Pazy [**100**].

We mention here, that a semilinear theory may also be established in case of time-dependent domains of definition $D(A(t))$. It is important to note that, in the case of parabolic initial-boundary value problems, this enables us to deal with nonlinearities acting on the boundary. This time dependent theory can be developed in the setting of interpolation and extrapolations setting in the Notes and References at the end of Section 5. Let it be said that the basic idea is to reduce the problem with time dependent domains to one involving constant domains at the cost of working in the setting of a weaker Banach space. Hence, the theory described in our book may be considered as a preparation for the more general case. Under suitable assumptions one may obtain results on the original Banach space by a 'lifting' procedure.

## 16. Existence and continuous dependence

In this section we shall establish the existence of mild and classical solutions of the initial value problem (15.1). In order to deal with continuous dependence of the solutions on the initial data we also derive some useful Gronwall-type inequalities.

*We assume throughout that* $(G0)$ *or* $(G0')$ (*from Section* 15 ) *holds.*

**A. Existence of maximal solutions:** We start by proving a local existence result by the already classical method based on Banach's contraction mapping principle.

**16.1 Lemma**

*To every $\rho > 0$ there exists a number $T_1 := T_1(\alpha, \rho) > 0$, such that for any $(s, x) \in [0, T) \times \overline{\mathbb{B}}_{X_\alpha}(0, \rho)$, the initial value problem* (15.1) *has a unique mild solution $u\colon I \to X_\alpha$. Here we have set $I := [s, \min\{s + T_1, T\}]$.*

*If $(G0)$ or $(G1)$ holds, then this solution is also a classical solution.*

**Proof**

We prove the result under assumption $(G0')$. The case where $(G0)$ holds is similar but easier. Observe also that by Proposition 15.3 whenever $(G0)$ or $(G1)$ holds any mild solution will also be classical. Hence, we may concentrate on the first part of the assertion.

Let $\rho < 0$ and $(s, x) \in [0, T) \times \overline{\mathbb{B}}_{X_\alpha}(0, \rho)$ be given. For $u \in C\big([s, T], X_\alpha\big)$ and $t \in [s, T]$ we set

$$G(u)(t) := U(t, s)x + \int_s^t U(t, \tau) g\big(\tau, u(\tau)\big)\, d\tau.$$

By Lemma 5.3 and 5.5 we see that for each $T_1 \in (s, T]$ the mapping $G$ takes $C\big([s, T_1], X_\alpha\big)$ into itself. We shall now prove that $G$ defines a contraction on a suitable closed subset of $C\big([s, T], X_\alpha\big)$.

The boundedness of $\{\|U(t, r)\|_{\alpha,\alpha};\ (t, r) \in \Delta_T\}$ implies the following inequality:

$$\|U(t, r)x - x\|_\alpha \le c_1(\alpha)\|x\|_\alpha \qquad \text{for } (t, r) \in \Delta_T \tag{16.1}$$

We set:

$$\varepsilon := \varepsilon(\rho) := c_1(\alpha)\rho + 1. \tag{16.2}$$

Fix now some $x \in \mathbb{B}_{X_\alpha}(0, \rho)$ and let $u \in C\big([s, T], \overline{\mathbb{B}}_{X_\alpha}(x, \varepsilon)\big)$. For every $T_1 \in (0, T]$ and $t \in [s, T_1]$ we have by Lemma 5.2(c):

$$\begin{aligned} \Big\| \int_s^t U(t, \tau) g\big(\tau, u(\tau)\big)\, d\tau \Big\|_\alpha &\le \int_s^t \|U(t, \tau)\|_{0,\alpha}\, d\tau \sup_{\tau \in [s, T]} \big\|g\big(\tau, u(\tau)\big)\big\|_0 \\ &\le c_2(\alpha, \rho)(t - s)^{1-\alpha} \le c_2(\alpha, \rho) T_1^{1-\alpha}, \end{aligned} \tag{16.3}$$

where we have used that, by $(G0')$, the set $\big\{\|g(r, u(r))\|_0;\ r \in [s, T]\big\}$ is bounded.

Choose now $T_1 := T_1(\alpha, \rho) > 0$, such that

$$c_2(\alpha, \rho) T_1^{1-\alpha} \le 1, \tag{16.4}$$

and define

$$\mathcal{M}_{T_1} := C\big([s, T_1], \overline{\mathbb{B}}_{X_\alpha}(x, \varepsilon)\big).$$

$\mathcal{M}_{T_1}$ is obviously a closed subset of $C\big([s,T],X_\alpha\big)$, and as such a complete metric space. From (16.1)–(16.4) it follows that $G$ is a selfmapping of $\mathcal{M}_{T_1}$. Furthermore, we have by $(G0')$, that for any two functions $u$ and $v$ lying in $\mathcal{M}_{T_1}$

$$\begin{aligned}\|G(u)(t)-G(v)(t)\|_\alpha &\le \int_s^t \|U(t,\tau)\|_{0,\alpha}\|g\big(\tau,u(\tau)\big)-g\big(\tau,v(\tau)\big)\|_0\,d\tau\\ &\le \kappa(\rho)T_1^{1-\alpha}\|u-v\|_{C([s,T],X_\alpha)}.\end{aligned}$$

Thus, after possibly making $T_1$ smaller, we obtain that $G$ is a contraction on $\mathcal{M}_{T_1}$. The lemma now follows by the contraction mapping principle and Proposition 15.3. $\square$

Now that we have established local existence, we may proceed to prove the existence of a unique maximal mild solution.

**16.2 Theorem**
*For any $(s,x)\in[0,T)\times X_\alpha$, the semilinear initial value problem has a unique maximal (mild) solution $u(\cdot\,;s,x)\in C\big(J(s,x),X_\alpha\big)$. The maximal existence interval $J(s,x)$ is an interval of one the following types: $[s,T]$, or $[s,\tilde T)$, for $\tilde T\in(s,T]$. The number*

$$t^+(s,x):=\sup(J(s,x))$$

*is called the positive escape time of $u(\cdot\,;s,x)$.*

**Proof**
The existence of a maximal mild solution may be obtained, for instance, by a standard application of Zorn's Lemma. To see that $J(s,x)$ is an interval of the claimed type, we need only observe that if $t^+(s,x)<T$, then $J(s,x)$ must be equal to $\big[s,t^+(s,x)\big)$, since otherwise, by Lemma 16.1 applied to the pair $\big(t^+(s,x),u(t^+(s,x);s,x)\big)$, we could extend $u(\cdot\,;s,x)$ to a strictly larger subinterval of $[s,T]$, contradicting the maximality of $u$. $\square$

The following corollary is instrumental in proving global existence results:

**16.3 Corollary**
*For each $(s,x)\in[0,T)\times X_\alpha$ one of the following alternatives holds:*
 *(i) $J(s,x)=[0,T]$.*
 *(ii) The positive orbit*

$$\gamma^+(s,x):=\big\{u(t;s,x);t\in J(s,x)\big\}$$

*is an unbounded set in $X_\alpha$.*

**Proof**
If $\gamma^+(s,x)$ is bounded in $X_\alpha$, there exists $\rho>0$, such that $\|u(t;s,x)\|_\alpha<\rho$ for all $t\in J(s,x)$.

We may now apply Lemma 16.1 consecutively on the intervals

$$\big[s, T_1(\alpha,\rho)\big], \big[T_1(\alpha,\rho), 2T_1(\alpha,\rho)\big], \dots .$$

Then, we obtain a global mild solution. □

Usually, it is rather difficult to show directly that an orbit is bounded in $X_\alpha$. In the next section we shall show how to obtain an $X_\alpha$-bound provided we know an estimate in a weaker norm. This, in conjunction with the above corollary, constitutes a powerful method to prove global existence in many practical instances, as we shall see in Chapter VI.

**16.4 Examples**

(a) Consider the semilinear initial-boundary value problem (15.11) on the bounded subdomain $\Omega$ of $\mathbb{R}^n$ under the same assumptions as in Section 15.C. In particular, the nonlinearity $f$ is assumed to satisfy either (15.12) or (15.17). Then, we obtain from Theorem 16.2 the following result for (15.21), the abstract formulation of (15.11):

> *Let $\frac{1}{2} + \frac{n}{2p} < \alpha < 1$ be fixed. Then, for each initial value $u_0 \in X_\alpha$, equation* (15.21) *posesses a unique maximal classical solution*
>
> $$u(\cdot\,; u_0) \in C\big(J(0,u_0), X_\alpha\big) \cap C^1\big(\dot{J}(0,u_0), L_p(\Omega)\big).$$
>
> *If the nonlinearity $f$ does not depend on $\nabla u$ explicitly, we may choose $\frac{n}{2p} < \alpha < 1$. Furthermore, in case that* (15.17) *holds $\alpha$ can be chosen to equal* 1*.*

Suppose now that $u_0 \in X_1$. For any $x \in \bar{\Omega}$ we set

$$u(x,t;u_0) := u(t;u_0)(x)$$

This definition makes sense since from the variation-of-constants formula and Corollary 5.6 we have that $u(t;u_0) \in X_1 \hookrightarrow C^1(\bar{\Omega})$ for all $t \geq 0$. A very natural question is now the following: What does the function $(x,t) \mapsto u(x,t;u_0)$ have to do with the initial-boundary value problem (15.11)? Is it by any chance a solution in the sense of classical differential equations? It turns out that the answer to this question is indeed affirmative, a fact which shall be proved in Chapter VI.

(b) We may also consider the semilinear initial value problem on $\mathbb{R}^n$ given by (15.21) under the assumptions of Section 15.D, with the nonlinearity satisfying either (15.22) or (15.23). Again, we obtain a solvability result for (15.28), the abstract formulation of

(15.21). Note that if $X_0 = C_0(\mathbb{R}^n)$ we also assume (15.26).

*Let $\frac{1}{2} < \alpha < 1$ be fixed. Then, for each initial value $u_0 \in X_\alpha$, equation* (15.21) *posesses a unique maximal classical solution*

$$u(\cdot\,;u_0) \in C\big(J(0,u_0), X_\alpha\big) \cap C^1\big(\dot{J}(0,u_0), X_0\big).$$

*If the nonlinearity $f$ does not depend on $\nabla u$ explicitly, we may choose any $0 \le \alpha < 1$. Furthermore, in case that* (15.23) *holds $\alpha$ can be chosen to equal* 1.

Again, for $u_0 \in X_1$ we may set

$$u(x,t;u_0) := u(t;u_0)(x)$$

for any $x \in \mathbb{R}^n$, and pose the question: Is the function $(x,t) \mapsto u(x,t;u_0)$ a solution in the sense of classical differential equations? And again the answer is affirmative. This shall also be proved in Chapter VI. □

Let now $s \in [0,T)$ and set

$$\mathcal{D}_s := \{(t,x) \in [s,T) \times X_\alpha;\ t \in J(s,x)\}.$$

Note that in the definition of $\mathcal{D}_s$ we have excluded points of the form $(T,x)$. It is our aim to study the continuity properties of the mappings:

$$u(\cdot\,;s,\cdot)\colon \mathcal{D}_s \to X_\alpha,$$

and

$$t^+(s,\cdot)\colon X_\alpha \to (0,T].$$

But before doing so we shall need to develop a method for obtaining estimates for the solution operator.

**B. Gronwalls inequality:** A very powerful instrument when dealing with integral equations is Gronwall's inequality. We shall give here a generalization of this classical result which shall enable us to cope with the singularities of the evolution operator $U\colon \dot{\Delta}_T \to \mathcal{L}(X_\alpha, X_\beta)$ for $0 \le \alpha < \beta \le 1$. Let us first define for every $\alpha \in [0,1)$ the analytic function, $m_\alpha\colon \mathbb{R} \to \mathbb{R}$, by setting

$$m_\alpha(\xi) := \sum_{k=1}^{\infty} \frac{\big[\Gamma(1-\alpha)\big]^k}{\Gamma\big(k(1-\alpha)\big)}\, \xi^{k-1}$$

for all $\xi \in \mathbb{R}$. Here, $\Gamma: \mathbb{C} \to \mathbb{C}$ denotes the Gamma-function. Observe that $m_\alpha(\xi) > 0$, whenever $\xi > 0$ and that $m_0(\xi) = e^\xi$ for all $\xi \in \mathbb{R}$.

**16.5 Lemma**
*Let $0 \le \alpha < 1$ and assume that $h \in L_1\big((0,T)\big)$ is almost everywhere positive. Furthermore, suppose that $w \in L_1\big((0,T)\big)$ satisfies the following integral inequality*

$$w(t) \le h(t) + c_0 \int_0^t (t-\tau)^{-\alpha} w(\tau)\, d\tau \tag{16.5}$$

*for almost all $t \in (0,T)$ and some constant $c_0 > 0$. Then,*

$$w(t) \le h(t) + c_0 \int_0^t (t-\tau)^{-\alpha} m_\alpha\big(c_0(t-\tau)^{1-\alpha}\big) h(\tau)\, d\tau \tag{16.6}$$

*for almost all $t \in (0,T)$.*

**Proof**
Consider the integral equation

$$v(t) = h(t) + c_0 \int_0^t (t-\tau)^{-\alpha} v(\tau)\, d\tau \tag{16.7}$$

in the Banach space $E := L_1\big((0,T)\big)$. Equation (16.7) is a Volterra integral equation in $E$ of type (9.2). In Section 9 we have seen how to deal with this kind of equations. One defines an integral operator $Q \in \mathcal{L}(E)$ by setting

$$[Qu](t) := c_0 \int_0^t (t-\tau)^{-\alpha} u(\tau)\, d\tau$$

for all $u \in E$ and $t \in (0,T)$. One then constructs the resolvent kernel of (6.6) by means of (9.7) and (9.11). In our special case, using (9.9) and (9.10), it is not difficult to see that it is given by

$$h(t,s) := c_0(t-s)^{-\alpha} m_\alpha\big((t-s)^{1-\alpha} c_0\big).$$

Hence, the solution $v := (1-Q)^{-1}h$ of (16.7) can be expressed by the right hand side of (16.6). Moreover, note that $(1-Q)^{-1}$ maps nonnegative functions into nonnegative functions.

Since, by assumption,

$$(1-Q)w(t) \le h(t)$$

holds for almost all $t \in (0,T)$, it follows from the above considerations that

$$w(t) \le (1-Q)^{-1} h(t)$$

holds for almost all $t \in (0,T)$, proving the desired inequality. $\square$

A simple consequence of this result is the following

**16.6 Corollary**

*Suppose that the nonnegative function $w \in L_1\big((0,T)\big)$ satisfies*

$$w(t) \le c_0 t^{-\beta} + c_1 \int_0^t (t-\tau)^{-\alpha} w(\tau)\, d\tau \tag{16.8}$$

*for almost all $t \in (0,T)$, where $c_0$ and $c_1$ are nonnegative constants and $\alpha, \beta \in [0,1)$. Then, there exists a nonnegative constant $c(\alpha,\beta,c_1,T)$ such that*

$$w(t) \le c_0 c(\alpha,\beta,c_1,T) t^{-\beta}$$

*holds for almost all $t \in (0,T)$.*

**Proof**

The preceding lemma implies the estimate

$$w(t) \le c_0 t^{-\beta} + c(\alpha,c_1,T) \int_0^t (t-\tau)^{-\alpha} \tau^{-\beta}\, d\tau$$

for almost all $t \in (0,T)$. From (9.10) we readily obtain

$$\int_0^t (t-\tau)^{-\alpha} \tau^{-\beta}\, d\tau = t^{-\beta} t^{1-\alpha}\, \mathrm{B}(1-\alpha, 1-\beta),$$

where $\mathrm{B}(\cdot\,,\cdot)$ is the Beta-function. Hence, the assertion follows. $\square$

**C. Continuous dependence:** We now resume the study of the initial value problem (15.1) under the same assumptions as in Subsection A. In particular we assume that $(G0)$ or $(G1)$ holds. We start with a simple lemma:

**16.7 Lemma**

*Let $s \in [0,T)$, $0 \le \beta \le \alpha \le 1$ be given, where $\beta > 0$ if $\alpha = 1$ and $\alpha < 1$ if $(G0')$ holds. Furthermore, suppose that $\rho > 0$. Then there exists a constant $L := L(\alpha,\beta,\rho,T) > 0$, such that for any $x, y \in X_\alpha$ satisfying*

$$\|u(t;s,x)\|_\alpha, \|u(t;s,y)\|_\alpha \le \rho \qquad \text{for all } t \in I,$$

*where $I$ is any subinterval of $J(s,x) \cap J(s,y)$ containing $s$, the following inequality holds:*

$$\|u(t;s,x) - u(t;s,y)\|_\alpha \le L(t-s)^{\beta-\alpha} \|x-y\|_\beta$$

*for all $t \in I$.*

**Proof**
Set $u := u(\cdot\,; s, x)$, $v := u(\cdot\,; s, y)$, and $w := u - v$. Then, using $(G0')$ or $(G0)$ we see that

$$\begin{aligned}\|w(t)\|_\alpha &\le \|U(t,s)\|_{\beta,\alpha}\|x-y\|_\beta + \int_s^t \|U(t,\tau)\|_{0,\alpha}\|g\big(\tau,u(\tau)\big) - g\big(\tau,v(\tau)\big)\|_0\,d\tau \\ &\le c(\alpha,\beta)(t-s)^{\beta-\alpha}\|x-y\|_\beta + \kappa(\rho)c(0,\alpha)\int_s^t (t-\tau)^{-\alpha}\|w(\tau)\|_\alpha\,d\tau\end{aligned}$$

holds for all $t \in I$.

Applying Gronwalls inequality we obtain:

$$\|w(t)\|_\alpha \le L(\alpha,\beta,\rho,T)(t-s)^{\beta-\alpha}\|x-y\|_\beta \qquad \text{for all } t \in I,$$

for a suitable constant $L(\alpha,\beta,\rho,T) > 0$, as claimed. □

Recall that at the end of Subsection A we had asked how regular the mappings

$$u(\cdot\,; s, \cdot) \colon \mathcal{D}_s \to X_\alpha,$$

and

$$t^+(s,\cdot) \colon X_\alpha \to (0,T]$$

were. The next result gives a first answer to this question. In particular it establishes the Lipschitz-continuous dependence of mild solutions on the initial data.

**16.8 Theorem**
*The following statements are true for each $s \in [0,T)$:*

*(i) $\mathcal{D}_s$ is open in $[0,T] \times X_\alpha$.*

*(ii) $t^+(s,\cdot)$ is a lower semicontinuous mapping from $X_\alpha$ to $(0,T]$.*

*(iii) $u(\cdot\,; s,\cdot) \in C^{0,1-}(\mathcal{D}_s, X_\alpha)$. More precisely: to every $(\bar t,\bar x) \in \mathcal{D}_s$, there exist constants $\rho := \rho(\alpha,\bar t,\bar x) > 0$, and $L := L(\alpha,\bar t,\bar x,\rho) > 0$, such that*

$$t^+(s,x), t^+(s,y) > \bar t,$$

*and*

$$\|u(t;s,x) - u(t;s,y)\|_\alpha \le L\|x-y\|_\alpha, \tag{16.9}$$

*for all $s \le t \le \bar t$ , whenever $x, y \in \overline{\mathbb{B}}_{X_\alpha}(\bar x,\rho)$.*

*If $(G0)$ holds we may choose $\alpha$ to equal $1$.*

**Proof**
(*ii*) Set $t^+(\cdot) := t^+(s,\cdot)$, and take $\bar{x} \in X_\alpha$. Let $t_1 < t^+(\bar{x})$. We have to show that there exists an $\varepsilon > 0$, such that

$$t_1 < t^+(x) \qquad \text{for all } x \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x},\varepsilon) \tag{16.10}$$

By the compactness of $\{u(t;s,\bar{x}); t \in [s,t_1]\}$ in $X_\alpha$, we find to every $R > 0$ a constant $\rho := \rho(\bar{x},t_1,R) > 0$, such that $\|x\|_\alpha \le \rho$ for all $x \in B_R := \overline{\mathbb{B}}_{X_\alpha}\big(u([s,t_1];s,\bar{x}),R\big)$. Applying Lemma 16.7, with $L := L(\alpha,\alpha,\rho,T)$, we obtain:

$$\|u(t;s,x) - u(t;s,y)\|_\alpha \le L\|x-y\|_\alpha, \tag{16.11}$$

as long as $u(t;s,x)$ and $u(t;s,y)$ remain in $B_R$.

We now set $\varepsilon := \frac{R}{2L}$, and show that with this $\varepsilon$, (16.10) holds.

First assume that there is an $x \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x},\varepsilon)$ which does not stay in the interior of $B_R$ for all $t \in J(s,x) \cap [s,t_1]$. It is easily seen that this implies that there exists a time $t_2 \in [s,t_1]$ such that $\|u(t_2;s,\bar{x}) - u(t_2;s,x)\|_\alpha = R$ but $u(t;s,x) \in B_R$ for all $t \in [s,t_2]$. In this case (16.11) would imply that:

$$R = \|u(t_2;s,\bar{x}) - u(t_2;s,x)\|_\alpha \le L\|\bar{x}-x\|_\alpha \le \frac{R}{2},$$

which is impossible. We thus have for any $x \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x},\varepsilon)$:

$$u(t;s,x) \in B_R \qquad \text{for all } t \in [s,t_1] \cap J(s,x). \tag{16.12}$$

Assume now that (16.10) does not hold. Then $t_1 \ge t^+(x)$ for an $x \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x},\varepsilon)$. By (16.12) and Corollary 16.3 we have then $t^+(x) = T$, contradicting $t^+(x) \le t_1 < T$. This proves (16.10), and thus (*ii*).

(*i*) Take now $(\bar{t},\bar{x}) \in \mathcal{D}_s$, and let $t_1 \in \big[\bar{t}, t^+(s,\bar{x})\big)$. By (*ii*), we can choose $\varepsilon > 0$, such that $t^+(s,x) > t_1$ for all $x \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x},\varepsilon)$. But then we have

$$(\bar{t},\bar{x}) \in (s,t_1) \times \overline{\mathbb{B}}_{X_\alpha}(\bar{x},\varepsilon) \subset \mathcal{D}_s,$$

proving the openness of $\mathcal{D}_s$ in $[s,T] \times X_\alpha$.

(*iii*) Take $(\bar{t},\bar{x}) \in \mathcal{D}_s$. Note that, for each $x \in X_\alpha$, the function $u(\cdot\,;s,x)$ is continuous from $J(s,x)$ to $X_\alpha$. Furthermore, (16.11) implie that (16.9) holds. These two facts imply that $u(\cdot\,;s,\cdot) \in C^{0,1-}(\mathcal{D}_s,X_\alpha)$, proving the theorem. □

**16.9 Remark**
Note that the constant $L(\alpha,\beta,\rho,T)$ of Lemma 16.7, depends only on $\kappa(\rho)$ of assumptions $(G0)$ or $(G1)$, and of the constants $c(\alpha,\beta)$ and $c(0,\alpha)$ of Lemma 5.2. This will be important when dealing with parameter dependent problems. □

We conclude this section by stating a theorem on and giving some examples of differentiable dependence of the mild solution on the initial data, if the nonlinearity $g$ is continuously differentiable with respect to the second argument. We shall refrain from giving the proof since it constitutes a very special case of a more general result on parameter dependent problems we shall prove in Section 18.

**16.10 Theorem**
*Assume that $g \in C^{0,1}\big([0,T] \times X_\alpha, X_0\big)$, uniformly on bounded subsets. Then, we have:*

$$u(\cdot\,; s, \cdot) \in C^{0,1}(\mathcal{D}_s, X_\alpha),$$

*and $w(t; s, x) := \partial_3 u(t; s, x)$ satisfies:*

$$w(t; s, x) = U(t, s) + \int_s^t U(t, \tau)\partial_2 g\big(\tau, u(\tau; s, x)\big) w(\tau; s, x)\, d\tau, \tag{16.13}$$

*for all $t \in J(s, x)$, in $\mathcal{L}(X_\alpha)$.*

**16.11 Examples**
(a) We carry on with Example 16.4(a). In particular, the nonlinearity $f$ satisfies either (15.12) or (15.17) .We may conclude with Theorem 16.8 that the following holds

> *For each $s \in [0, T)$ the mapping $[(t, u_0) \mapsto u(t; s, u_0)]$ is continuous from its domain of definition $\mathcal{D}_s \subset [s, T] \times X_\alpha$ into $X_\alpha$. Furthermore, we have that it is Lipschitz continuous in the second argument, uniformly on bounded subsets.*

Observe that we may choose $\alpha = 1$ if $(G0)$ holds. If we assume that the nonlinearity $f$ satisfies (15.17) we conclude from Proposition 15.13 and Theorem 16.8 that the above mapping is even continuously differentiable in the second argument.

(b) If we consider Example 16.4(b) we get similar assertions by replacing (15.12) and (15.17) by (15.23) and (15.24), respectively. □

**Notes and references:** The use of Banach's contraction mapping principle in proving local existence is by now the standard method for obtaining this kind of results. Its origins lie of course in Picard-Lindelöff's theorem from the theory of ordinary differential equations. This section makes evident what one means when saying that the theory presented here 'imitates' the theory of ordinary differential equations. We have relied mainly on Amann's papers [**7**] and [**10**] as well as on Henry's lecture notes [**66**].

## 17. Global solutions

In this section we exhibit a simple method for proving globality of solutions of (15.1) in the case that we have only a weak a priori estimate at our disposal.

**A. When does a weak a priori bound imply a strong one?** Let $(s,x) \in [0,T) \times X_\alpha$. By Corollary 16.3, the solution $u(\cdot\,; s,x)$ of (15.1) is a global solution if we can find an *a priori estimate* for $\|u(\cdot\,; s,x)\|_\alpha$, that is if we can find a constant $\rho > 0$, such that

$$\|u(t;s,x)\|_\alpha \le \rho, \tag{17.1}$$

for every $t \in J(s,x)$.

Finding such a priori estimates in the $X_\alpha$-norm directly, is a difficult task, but sometimes one can bound $u(\cdot\,; s,x)$ a priori, in a weaker norm, that is we find a constant $\varepsilon > 0$ such that

$$\|u(t;s,x)\|_Z \le \varepsilon \tag{17.2}$$

holds for all $t \in J(s,x)$. Here $Z$ is a Banach space satisfying $X_\alpha \hookrightarrow Z \hookrightarrow X_0$. In this section we give a condition on the nonlinearity $g$, such that (17.1) follows from (17.2).

We assume throughout, that $(G0)$ from Section 15 holds and we additionally require that

$(G2)$ $Z$ is a Banach space satisfying $X_\alpha \hookrightarrow Z \hookrightarrow X_0$, and there exists an increasing function
$\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $N \ge 0$, such that for any $\rho > 0$

$$\|g(t,y)\|_0 \le \lambda(\rho)(N + \|y\|_\alpha)$$

for all $t \in [0,T]$, and $y \in X_\alpha$, satisfying $\|y\|_Z \le \rho$,

is met.

**17.1 Lemma**
*Let $\rho > 0$ be given. Then there exists a constant $c(\alpha,\rho) > 0$, such that for any $(s,x) \in [0,T) \times X_\alpha$, satisfying $\|u(t;s,x)\|_Z \le \rho$, for all $t \in J(s,x)$, the following inequality holds:*

$$\|u(t;s,x)\|_\alpha \le \big(m_\alpha \|x\|_\alpha + \lambda(\rho) N T^{1-\alpha}\big) c(\alpha,\rho),$$

*for all $t \in J(s,x)$. Here we have set $m_\alpha := \sup_{(t,r)\in\Delta_T} \|U(t,r)\|_\alpha$.*

**Proof**
Let $(s,x) \in [0,T)$, and set $u := u(\cdot\,; s,x)$. Suppose that $\|u(t)\|_Z \le \rho$ for all $t \in J(s,x)$. By $(G2)$ we then have:

$$\begin{aligned}
\|u(t)\|_\alpha &\le \|U(t,s)\|_{\alpha,\alpha}\|x\|_\alpha + \int_s^t \|U(T,\tau)\|_{0,\alpha} \|g\big(\tau,u(\tau)\big)\|_0 \, d\tau \\
&\le m_\alpha \|x\|_\alpha + \lambda(\rho) \int_s^t (t-\tau)^{-\alpha}(N + \|u(\tau)\|_\alpha)\, d\tau \\
&\le m_\alpha \|x\|_\alpha + \lambda(\rho) N T^{1-\alpha} + \lambda(\rho) \int_s^t (t-\tau)^{-\alpha} \|u(\tau)\|_\alpha \, d\tau,
\end{aligned}$$

for all $t \in J(s,x)$. The lemma follows now by a direct application of Gronwall's inequality (Corollary 16.6). □

The main result of this section is the following simple consequence of the above lemma.

**17.2 Corollary**
*Let $\rho > 0$ and $\varepsilon > 0$ be given. Then there exists a constant $c(\alpha, \rho, \varepsilon)$, such that for any $(s,x) \in [0,T) \times \overline{\mathbb{B}}_{X_\alpha}(0,\varepsilon)$, satisfying $\|u(t;s,x)\|_Z \le \rho$, for $t \in J(s,x)$, we have:*

$$\|u(t;s,x)\|_\alpha \le c(\alpha,\rho,\varepsilon) \qquad \text{for all } t \in J(s,x).$$

*In this case we have $J(s,x) = [s,T]$.* □

Observe that this implies that an a priori $Z$-norm bound for a solution forces an a priori $X_\alpha$-norm bound and, hence globality of the solution.

**B. $L_\infty$-bounds and globality for parabolic equations:** Here we deduce some consequences of the above results which have proved to be important in concrete applications.

1) Consider first the initial-boundary value problem (15.11) under the same assumptions as in Subsection 15.D. In Example 16.4(a) we established the existence of a maximal classical solution, $u(\cdot\,; u_0)$, of the abstract $L_p$-formulation, (15.21), whenever $x \in X_\alpha$, where we shall assume throughout that in addition to the restrictions in Subsection 15.D $\alpha < 1$ holds. We have already mentioned that in order to establish the globality of $u(\cdot\,; u_0)$ it is necessary to have an a priori bound for it in the $X_\alpha$-norm. In general this is a rather difficult task, since $X_\alpha \hookrightarrow C^1(\overline{\Omega})$ (or at least $X_\alpha \hookrightarrow C^\nu(\overline{\Omega})$ for some $\nu \in (0,1)$) would mean, loosely speaking, that we have to bound not only the function itself but also its derivatives. On the other hand, by using the parabolic maximum principle, it is sometimes very easy to establish bounds for the $L_\infty$-norm of a solution. Thus, the question arises as to when such an a priori bound implies a bound in the $X_\alpha$-norm. We shall give a very easy result in this direction where we allow the nonlinearity to grow utmost linearly in the gradient of $u$. More precisely we shall assume;

$$(17.3) \qquad \begin{cases} \text{For each } R > 0 \text{ there exists a constant } c(R) > 0 \text{ such that} \\ \qquad |f(x,t,\xi,\zeta)| \le c(R)(1+|\zeta|) \\ \text{for all } (x,t) \in \overline{\Omega} \times [0,T] \text{ and } \zeta \in \mathbb{R}^n \text{ satisfying } |\zeta| \le R. \end{cases}$$

It is an easy exercise to deduce from (17.3) that the Nemitskii-operator $g_f$ must then satisfy

$$(17.4) \qquad \|g_f(t,u)\| \le c(R)(1 + \|u\|_\alpha)$$

for every $(t,u) \in [0,T] \times X_\alpha$ with $\|u\|_\infty \le R$. From (17.4) , Corollaries 16.3 and 17.2 we immediately obtain

**17.3 Proposition**
*Assume that* (15.12) *holds. Moreover, suppose that* $f$ *additionally satisfies* (17.3)*. Then, any solution of* (15.21) *which is bounded in the* $L_\infty$*-norm is a global solution. Moreover, its orbit is a relatively compact subset of* $X_\alpha$ *if* $0 \le \alpha < 1$.

Observe in particular that (17.3) is certainly satisfied if $f$ does not depend on the gradient of $u$. Hence, in this situation we always get globality from a priori estimates in the $L_\infty$-norm.

2) If we now consider the initial value problem (15.22) under the same assumptions as in Subsection 15.E. In Example 16.4(b) we established the existence of a maximal classical solution $u(\cdot\,; u_0)$, of the abstract $X_0$-formulation, (15.28), whenever $x \in X_\alpha$. Again, as in the bounded domain case, in order to establish the globality of $u(\cdot\,; u_0)$ it is necessary to have an a priori bound for it in the $X_\alpha$-norm, which ,by the same arguments as in Example (a), is a difficult task. On the other hand, by using the Phragmen-Lindeloff principle (the unbounded domain version of the parabolic maximum principle), it is sometimes very easy to establish bounds for the $L_\infty$-norm of a solution. Hence, we also ask in this context when such an a priori bound implies a bound in the $X_\alpha$-norm. Basically the same result as in the bounded domain case applies here. We shall assume;

$$(17.5)\qquad \begin{cases} \text{For each } R > 0 \text{ there exists a constant } c(R) > 0 \text{ such that} \\ \qquad |f(x,t,\xi,\zeta)| \le c(R)(1 + |\zeta|) \\ \text{for all } (x,t) \in \mathbb{R}^n \times [0,T] \text{ and } \zeta \in \mathbb{R}^n \text{ satisfying } |\zeta| \le R. \end{cases}$$

From (17.3) we may easily conclude that the Nemitskii-operator $g_f$ must then satisfy (17.4). From (17.6), Corollaries 16.3 and 17.2 we immediately obtain:

**17.4 Proposition**
*Assume that* (15.23) *or* (15.24) *hold. Moreover, suppose that* $f$ *additionally satisfies* (17.5)*. Then, any solution of* (15.28) *which is bounded in the* $L_\infty$*-norm is a global solution.*

Observe in particular that (17.5) is certainly satisfied if $f$ does not depend on the gradient of $u$. Hence, in this situation we always get globality from a priori estimates in the $L_\infty$-norm. In this case we could argue that if there is no gradient dependency we may choose $\alpha$ to equal 0, so that an $L_\infty$-bound is just an $X_\alpha$-bound.

**Notes and references:** Conditions of type ($G2$) are quite common in the literature. Our proof follows Amann [**7**]. Note that in our examples we basically obtain bounds in the $C^1$-norm. By different techniques it is also possible to prove that $L_\infty$-bounds for parabolic initial-boundary value problems force $C^2$-bounds. These results exploit the estimates by Schauder theory. For details consult Reidlinger [**103**], [**104**].

In our examples we have treated the case in which the nonlinearity grows utmost linearly in the gradient. By using essentially the same technique it is also possible to admit 'almost' quadratic growth (see [**7**] or [**104**]). By completely different means it is actually possible to deal with nonlinearities which grow quadratically in the gradient. This has been accomplished by Amann [**6**] using techniques going back to Tomi [**120**] and von Wahl [**123**].

## 18. Parameter dependent problems

In many applications it is necessary to consider equations depending on a parameter. In these instances one is interested in the behaviour of solutions as the parameter varies. A precise knowledge of this information plays a fundamental rôle in the study of bifurcation phenomena or in establishing continuation theorems. It is the purpose of this section to give a rather general theorem on parameter dependence which should provide the basis for a wide range of applications. Recall that in Section 16 we proved the existence of solutions to semilinear evolution equations by resorting to the Banach contraction mapping principle. Therefore, it seems only natural to try to establish regular dependence on parameters for fixed points of parameter dependent contractions, and to subsequently apply this result to differential equations.

**A. The contraction mapping principle:** Assume that $E$ and $F$ are Banach spaces and that $U \subset E$ and $\Lambda \subset F$ are open. Consider a map

$$F\colon \overline{U} \times \Lambda \to \overline{U}$$

satisfying the following condition

$$(18.1)\qquad \begin{cases} \text{There exists a constant } \kappa \in (0,1) \text{ such that} \\ \qquad\qquad \|F(x,\lambda) - F(y,\lambda)\| \le \kappa \|x - y\| \\ \text{holds for all } x, y \in \overline{U} \text{ and all } \lambda \in \Lambda. \end{cases}$$

Such a map is said to be a *uniform contraction on* $\overline{U}$.

Since for each $\lambda \in \Lambda$ the mapping $F(\cdot,\lambda)\colon \overline{U} \to \overline{U}$ is a contraction, the contraction mapping principle implies that there exists a unique fixed point, i.e. there exists a unique $x_0(\lambda) \in \overline{U}$ such that

$$F(x_0(\lambda),\lambda) = x_0(\lambda)$$

holds. The following proposition gives information on the regularity of the mapping $x_0 \colon \Lambda \to \overline{U}$.

**18.1 Proposition**
*Suppose that*

$$F \colon \overline{U} \times \Lambda \to \overline{U}$$

*is a uniform contraction on* $\overline{U}$. *Furthermore, assume that*

$$F \in C^k(\overline{U} \times \Lambda, \overline{U})$$

*for some fixed* $k \in \mathbb{N}$. *Then,*

$$[\lambda \mapsto x_0(\lambda)] \in C^k(\Lambda, \overline{U}).$$

*If* $k \geq 1$ *we have that for each* $\lambda \in \Lambda$ *the linear operator* $w(\lambda) := Dx_0(\lambda) \in \mathcal{L}(F, E)$ *satisfies the operator equation*

$$w(\lambda) = D_1 F\big(x_0(\lambda), \lambda\big) w(\lambda) + D_2 F\big(x_0(\lambda), \lambda\big).$$

*Moreover, if* $F$ *is analytic on a neighbourhood of* $\overline{U} \times \Lambda$ *then,* $x_0 \colon \Lambda \to \overline{U}$ *is also analytic.*

**18.2 Remarks**

(a) A proof of the above result may be found in [**66**], Section 1.2.6 or in [**125**], Satz 3.2.14.

(b) Of course, if $k = 0$ the assertion of the proposition remains valid if we replace $\overline{U}$ and $\Lambda$ by arbitrary metric spaces $M$ and $N$, respectively, with the only restriction being that $M$ be complete.

(c) One may also prove $C^{k+\nu}$ versions of the parameter dependent contraction mapping principle ([**66**], Exercise 4 in Section 1.2.6). □

**B. Parameter dependent evolution equations:** We now turn to study semilinear evolution equations, where the nonlinearity depends on a parameter. We assume as usual that $X_0$ and $X_1$ are Banach spaces satisfying $X_1 \overset{d}{\hookrightarrow} X_0$ and that $(A(t))_{0 \leq t \leq T}$ is a family of closed linear operators in $X_0$ satisfying (A1)–(A3) of Section 2. In particular, $D(A) \doteq X_1$.

Furthermore, we take $\Lambda$ to be an open subset of a Banach space $F$, and consider a function

$$g \colon [0, T] \times X_\alpha \times \Lambda \to X_\gamma$$

where $\alpha, \gamma \in [0, 1]$, are such that $\gamma > 0$ if $\alpha = 1$ and $\alpha < 1$ if $\gamma = 0$. We make the following regularity assumption

$(\overline{G})\quad g \in C^{0,1-,0}\big([0,T] \times X_\alpha \times \Lambda, X_\gamma\big) \cap C^{0,r}\big([0,T] \times (X_\alpha \times \Lambda), X_\gamma\big)$

for some fixed $r \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$ and uniformly on subsets of the form $[0,T] \times \mathbb{B}_{X_\alpha}(0,\rho) \times \Lambda$, $\rho > 0$.

**18.3 Remark**

Assumption $(\overline{G})$ implies for each $\rho > 0$ the existence of a constant $\kappa(\rho) > 0$ such that

$$\|g(t,x,\lambda) - g(t,y,\lambda)\|_\gamma \leq \kappa(\rho)\|x - y\|_\alpha \tag{18.1}$$

holds whenever $t \in [0,T]$, $x, y \in \mathbb{B}_{X_\alpha}(0,\rho)$, and $\lambda \in \Lambda$. □

The equation to which we shall devote our attention is the following *parameter dependent semilinear evolution equation*

$$\begin{cases} \partial_t u + A(t)u = g(t,u,\lambda) & \text{for } t \in (s,T] \\ u(s) = x \end{cases} \tag{18.2}$$

where $s \in [0,T)$ is the *initial time*, $x \in X_\alpha$ is the *initial value* and $\lambda \in \Lambda$ is the *parameter*.

Now, by the theory of Section 16, for any choice of $(s,x,\lambda) \in [0,T) \times X_\alpha \times \Lambda$ the initial value problem (18.2) posesses a unique maximal mild solution

$$u(\cdot;s,x,\lambda)\colon J(s,x,\lambda) \to X_\alpha,$$

where $J(s,x,\lambda)$ denotes the maximal interval of existence. The positive escape time of $u(\cdot;s,x,\lambda)$ shall be denoted by $t^+(s,x,\lambda)$, i.e. $t^+(s,x,\lambda) := \sup J(s,x,\lambda)$.

Before stating the problem we need a few technical preparations.

**18.4 Lemma**

*Let $s \in [0,T)$ be given. Then,*

*(i)* $[x \to U(\cdot,s)x] \in \mathcal{L}(X_\alpha, C([s,T],X_\alpha))$,

*(ii)For $g \in L_\infty\big((s,T),X_0\big)$ and $t \in [s,T]$ set*

$$H_s(g)(t) := \int_s^t U(t,\tau)g(\tau)\,d\tau.$$

*Then,*

$$H_s \in \mathcal{L}\big(L_\infty\big((s,T),X_0\big), C\big([s,T],X_\beta\big)\big) \cap \mathcal{L}\big(L_\infty\big((s,T),X_\gamma\big), C\big([s,T],X_1\big)\big)$$

*whenever $\beta \in [0,1)$ and $\gamma \in (0,1]$.*

**Proof**
The first assertion is a simple consequence of Lemma 5.2($b$) and the second one is part of Lemma 5.5. □

**18.5 Lemma**
*Let $g$ satisfy $(\overline{G})$ and define for each $u \in C([s,T], X_\alpha)$ and $\lambda \in \Lambda$ a function $\Phi(u,\lambda)$ by*

$$\Phi(u,\lambda)(t) := g(t, u(t), \lambda)$$

*for all $t \in [s,T]$. Then, $\Phi(u,\lambda) \in C([s,T], X_\gamma)$ and*

$$\Phi \in C^r(C([s,T], X_\alpha) \times \Lambda, C([s,T], X_\gamma)).$$

*Moreover, if $r \geq 1$ we have that for any $u_0 \in C([s,T], X_\alpha)$ and $\lambda_0 \in \Lambda$ $D_1\Phi(u_0,\lambda_0) \in \mathcal{L}\big(C([s,T], X_\alpha), C([s,T], X_\gamma)\big)$ acts on a function $h \in C\big([s,T], X_\alpha\big)$ in the following way:*

$$\big[D_1\Phi(u_0,\lambda_0)h\big](t) = D_2 g\big(t, u_0(t), \lambda_0\big)h(t)$$

*for $t \in [s,T]$. Analogously, $D_2\Phi(u_0,\lambda_0) \in \mathcal{L}\big(F, C([s,T], X_\gamma)\big)$ acts on $\lambda \in \Lambda$ in the following way:*

$$\big[D_2\Phi(u_0,\lambda_0)\lambda\big](t) = D_3 g(t, u_0(t), \lambda_0)\lambda$$

*for $t \in [s,T]$.*

**Proof**
If $\Phi$ is not continuous at some point $(u,\lambda) \in C\big([s,T], X_\alpha\big) \times \Lambda$, we can find an $\varepsilon > 0$ and a sequence $(u_n, \lambda_n) \to (u,\lambda)$ in $C\big([s,T], X_\alpha\big) \times \Lambda$ as $n \to \infty$, such that

$$\|\Phi(u_n,\lambda_n) - \Phi(u,\lambda)\| \geq \varepsilon$$

holds for all $n \in \mathbb{N}$. Therefore, there exists a sequence $(t_n)$ in $[s,T]$ such that

$$\|g\big(t_n, u_n(t_n), \lambda_n\big) - g\big(t_n, u(t_n), \lambda\big)\| \geq \varepsilon \tag{18.3}$$

holds for all $n \in \mathbb{N}$. Since $[s,T]$ is compact we may extract a convergent subsequence $t_{n_k} \to t$ from $(t_n)$. But then, the sequences $\big(u_{n_k}(t_{n_k})\big)$ and $\big(u(t_{n_k})\big)$ both converge to $u(t)$ as $k \to \infty$. This together with (18.3) contradicts the continuity of $g$.

If $r = 1$ we may proceed in the following way. Take $(u_0, \lambda_0) \in C([s,T], X_\alpha) \times \Lambda$ and $h \in C\big([s,T], X_\alpha\big)$. Using the Mean Value Theorem we see that

$$g\big(t, u_0(t) + h(t), \lambda_0\big) - g\big(t, u_0(t), \lambda_0\big) - D_2 g\big(t, u_0(t), \lambda_0\big)h(t)$$
$$\int_0^1 \big(D_2 g(t, u_0(t) + \tau h(t), \lambda_0) - D_2 g(t, u_0(t), \lambda_0)\big)\, d\tau h(t)$$

holds for all $t \in [s,T]$. By the continuity of $D_2 g$ we conclude that the integral term converges to zero uniformly in $t \in [s,T]$ as $h \to 0$ in $C([s,T],X_\alpha)$. This readily implies the existence and continuity of $(u_0,\lambda_0) \mapsto D_1\Phi(u_0,\lambda_0)$. Moreover, it gives the claimed formula for $D_1\Phi(u_0,\lambda_0)$.

The existence and continuity of $(u_0,\lambda_0) \mapsto D_2\Phi(u_0,\lambda_0)$ is proved in a similar way. Putting these two facts together we get that $\Phi \in C^1\big(C([s,T],X_\alpha)\times\Lambda, C([s,T],X_\gamma)\big)$.

The case $r > 1$ is proved by induction.

The analytic case may be proved by using the analyticity of $g$ and Theorem A1.5 to estimate the derivatives of $\Phi$ in a straightforward manner and aplying the same theorem, but now backwards, to obtain the analyticity of $\Phi$. □

**18.6 Corollary**

*For each $(x,u,\lambda,t) \in X_\alpha \times C([s,T],X_\alpha)\times\Lambda\times[s,T]$ set*

$$G(x,u,\lambda)(t) := U(t,s)x + \int_s^t U(t,\tau)g\big(\tau,u(\tau),\lambda\big)\,d\tau.$$

*Then*

$$G \in C^r\big(X_\alpha \times C([s,T],X_\alpha)\times\Lambda, C([s,T],X_\alpha)\big).$$

*Moreover, if $r \geq 1$, we have that for each $(x_0,u_0,\lambda_0) \in X_\alpha \times C\big([s,T],X_\alpha\big)\times\Lambda$ the partial derivatives are given by*

$$\big[D_1G(x_0,u_0,\lambda_0)x\big](t) = U(t,s)x \tag{18.4}$$

*for $x \in X_\alpha$ and $t \in [s,T]$,*

$$\big[D_2G(x_0,u_0,\lambda_0)h\big](t) = \int_s^t U(t,\tau)D_2g\big(\tau,u_0(\tau),\lambda_0\big)h(\tau)\,d\tau \tag{18.5}$$

*for $h \in C\big([s,T],X_\alpha\big)$ and $t \in [s,T]$*

$$\big[D_3G(x_0,u_0,\lambda_0)\lambda\big](t) = \int_s^t U(t,\tau)D_3g\big(\tau,u_0(\tau),\lambda_0\big)\lambda\,d\tau \tag{18.6}$$

*for $\lambda \in \Lambda$ and $t \in [s,T]$.*

**Proof**

The proof is trivial by Lemma 18.4 and 18.5 if one observes that

$$G(x,u,\lambda) = U(\cdot,s)x + H_s \circ \Phi(u,\lambda)$$

holds for all $(x,u,\lambda) \in X_\alpha \times C\big([s,T],X_\alpha\big)\times\Lambda$. □

The last technical result is the following simple lemma:

**18.7 Lemma**
*Suppose that $E$ and $F$ are Banach spaces and that $a < b$ are given real numbers. Furthermore, let $V$ be an open subset of $E$ and consider a function*

$$f\colon V \to C\big([a,b],F\big).$$

*Define now for each $t \in [a,b]$ and $\delta > 0$ the (possibly empty) interval $I(t,\delta) := [a,b] \cap [t, t+\delta]$ and the function*

$$f_{t,\delta}\colon V \to C\big(I(t,\delta),F\big),\ x \mapsto f\restriction_{I(t,\delta)}.$$

*Then, for any $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, the function $f$ is of class $C^k$ if and only if for each $t \in [s,T]$ there exists a $\delta_t > 0$ such that $f_{t,\delta_t}$ is of class $C^k$.*

*If $k = 0$ the result is also true if $V$ is any topological space.*

**Proof**
Let $f$ satisfy the assumptions of the proposition. By compactness of $[a,b]$ it is possible to choose a partition $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ of $[a,b]$ such that $f_{t_i,\delta_{t_i}} \in C^k\big(V, C(I(t_i,\delta_{t_i}),F)\big)$ for all $i = 0,\ldots,n-1$, where $\delta_{t_i} = t_{i+1} - t_i$. Equip the space

$$X = C\big([a,t_1],F\big) \times C\big([t_2,t_3],F\big) \times \ldots \times C\big([t_{n-1},b],F\big)$$

with the equivalent norm

$$\|(u_1,u_2,\ldots,u_n)\| := \max_{1\le i\le n} \|u_i\|_{C([t_{i-1},t_i],F)}.$$

Then, the mapping $P\colon C\big([a,b],F\big) \to X$ given by

$$Pu := \big(u\restriction_{[a,t_1]}, u\restriction_{[t_1,t_2]}, \ldots, u\restriction_{[t_{n-1},b]}\big)$$

is a linear isometry. Hence, it is an isometric isomorphism onto its image $\operatorname{im}(P)$. We thus obtain the following commutative diagramm

$$\begin{array}{ccc} C\big([0,T],F\big) & \xrightarrow{\quad P \quad} & \operatorname{im}(P) \\ & {}_{f}\nwarrow \qquad \nearrow_{(f_{t_0,\delta_{t_0}},\ldots,f_{t_{n-1},\delta_{t_{n-1}}})} & \\ & V & \end{array}$$

from which the assertion follows. □

In order to give a precise formulation of the result we have in mind we need to introduce some notation in a slightly different way as at the end of Subsection 16.A.

For fixed $s < t_0 \leq T$ we define

$$\mathcal{D}_s := \big\{(t, x, \lambda) \in [s, T) \times X_\alpha \times \Lambda;\ t \in J(s, x, \lambda)\big\} \subset X_\alpha \times \Lambda \times [s, T)$$

and for any $t_0 \in (s, T]$

$$\mathcal{D}_{s,t_0} := \big\{(x, \lambda) \in X_\alpha \times \Lambda;\ t_0 \in J(s, x, \lambda)\big\} \subset X_\alpha \times \Lambda.$$

We shall establish a theorem on the regularity properties of the mappings:

$$\big[(t, x, \lambda) \mapsto u(t; s, x, \lambda)\big]\colon \mathcal{D}_s \to X_\alpha,$$

and

$$t^+(s, \cdot, \cdot)\colon X_\alpha \times \Lambda \to (0, T].$$

We start with the following lemma.

**18.8 Lemma**

*Let $s \in [0, T)$ and $t_0 \in [s, T]$ be given. Moreover, suppose that $\mathcal{M}$ is a subset of $\mathcal{D}_{s,t_0} \neq \emptyset$ such that $\|u(t; s, x, \lambda)\|_\alpha \leq \rho$ holds for all $(x, \lambda) \in \mathcal{M}$ and $t \in [s, t_0]$. Then,*

$$\big[(x, \lambda) \mapsto u(\cdot\,; s, x, \lambda)\big] \in C\big(\mathcal{M}, C([s, t_0], X_\alpha)\big)$$

*holds.*

**Proof**

Fix $s_0 \in [s, t_0]$ and set

$$\mathcal{M}_{s_0} := \{(u(s_1; s, x, \lambda), \lambda);\ (x, \lambda) \in \mathcal{M}\}.$$

For each $(x, \lambda) \in \mathcal{M}_{s_0}$ and $u \in C([s_0, t_0], X_\alpha)$ we define

$$G_{x,\lambda,s_0}(u)(t) := U(t, s_0)x + \int_{s_0}^{t} U(t, \tau) g\big(\tau, u(\tau)\big)\, d\tau$$

whenever $t \in [s_0, t_0]$.

Observe that per definition of $\mathcal{M}_{s_0}$ and our assumptions $\|x\| \leq \rho$ holds for all $(x, \lambda) \in \mathcal{M}_{s_0}$. With this, a close inspection of the proof of Lemma 16.1 reveals that there exists a number $T_1 > 0$, depending only on $\rho$ and $\alpha$ (and in particular not on $s_0 \in [s, t_0]$), such that

$$G_{x,\lambda,s_0}\colon C([s_0, s_0 + T_1], X_\alpha) \to C([s_0, s_0 + T_1], X_\alpha)$$

is a contraction which is uniform in $(x,\lambda)\in\mathcal{M}_{s_0}$. By Proposition 18.1 and Corollary 18.6 we immediately conclude that

$$\big[(x,\lambda)\mapsto u(\cdot\,;s_0,x,\lambda)\big]\in C\big(\mathcal{M}_{s_0},C([s_0,s_0+T_1],X_\alpha)\big)$$

is a continuous mapping. But this easily implies that

$$\mathcal{M}\to\mathcal{M}_{s_0},\quad (x,\lambda)\mapsto u(s_0\,;s,x,\lambda)$$

is also continuous. Putting these two facts together we obtain that for each $s_0\in[s,t_0]$

$$\big[(x,\lambda)\mapsto u(\cdot\,;s_0,x,\lambda)\big]\in C\big(\mathcal{M},C([s_0,s_0+T_1],X_\alpha)\big).$$

The assertion now follows by Lemma 18.7. □

**18.9 Lemma**

*The following statements are true for any $s\in[0,T)$:*

*(i) $\mathcal{D}_s$ is open in $[0,T)\times X_\alpha\times\Lambda$.*

*(ii) $t^+(s,\cdot,\cdot)\colon X_\alpha\times\Lambda\to(0,T]$ is lower semicontinuous.*

*(iii) For each $(\bar t,\bar x,\bar\lambda)\in\mathcal{D}_s$ there exists an $\varepsilon>0$ and an $R>0$, such that $(\bar t,x,\lambda)\in\mathcal{D}_s$ and $\|u(t;s,x,\lambda)\|\le R$ holds for all $t\in[s,\bar t]$ and $(x,\lambda)\in\overline{\mathbb{B}}_{X_\alpha}(\bar x,\varepsilon)\times\overline{\mathbb{B}}_\Lambda(\bar\lambda,\varepsilon)$.*

**Proof**

*(ii)* Set $t^+(\cdot\,,\cdot):=t^+(s,\cdot\,,\cdot)$, and take $(\bar x,\bar\lambda)\in X_\alpha\times\Lambda$. Let $t_1<t^+(\bar x,\bar\lambda)$ be given. We shall show that there exists an $\varepsilon>0$, such that

(18.7) $$t_1<t^+(x,\lambda)\qquad\text{for all}\quad (x,\lambda)\in\overline{\mathbb{B}}_{X_\alpha}(\bar x,\varepsilon)\times\overline{\mathbb{B}}_\Lambda(\bar\lambda,\varepsilon).$$

By the compactness of $\{u(t;s,\bar x,\bar\lambda);t\in[s,t_1]\}$ in $X_\alpha$, we find to every $R>0$ a constant $\rho:=\rho(\bar x,t_1,R)>0$, such that $\|x\|_\alpha\le\rho$ for all $x\in B_R:=\overline{\mathbb{B}}_{X_\alpha}(u([s,t_1];s,\bar x),R)$. Similar as in the proof of the previous lemma we find $T_1>0$ independent of $s_0\in[s,t_1]$, $x\in B_R$ and $\lambda\in\Lambda$ such that

(18.8) $$t^+(s_0,x,\lambda)>s_0+T_1$$

holds.

Suppose now that there exist sequences $(x_n)$ in $X_\alpha$ and $(\lambda_n)$ in $\Lambda$ with $\lim_{n\to\infty}x_n=\bar x$ and $\lim_{n\to\infty}\lambda_n=\bar\lambda$, such that $u(t;s,x_n,\lambda_n)$ does not stay in the interior of $B_R$ for all $t\in[s,s+T_1]$. As easily seen, this implies the existence of a sequence $(t_n)$ in $[s,t_1]$ such that

(18.9) $$\|u(t_n;s,\bar x,\bar\lambda)-u(t_n;s,x_n,\lambda_n)\|_\alpha=R$$

and $u(t;s,x_n,\lambda_n) \in B_R$ for all $t \in [s,t_n]$. Selecting a suitable subsequence we may assume without loss of generality that $\lim_{n\to\infty} t_n = \bar{t}$ holds for some $\bar{t} \in [s, s+T_1]$. Hence, by Lemma 18.8

$$\lim_{n\to\infty} \|u(t_n;s,\bar{x},\bar{\lambda}) - u(t_n;s,x_n,\lambda_n)\|_\alpha = 0 \tag{18.10}$$

which contradicts (18.9). We thus have that there must exist an $\varepsilon_1 > 0$ such that for any $x \in \overline{\mathbb{B}}_{X\alpha}(\bar{x},\varepsilon_1)$ and $\lambda \in \overline{\mathbb{B}}_\Lambda(\bar{\lambda},\varepsilon_1)$:

$$u(t;s,x,\lambda) \in B_R \qquad \text{for all } t \in [s,s+T_1]$$

By (18.8) we may now repeat the argument on the interval $[s,s+2T_1]$ and obtain the existence of an $\varepsilon_2$ such that $u(t;s,x,\lambda) \in B_R$ for all $t \in [s,s+2T_1]$, $x \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x},\varepsilon_2)$ and $\lambda \in \overline{\mathbb{B}}_\Lambda(\bar{\lambda},\varepsilon_2)$. Since by this procedure we reach $t_1$ in finitely many steps we may assume that there exists an $\varepsilon > 0$ such that $u(t;s,x,\lambda)$ remains in $B_R$ for all $(t,x,\lambda) \in [0,t_1] \times \overline{\mathbb{B}}_{X\alpha}(\bar{x},\varepsilon) \times \overline{\mathbb{B}}_\Lambda(\bar{\lambda},\varepsilon)$.

Assume now that (18.7) does not hold. Then $t_1 \geq t^+(x,\lambda)$ for an $x \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x},\varepsilon)$ and $\lambda \in \overline{\mathbb{B}}_\Lambda(\bar{\lambda},\delta)$. By (18.10) and Corollary 16.3 we have then that $t^+(x,\lambda) = T$ holds, contradicting $t^+(x,\lambda) \leq t_1 < T$. This proves (18.6), and thus (*ii*).

(*iii*) is shown in the proof of (*ii*) above.

(*i*) Take now $(t_1,\bar{x},\bar{\lambda}) \in \mathcal{D}_s$. By definition of $\mathcal{D}_s$ we have that $t_1 < t^+(\bar{x},\bar{\lambda})$. Hence by (*ii*) we find $\varepsilon > 0$ such that (18.7) holds. But this implies that

$$[s,t_1] \times \overline{\mathbb{B}}_{X_\alpha}(\bar{x},\varepsilon) \times \overline{\mathbb{B}}_\Lambda(\bar{\lambda},\delta) \subset \mathcal{D}_s$$

holds, proving the openness of $\mathcal{D}_s$. □

In the following theorem we formulate the main result of this section.

**18.10 Theorem**
*Suppose that* $(\overline{G})$ *holds. Then, the mapping*

$$\big[(t,x,\lambda) \mapsto u(t;s,x,\lambda)\big] \in C^{0,r,r}(\mathcal{D}_s \to X_\alpha).$$

*If* $r \geq 1$ *we set for any* $(x,\lambda) \in \mathcal{D}_s$

$$v(t) := \partial_x u(t;s,x,\lambda) \qquad \text{and} \qquad w(t) := \partial_\lambda u(t;s,x,\lambda).$$

*Then,* $v\colon [s,t_0] \to \mathcal{L}\big(X_\alpha, C([s,t_0],X_\alpha)\big)$ *and* $w\colon [s,t_0] \to \mathcal{L}\big(F, C([s,t_0],X_\alpha)\big)$ *satisfy*

$$v(t) = U(t,s) + \int_s^t U(t,\tau) D_2 g(\tau, u(\tau;s,x,\lambda),\lambda) v(\tau)\,d\tau \tag{18.11}$$

*and*

$$w(t) = \int_s^t U(t,\tau)D_2g(\tau, u(\tau; s, x, \lambda), \lambda)w(\tau)\,d\tau + \int_s^t U(t,\tau)D_3g(\tau, u(\tau; s, x, \lambda), \lambda)\,d\tau \tag{18.12}$$

**Proof**
Suppose that $(\bar{t}, \bar{x}, \bar{\lambda}) \in \mathcal{D}_s$ is fixed. By the previous lemma there exists an $\varepsilon > 0$ and an $R > 0$, such that $(\bar{t}, x, \lambda) \in \mathcal{D}_s$ and $\|u(t; s, x, \lambda)\| \le R$ holds for all $t \in [s, \bar{t}]$ and $(x, \lambda) \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x}, \varepsilon) \times \overline{\mathbb{B}}_\Lambda(\bar{\lambda}, \varepsilon)$.

Now the theorem may be proved using exactly the same arguments as in the proof of Lemma 18.8. The assertions about the derivatives are obtained from Proposition 18.1 in conjunction with formulas (18.4)–(18.6) in Corollary 18.6. □

**Notes and references:** The results of this section are standard in this nature. The idea of using the parameter dependent version of Banachs contraction mapping principle was taken from Henry [**66**]. As already mentioned theorems on parameter dependence are extremely important in as much they allow to use implicit function theorems and bifurcation theory.

# V. Semilinear periodic evolution equations

In this chapter we consider semilinear periodic evolution equations of parabolic type. We are mainly interested in establishing the principles of linearized stability and instability in the neighbourhood of a periodic solution. We do this by first showing that we can reduce the problem of Ljapunov-stability of such a solution $\bar{u}$, to the problem of stability of $\bar{x} := u(0)$ as a fixed-point of the period-map.

## 19. Equilibria in autonomous equations

The purpose of this section is to bring back to memory some basic features which one has come to expect from a foundation of a geometric theory for evolution equations. We shall accomplish this by looking at autonomous semilinear equations of parabolic type. A geometric theory for this type of equation has been developed in great detail in Henry's classic lecture notes [**66**], and is by now well known. The results for time-independent equations shall serve as models for those in the time-periodic context. We emphasize though that our work does not aim at being as exhaustive as D. Henry's book, but only at providing the sound foundations for further development. Finally, we remark that since this material has a purely motivating character, we shall not take pains to achieve neither the greatest possible generality nor the utmost precision.

**A. Autonomous equations define semiflows:** Let $X_0$ and $X_1$ be Banach spaces satisfying $X_1 \overset{d}{\hookrightarrow} X_0$. Suppose that $A\colon D(A) \subset X_0 \to X_0$ is a given closed operator, such that $-A$ is the infinitesimal generator of an analytic $C_0$-semigroup, and such that $D(A) = X_1$, up to equivalent norms. Assume, moreover, that $\alpha \in (0,1]$ is fixed and that a function

$$g \in C^{1-}(X_\alpha, X_0),$$

uniformly on bounded sets, is given.

Consider the autonomous semilinear initial value problem

$$\begin{cases} \dot{u} + Au = g(u) & \text{for } t > 0 \\ u(0) = x, \end{cases} \tag{19.1}$$

where the *initial value* $x$ lies in $X_\alpha$. By the results in Section 16 this problem posesses a unique maximal solution

$$u(\cdot;x) := u(\cdot;0,x) : J(0,x) \to X_\alpha.$$

We set

$$D(\varphi) := \bigcup_{x\in X_\alpha} J(0,x)\times\{x\} \subset \mathbb{R}_+ \times X_\alpha \qquad \text{and} \qquad \varphi(t,x) := u(t,x)$$

for $(t,x) \in D(\varphi)$. Again by the results in Section 16 the mapping

$$\varphi\colon D(\varphi) \to X_\alpha$$

enjoyes the following properties:

$(SF1)$ $D(\varphi)$ is open in $\mathbb{R}_+ \times X_\alpha$,

$(SF2)$ $\varphi \in C^{1-}(D(\varphi), X_\alpha)$ uniformly on bounded sets,

$(SF3)$ $\varphi(0,x) = x$ holds for all $x \in X_\alpha$,

$(SF4)$ If $(t,x)$ and $(s,\varphi(t,x))$ are in $D(\varphi)$, then $(t+s,x)$ lies also in $D(\varphi)$ and

$$\varphi(t+s,x) = \varphi(s,\varphi(t,x))$$

holds.

Properties $(SF1)$–$(SF4)$ tell us that $\varphi$ is a (local) semiflow on $X_\alpha$. To know this represents a great advantage since it allows the use of the conceptual and technical machinery of abstract semiflow theory. Observe that for the validity of $(SF4)$ it is essential that neither $A$ nor $g$ depend on time. Moreover, if for some $\gamma \in (0,1]$ we have that $g \in C^{1-}(X_\alpha, X_\gamma)$, uniformly on bounded sets, we may allow the case $\alpha = 1$.

**B. Stability of equilibria:** A *critical point*, or *equilibrium solution* of

$$\dot{u} + Au = g(u) \qquad \text{for } t > 0 \tag{19.2}$$

is a time-independent solution, i.e. a point $x_0 \in X_\alpha$, such that $\varphi(t,x_0) = x_0$ holds for all $t \geq 0$. This means that $x_0$ satisfies the *abstract elliptic equation*

$$Ax_0 = g(x_0). \tag{19.3}$$

Observe that by the smoothing property of parabolic equations, or by (19.3), any equilibrium solution actually lies in $X_1$.

Equilibrium solutions of (19.2) are the simplest possible solutions since they represent states in which the system remains at rest for all times. Of particular interest are stable equilibrium solutions. It is to this type of solutions that a well-behaved system tends to evolve as time progresses.

**19.2 Definitions**
An equilibrium solution, $x_0$, of (19.2) is called (*Ljapunov*) *stable* if to every $\varepsilon > 0$ there exists a $\delta > 0$ such that $t^+(0, x) = \infty$ and

$$\|\varphi(t, x) - x_0\| \leq \varepsilon$$

hold for all $t \geq 0$ whenever $\|x - x_0\|_\alpha \leq \delta$. It is called *asymptotically stable* if it is stable and there exists a $\delta > 0$, such that

$$\lim_{t\to\infty} \varphi(t, x) = x_0$$

whenever $\|x - x_0\|_\alpha \leq \delta$. It is called *exponentially stable* if there exist a $\delta > 0$, an $M \leq 1$, and an $\omega > 0$, such that

$$\|\varphi(t, x) - x_0\| \leq Me^{-t\omega}$$

holds for all $t \geq 0$ whenever $\|x - x_0\|_\alpha \leq \delta$. Finally. it is called *unstable* if it is not stable. □

**C. Stability via linearization:** In general it is no easy task to decide whether an equilibrium solution has one of the various stability properties or not. But if we require slightly more regularity of our nonlinearity, namely

$$g \in C^1(X_\alpha, X_0),$$

uniformly on bounded sets, we are suddendly in a position to use one of the most powerful tools which analysis has to offer: Linearization. The simple fact that a differentiable function in a small enough neighbourhood of a point behaves more or less like its derivative at that point, has an important counterpart in stability considerations in differential equations: The principles of linearized stability and instability.

Suppose that $x_0 \in X_1$ is an equilibrium solution of (19.2) and consider the linearization of (19.2) at $x_0$, i.e. the linear equation

$$\dot{v} + Av = Dg(x_0). \tag{19.4}$$

Since $Dg(x_0) \in \mathcal{L}(X_\alpha, X_0)$, we see by the perturbation Theorem 1.3, that $-(A - Dg(x_0))$ generates an analytic $C_0$-semigroup on $X_0$. Therefore, one may study the stability properties of the zero solution of (19.4). Recall that the stability properties of the zero solution of linear homogeneous equations were studied in Section 6 for the time-periodic case. There they were related to the spectrum of the period-map. In the autonomous case one may relate them to the spectrum of the generator $A$. The reason for this is the validity of a the spectral-mapping theorem (see e.g. [**98**], Corollary A-III.6.7.

**19.2 Definitions**
The equilibrium solution, $x_0$, of (19.2) is called *linearly stable* if the spectrum of $-\big(A - Dg(x_0)\big)$ lies entirely in the open half plane $[\operatorname{Re}\lambda > 0]$. It is called *linearly unstable* if $\sigma\big(-(A - Dg(x_0))\big) \cap [\operatorname{Re}\lambda < 0]$ contains a nonempty spectral set. Finally, it is called *neutrally stable* if it is neither linearly stable nor linearly unstable. $\square$

Observe that since we are dealing with generators of analytic $C_0$-semigroups linear stability implies the existence of an $\varepsilon > 0$ such that the spectrum of $-\big(A - Dg(x_0)\big)$ lies entirely in the halfplane $[\operatorname{Re}\lambda > \varepsilon]$. Moreover, the linear stability of $x_0$ is in fact equivalent to the exponential stability of the zero solution of (19.4).

The *principle of linearized stability* may be phrased as

*If $x_0$ is linearly stable, then it is also exponentially stable.*

On the other hand the principle of linearized instability reads

*If $x_0$ is linearly unstable, then it is unstable.*

Note that nothing whatsoever is said in the neutrally stable case. In fact, no general conclusion is possible since in this case $x_0$ may have any of the stability properties, as is well known even for finite dimensional ordinary differential equations.

**D. Compactness of orbits:** For many semiflows the fact that bounded orbits are relatively compact implies that solutions which remain uniformly bounded for all times will tend to the set of equilibria, or sometimes even to a single equilibrium, as time approaches infinity. This makes it an imperative to try to establish criteria guaranteeing that bounded orbits are relatively compact.

## 20. The period-map

Let $T > 0$ be fixed. Assume that $(A(t))_{t\in\mathbb{R}}$ is a family of closed linear operators satisfying $(A0)$ from Section 6, and $(A1)$–$(A3)$ from Section 2. For the properties of the associated evolution operator we refer to Section 5. In this section we introduce the important concept of the period-map corresponding to a semilinear time-periodic evolution equation.

**A. Semilinear periodic problems:** The object of our considerations will be the following semilinear periodic evolution equation:

$$\partial_t u + A(t)u = g\big(t, u(t)\big) \qquad \text{for } t \in \mathbb{R}, \tag{20.1}$$

Here, the nonlinearity $g$ satisfies one of the following regularity and periodicity conditions:

$G3$) $g: \mathbb{R} \times X_\alpha \to X_\gamma$, and
$g(t+T, y) = g(t, y)$ for all $(t, y) \in \mathbb{R} \times X_\alpha$,
as well as $(G0)$ of Section 15,

or

$(G3')$ $g: \mathbb{R} \times X_\alpha \to X_0$, and
$g(t+T, y) = g(t, y)$ for all $(t, y) \in \mathbb{R} \times X_\alpha$,
as well as $(G0')$ of Section 15.

We shall also consider the corresponding semilinear initial value problem:

$$\begin{cases} \partial_t u + A(t) = g\big(t, u(t)\big) & \text{for } t > 0 \\ \qquad u(0) = x, \end{cases} \tag{20.2}$$

where $x \in X_\alpha$. It is clear how *local, maximal, and global solutions* of (20.2) should be defined (compare with Definition 15.1). For convenience we state the precise result on existence and continuous dependence on the initial data:

**20.1 Theorem**

*For each $x \in X_\alpha$, there exists a unique maximal solution*

$$u(\cdot\,; x) \in C\big(J(x), X_\alpha\big) \cap C^1\big(\dot{J}(x), X_0\big),$$

*where the maximal existence interval $J(x)$,is of the form*

$$J(x) = \big[0, t^+(x)\big),$$

*with a lower semicontinuous function $t^+: X_\alpha \to \mathbb{R} \cup \{\infty\}$. Furthermore*

$$\Omega_0 := \big\{(t, x) \in \mathbb{R}_+ \times X_\alpha; t \in J(x)\big\},$$

*is open in $\mathbb{R}_+ \times X_\alpha$, and*

$$u \in C^{0,1-}(\Omega_0, X_\alpha).$$

*More precisely: to every $(\bar{t}, \bar{x}) \in \Omega_0$, there exist constants $\rho := \rho(\bar{t}, \bar{x}) > 0$, and $L := L(\bar{t}, \bar{x}, \rho) > 0$, such that*

$$t^+(x), t^+(y) > \bar{t},$$

*and*

$$\|u(t; x) - u(t; y)\|_\alpha \le L\|x - y\|_\alpha,$$

*for all $0 \le t \le \bar{t}$, whenever $x, y \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x}, \rho)$.*

In the study of nonautonomous periodic semilinear evolution equations, $T$-periodic solutions of (20.1) have quite the same importance, as equilibrium points in the study of autonomous problems. Here, a *$T$-periodic solution* of (20.1) is a function

$$u \in C(\mathbb{R}, X_\alpha) \cap C^1(\mathbb{R}, X_0),$$

satisfying

$$u(t+T) = u(t)$$

for all $t \in \mathbb{R}$.

The one concept that pervades most of the literature concerning periodic equations, is that of the period-map associated to the given equation.

**20.2 Definition**
We set

$$D(S) := \{x \in X_\alpha; t^+(x) > T\},$$

and

$$S(x) := u(T; x),$$

for all $x \in D(S)$. The map $S: X_\alpha \supset D(S) \to X_\alpha$ is called the *period-map*, or *time-$T$-map*, associated to (20.1). $\square$

**B. Fixed points of the period map and $T$-periodic solutions:** The importance of the period-map for $T$-periodic problems lies in the following elementary result, which goes back to Poincaré in the case of ordinary differential equations.

**20.3 Proposition**
*If $x \in D(S)$ is a fixed-point of $S$, i.e. $S(x) = x$, then $u(\cdot\,; x)$ can be defined on all $\mathbb{R}$, and is a $T$-periodic solution of* (20.1)*. Conversely, if $u$ is a $T$-periodic solution of* (20.1)*, and we set $x := u(0)$, then $x \in D(S)$ and is a fixed-point of $S$.*

The proof of the above result is completly trivial and therefore ommited. Our next result concerns the continuity of $S$ and is an immediate consequence of Theorem 20.1.

**20.4 Proposition**
*$D(S)$ is open in $X_\alpha$, and $S: X_\alpha \supset D(S) \to X_\alpha$, is Lipschitz-continuous, uniformly on bounded sets.*

**20.5 Examples**

(a) Consider the semilinear initial-boundary value problem on a bounded subdomain of $\mathbb{R}^n$, $n \geq 1$, of class $C^\infty$ given by

$$(20.3) \quad \begin{cases} \partial_t u(x,t) + \mathcal{A}(x,t,D)u(x,t) = f(x,t,u(t,x),\nabla u(x,t)) & (x,t) \in \Omega \times (0,T] \\ \mathcal{B}(x,D)u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

under exactly the same assumptions as in Section 15.D, but additionally requiring that $\mathcal{A}(x,t,D)$ and $f(x,t,u,\nabla u)$ are defined for all $t \in \mathbb{R}$ and are $T$-periodic. Then, the $L_p$-formulation

$$(20.4) \quad \begin{cases} \partial_t u + A(t)u = g_f\big(t,u(t)\big) & \text{for } t \in (s,T] \\ u(0) = u_0, \end{cases}$$

on $X_\alpha$, where we choose $\frac{1}{2} + \frac{n}{2p} < \alpha \leq 1$ (or $\frac{n}{2p} < \alpha \leq 1$ if $f$ does not depend on $\nabla u$) fits into the abstract framework of this section.

(b) Analogously, one can consider $t$-periodic versions of semilinear initial value problems on the whole $\mathbb{R}^n$ as considered in Section 15.E. □

**Notes and references:** Although we are not aware of any precise reference, the material of this section is rather standard and belongs to mathematical folklore. We point out that, its simplicity notwithstanding, Proposition 20.3 is of paramount importance, since it enables to reduce the study of periodic solutions to the investigation of a fixed-point equation. This means that one may use any of the standard fixed-point theoretic approaches such as Banach's contraction mapping principle, Schauder's fixed-point theorem, degree and index theory, et cetera. This point of view is exploited succesfully in Hess [**67**], for instance. In Amann [**7**], the properties of the period map of initial-boundary value problems in Hölder spaces are investigated.

For more information on the period-map in case of ordinary differential equations consult [**82**], [**33**] and [**17**].

## 21. Stability of periodic solutions

Proposition 20.3 reduces the question of existence of $T$-periodic solutions of (20.1) to the question of existence of fixed-points of the period-map $S$. This correspondence gives rise to two different notions of stability of a $T$-periodic solution. We first discuss Ljapunov-stability.

**A. Ljapunov-stability:**

**21.1 Definitions**
Let $\bar{u} \in C^1(\mathbb{R}, X_\alpha)$ be a $T$-periodic solution of (20.1), and set $\bar{x} := u(0) \in D(S)$. Furthermore let $\mathcal{M}$ be a subset of $X_\alpha$.

($i$) $\bar{u}$ is said to be *(Ljapunov)-stable with respect to* $\mathcal{M}$, if to every $\varepsilon > 0$, there exists a $\delta > 0$, such that $t^+(x) = \infty$ and

$$\|u(t;x) - \bar{u}(t)\|_\alpha < \varepsilon \qquad \text{for all } t > 0,$$

whenever $x \in \mathcal{M}$ satisfies $\|x - \bar{x}\|_\alpha < \delta$.

($ii$) $\bar{u}$ is said to be *asymptotically stable with respect to* $\mathcal{M}$, if $\bar{u}$ is stable, and there exist a $\delta > 0$, such that

$$\|u(t;x) - \bar{u}(t)\|_\alpha \to 0 \qquad \text{for } t \to \infty,$$

for every $x \in \mathbb{B}_{X_\alpha}(\bar{x}, \delta) \cap \mathcal{M}$.

($iii$) $u$ is said to be *exponentially asymptotically stable with respect to* $\mathcal{M}$, if it is stable, and there exist constants $\delta > 0$, $M > 0$, and $\omega > 0$, such that

$$\|u(t;x) - \bar{u}(t)\|_\alpha \leq Me^{-t\omega} \qquad \text{for all } t > 0,$$

whenever $x \in \mathcal{M}$ satisfies $\|x - \bar{x}\|_\alpha < \delta$.

($iv$) Finally $u$ is *unstable with respect to* $\mathcal{M}$ if it is not stable with respect to $\mathcal{M}$.

In case that $\mathcal{M} = X_\alpha$, we suppress the reference to $X_\alpha$ in all of the above notions. □

In the next section we shall prove theorems giving sufficient conditions for a $T$-periodic solution to be exponentially asymptotically stable, or unstable, (principles of linearized stability and instability). But we now turn to discuss the other stability concept that we mentioned.

**B. Stability of fixed-points of the period-map:** For the rest of this section assume that $\bar{u}$ is a $T$-periodic solution and that $\bar{x} := u(0)$.

We have already seen that $\bar{x}$ is then a fixed-point of $S\colon X_\alpha \supset D(S) \to X_\alpha$. We thus have at our disposal the conceptual apparatus of asymptotic fixed-point theory and can talk about the stability of the fixed-point $\bar{x}$ of $S$. We proceed to give the relevant definitions:

**21.2 Definitions**
Let $E$ be a Banach space and $F\colon E \supset D(F) \to E$. Supose that $x_0 \in D(F)$ is a fixed-point of $F$, i.e $F(x_0) = x_0$.

$(i)$ $x_0$ is said to be a *stable fixed-point* of $F$ if to every $\varepsilon > 0$, there exists a $\delta > 0$, such that $T^n(x)$ is defined,and

$$\|F^n(x) - x_0\|_\alpha < \varepsilon \qquad \text{for all } n \in \mathbb{N},$$

whenever $x \in D(F)$ satisfies $\|x - x_0\|_\alpha < \delta$.

$(ii)$ $x_0$ is said to be an *asymptotically stable fixed point* of $F$ if it is stable, and there exists a $\delta > 0$, such that

$$\|F(x) - x_0\|_\alpha \to 0 \qquad \text{for } n \to \infty,$$

for all $x \in D(F)$ satisfying $\|x - x_0\|_\alpha$.

$(iii)$ $x_0$ is said to be an *exponentially asymptotically stable fixed-point* of $F$, if it is stable and there exist constants $\delta > 0$, $M > 0$, and $\omega > 0$, such that

$$\|F^n(x) - x_0\|_\alpha \leq Me^{-n\omega} \qquad \text{for all } n \in \mathbb{N},$$

whenever $x \in D(F)$ satisfies $\|x - x_0\|_\alpha < \delta$.

$(iv)$ $x_0$ is an *unstable fixed-point* of $F$, if it is not stable. □

While it is clear that the ((exponential) asymptotic) stability of the $T$-periodic solution $\bar{u}$ of (20.1), implies the ((exponential) asymptotic) stability of the fixed-point $\bar{x}$ of $S$, the converse is not quite so obvious. It is useful to know that this is nevertheless true:

**21.3 Theorem**

*Let $\bar{u}$, and $\bar{x}$ be as above. Then $\bar{u}$ is an ((exponentially) asymptotically) Ljapunov-stable $T$-periodic solution of* (20.1)*, if and only if, $\bar{x}$ is an ((exponentially) asymptotically) stable fixed-point of the period-map of* (20.1)*.*

**Proof**

By the remark preceding the statement of the theorem, we only have to prove that, if $\bar{x}$ has a specific stability property, the same is true of $\bar{u}$.

Let $\bar{x}$ be stable. By Theorem 20.1 there exists a $\rho > 0$ and an $L > 0$, such that

$$\|u(t;x) - u(t;y)\|_\alpha \leq L\|x - y\|_\alpha, \tag{21.1}$$

for all $x, y \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x}, \rho)$, and all $t \in [0, T]$.

Let now $\varepsilon > 0$ be given. By the stability of $\bar{x}$, we find a $\delta > 0$, such that:

$$\|S^n(x) - \bar{x}\|_\alpha < \min\{\varepsilon L^{-1}, \rho\},$$

for all $x \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x}, \delta)$. But then, using (21.1) we have:

$$\begin{aligned}\|u(t + nT; x) - u(t + nT; \bar{x})\|_\alpha &= \|u(t; S^n(x)) - u(t; \bar{x})\|_\alpha \\ &\leq L\|S^n(x) - \bar{x}\|_\alpha < \varepsilon,\end{aligned}$$

for all $t \in [0,T]$, $n \in \mathbb{N}$, and $x \in \overline{\mathbb{B}}_{X_\alpha}(\bar{x},\delta)$, establishing the stability of $\bar{u}$.

For (exponential) asymptotic stability we also use (21.1) analogously. □

**Notes and references:** Again, these results are well known and widely used but a precise reference is not known to us. For a few more aspects concerning the stability properties of fixed-points of maps consult for instance Zeidler [**128**]. For questions related to stability and involving the positivity of the mappings under consideration the reader is referred to Hess [**67**] and Dancer [**34**], [**35**].

## 22. Linearized stability and instability

We make the same general assumptions on $\big(A(t)\big)_{0\le t\le T}$ and $g$ as in Section 20. Furthermore we assume that $\bar{u}$ is a $T$-periodic solution of (20.1), and set $\bar{x} := \bar{u}(0)$.

**A. Stability via linearization:** In this section we study the stability properties of $\bar{u}$ by linearizing (20.1) at $u_0$. For this purpose we need to impose some additional regularity on the nonlinearity $g$:

$(G4)$ $\quad g \in C^{0,1}(\mathbb{R}\times X_\alpha, X_\gamma)$, and
$\big[t \mapsto \partial_2 g\big(t,\bar{u}(t)\big)\big]$ is Hölder-continuous.

if $(G3)$ holds, and,

$(G4')$ $\quad g \in C^{0,1}(\mathbb{R}\times X_\alpha, X_0)$, and
$\big[t \mapsto \partial_2 g\big(t,\bar{u}(t)\big)\big]$ is Hölder-continuous,

if $(G3')$ holds. Furthermore we set for $t\in\mathbb{R}$:

$$\overline{A}(t) := A(t) - \partial_2 g\big(t,\bar{u}(t)\big).$$

It is easily verified, that the family $\big(\overline{A}(t)\big)_{t\ge 0}$, of closed linear operators in $X_0$, satisfies $(A1)$, $(A2')$ and $(A3)$ of Section 2 (possibly with different constants as $(A(t))_{t\ge0}$) and the periodicity condition $(A0)$ of Section 6.

Let $\overline{U}\colon \Delta_\infty \to \mathcal{L}(X_0)$ be the evolution operator corresponding to (20.1), and set

$$\overline{K} := \overline{U}(T,0).$$

To study the stability of $\bar{u}$ by linearization means studying the stability of the zero-solution of the linear variational equation, and infering that $\bar{u}$ has the same stability properties. Here, the *linear variational equation of* (20.1) *with respect to* $\bar{u}$, is the following $T$-periodic homogeneous linear initial value problem:

$$\begin{cases} \partial_t v + \overline{A}(t)v = 0 & \text{for } t > 0 \\ \qquad\quad v(0) = x\,. \end{cases} \tag{22.1}$$

The next proposition follows immediately from Theorem 16.10.

**22.1 Proposition**
*The period-map* $S\colon X_\alpha \supset D(S) \to X_\alpha$ *is continuously differentiable on* $D(S)$*, and*

$$S'(\bar{x}) = \overline{K}.$$

We are now ready to formulate the main results of this section. The first theorem is the so called *principle of linearized stability* and the second one the *principle of linearized instability*. A great portion of the qualitative theory for solutions of differential equations relies heavily on this kind of result, as they reduce the complex question of stability of a nonlinear problem to the question of stability of a linear one, for which the analysis is generally easier to carry out. This two results are direct consequences of Theorem 21.3 and a theorem from asymptotic fixed-point theory for differentiable maps, which we shall prove in the next subsection.

**22.2 Theorem**
*Assume that* $\sigma(\overline{K})$ *lies within the open complex unit-disk. Then the* $T$*-periodic solution* $u$ *of* (20.1) *is exponentially asymptotically stable.*

**22.3 Theorem**
*Assume that*

$$\sigma(\overline{K}) \cap \{\mu \in \mathbb{C}; |\mu| > 1\}, \tag{22.2}$$

*is a nonempty spectral set of* $\overline{K}$*. Then, the* $T$*-periodic solution* $u$ *of* (20.1) *is unstable.*

**22.4 Remark**
If $A(0)$ has compact resolvent, $\overline{K}$ is a compact operator, so that (22.2) is equivalent to asking that $\overline{K}$ has an eigenvalue with modulus strictly greater than 1. □

**B. Asymptotic fixed-point theorems:** Let $E$ be a Banach space and $F\colon E \supset D(F) \to E$. Suppose that $x_0 \in D(F)$ is a fixed-point of $F$, i.e $F(x_0) = x_0$. We shall study the stability properties of this fixed-point via linearization. But before stating and proving the above mentioned result from asymptotic fixed-point theory, we shall need a technical lemma on renorming.

**22.5 Lemma**
*Assume that* $L \in \mathcal{L}(E)$ *is a given operator and that we have a spectral decomposition* $\sigma(L) = \sigma_1 \dot{\cup} \sigma_2$ *with*

$$s_1 := \sup\{|\lambda|; \lambda \in \sigma_1\} < s_2 := \inf\{|\lambda|; \lambda \in \sigma_2\}.$$

*Suppose that $E = E_1 \oplus E_2$ is the corresponding decomposition. Then, for any $\varepsilon > 0$ there exists an equivalent norm, $\|\cdot\|_\varepsilon$, on $E$, such that for any $x = x_1 \oplus x_2 \in E_1 \oplus E_2$ we have*

$$\|x\|_\varepsilon = \|x_1\|_\varepsilon + \|x_2\|_\varepsilon,$$

*and*

$$\|Lx_1\|_\varepsilon \leq (s_1 + \varepsilon)\|x_1\|_\varepsilon$$

*as well as*

$$\|Lx_2\|_\varepsilon \geq (s_2 - \varepsilon)\|x_2\|_\varepsilon$$

*hold.*

**Proof**

With the obvious notation we have

$$L = L_1 \oplus L_2 \colon E_1 \oplus E_2 \to E_1 \oplus E_2.$$

We first define a norm, $\|\cdot\|_{1,\varepsilon}$, on $E_1$ by setting

$$\|x_1\|_{1,\varepsilon} := \sup_{n \geq 0} \frac{\|L_1^n x_1\|}{(s_1 + \varepsilon)^n}.$$

Obviously, we get that

$$\|x_1\| \leq \|x_1\|_{1,\varepsilon} \leq \Big(\sup_{n \geq 0} \frac{\|L_1^n\|}{(s_1 + \varepsilon)^n}\Big)\|x_1\|$$

holds. But since $\lim_{n\to\infty} \|L_1^n\|^{\frac{1}{n}} = s_1$, we readily obtain that $\|\cdot\|$ and $\|\cdot\|_{1,\varepsilon}$ are equivalent norms on $E_1$. Moreover,

$$\|L_1 x_1\|_{1,\varepsilon} = \sup_{n \geq 0} \frac{\|L_1^{n+1} x_1\|}{(s_1 + \varepsilon)^n} = (s_1 + \varepsilon)\|x_1\|_{1,\varepsilon}.$$

Observing that $0 \notin \sigma_2$ we see that $L_2$ is invertible. So we may define the norm

$$\|x_2\|_{2,\varepsilon} := \sup_{n \geq 0} \frac{\|L_2^{-n} x_2\|}{(s_2 - \varepsilon)^{-n}}$$

on $E$. Now, the fact that $\lim_{n\to\infty} \|L_2^{-n}\|^{\frac{1}{n}} = \frac{1}{s_2}$ easily implies that $\|\cdot\|_{2,\varepsilon}$ and $\|\cdot\|$ are equivalent norms on $E_2$. Moreover, we have

$$\|L_2^{-1} x_2\|_{2,\varepsilon} \leq (s_2 - \varepsilon)\|x_2\|_{2,\varepsilon},$$

and hence,

$$\|L_2 x_2\|_{2,\varepsilon} \geq (s_2 - \varepsilon)\|x_2\|_{2,\varepsilon}.$$

Putting

$$\|x\|_\varepsilon := \|x_1\|_{1,\varepsilon} + \|x_2\|_{2,\varepsilon}$$

we get a norm with the asserted properties. □

**22.6 Theorem**

*Let $E$ be a Banach space and $F\colon E \supset D(F) \to E$, a continuously differentiable map defined on the open set $D(F)$. Supose that $x_0$ is a fixed-point of $F$. Finally set $L := F'(x_0)$. Then the following statements are true:*

*(i) If $\sigma(L)$ is contained within the open complex unit-disk, then $x_0$ is an exponentially stable fixed-point of $F$.*

*(ii) If*

$$\sigma(L) \cap \{\mu \in \mathbb{C}; |\mu| > 1\} \tag{22.3}$$

*is a nonempty spectral set of $L$, then $x_0$ is an unstable fixed-point of $F$.*

**Proof**

Without loss of generality we may assume that $0 \in D(F)$ and $x_0 = 0$. We set

$$L := DF(0).$$

(*i*) By the previous lemma we may choose an equivalent norm $|\cdot|$ on $E$ such that $|L| < 1$ holds. Since $F$ is of class $C^1$ we find for each $\varepsilon > 0$ a $\delta > 0$ such that whenever $|x| \le \delta$ holds, we have that $x \in D(F)$ and

$$F(x) = Lx + r(x)$$

with $|r(x)| \le \varepsilon|x|$. Hence, $|F(x)| \le (|L| + \varepsilon)|x|$. Choose $\varepsilon > 0$ so small that $|L| + \varepsilon < 1$ holds. This implies that $|F(x)| < \delta$ and hence $|F^2(x)| < (|L| + \varepsilon)^2|x|$. Set now $\omega := -\log(|L| + \varepsilon) > 0$. Iterating the argument above gives that

$$|F^n(x)| \le e^{-n\omega}$$

holds for all $x \in \mathbb{B}_E(0, \delta)$ and $n \in \mathbb{N}$, showing the exponential stability of $x_0 = 0$.

(*ii*) Observe that $\sigma(L) = \sigma_1 \dot\cup \sigma_2$ is a spectral decomposition as in our previous lemma. We shall use the same notation as in that lemma. In particular we denote by $x = x_1 + x_2 \in E_1 \oplus E_2$ a generic element in $E$.

Choose now $a > 1$ and $\rho > 0$ as to satisfy $s_1 < a - \rho < a < s_2$. By the lemma we may choose an equivalent norm $|\cdot|$ on $E$ such that

$$|x| = |x_1| + |x_2|, \tag{22.4}$$

$$|L_1 x_1| \leq (a-\rho)|x_1|, \tag{22.5}$$

$$|L_2 x_2| \geq a|x_2|, \tag{22.6}$$

all hold.

For each $q > 0$ we define the set

$$K_q := \{x \in E; |x_1| \leq q|x_2|\}.$$

Note that $K_q$ is a cone with nonempty interior.

Assume for a moment that we had shown that for $\delta > 0$ small enough there exists a constant $\hat{a}$ such that

$$\begin{cases} \text{For all } y_1, \dots, y_n \in \overline{\mathbb{B}}_E(0,s) \text{ and } x \in \big(K \cap \overline{\mathbb{B}}_E(0,s)\big) \setminus \{0\} \text{ and all} \\ \lambda_1, \dots, \lambda_n \geq 0 \text{ such that } \sum_{j=1}^n \lambda_j = 1 \text{ we have} \\ \qquad v := \displaystyle\sum_{j=1}^{n} \lambda_j DF(y_j) \in \operatorname{int}(K) \qquad \text{as well as} \qquad |v| \geq \hat{a}|x|. \end{cases} \tag{22.7}$$

Observe now that

$$F(x) = \int_0^1 DF(tx)x\,dt.$$

Approximating this integral by Riemann sums and using (22.7) we see that whenever $x \in \big(K_q \cap \overline{\mathbb{B}}_E(0,s)\big) \setminus \{0\}$ holds, $F(x)$ lies in $\operatorname{int}(K)$ and

$$|F(x)| \geq \hat{a}|x| \tag{22.8}$$

is satisfied. If $F^k(x)$ remains in $(K_q \cap \overline{\mathbb{B}}_E(0,s))$ for $k = 1, \dots, n-1$ then, by (22.8),

$$|F^n(x)| \geq \hat{a}^n |x|.$$

Since, $\hat{a} > 1$ we see that for some $m \in \mathbb{N}$ the iterate $F^m(x)$ is still in $K_q$ but must leave $\overline{\mathbb{B}}_E(0,s)$. This proves the instability of $x_0 = 0$ and, hence, the second assertion of the theorem.

We still have to show that (22.7) holds. In order to do this note that by definition of $K_q$, for each $x \in K_q \setminus \{0\}$ we have $|L_1 x_1| \leq (a-\rho)|x_1| \leq (a-\rho)q|x_2|$ and, hence, by (22.6),

$$|L_1 x_1| \leq \big(1 - \frac{\rho}{a}\big) q |L_2 x_2| \tag{22.9}$$

and, therefore, $Lx \in \operatorname{int}(K)$.

Moreover, (22.6) and $|x| \le (1+q)|x_2|$, imply

$$|Lx| \ge |L_2 x_2| \ge \frac{a}{1+q}|x|. \tag{22.10}$$

From now on we shall assume that

$$\frac{a}{1+q} > 1$$

holds, which can be achieved by choosing $q$ small enough.

Note that since $F$ is of class $C^1$, we have that to each $\delta > 0$ there exists an $s > 0$ such that

$$|L - DF(y)| \le \delta \tag{22.11}$$

holds whenever $y \in E$ satisfies $|y| \le s$.

Using (22.10) and (22.11) we see that

$$\begin{aligned} |v_2| &\ge |L_2 x_2| - \Big(\sum_{j=1}^{n} \lambda_j \big|DF(y_j) - L)\big|\Big)|x| \ge \big(\frac{a}{1+q} - \delta\big)|x| \\ &\ge \big(1 + \delta\frac{1+q}{a}\big)|L_2 x_2| \end{aligned}$$

holds. Moreover from (22.9) we obtain that

$$\begin{aligned} |v_1| &\le |L_2 x_2| + \Big(\sum_{j=1}^{n} \lambda_j \big|DF(y_j) - L)\big|\Big)|x| \\ &\le \Big((1 - \frac{\rho}{a})q + \delta\frac{1+q}{a}\Big)|L_2 x_2| \end{aligned}$$

is satisfied. If we choose $\delta > 0$ so small that $\delta(1+q)^2 < \rho q$, this two inequalities yield

$$|v_1| < q|v_2|,$$

proving that $v \in \mathrm{int}(K)$.

From (22.11) and (22.10) we obtain that

$$|v| \ge |Lx| - \sum_{j=1}^{n} \lambda_j \big|DF(y_j) - L)\big| \ge \big(\frac{a}{1+q} - \delta\big)|x| \tag{22.12}$$

holds. Setting

$$\hat{a} := \frac{a}{1+q} - \delta$$

and making $\delta > 0$ small enough we see that $\hat{a} > 1$ satisfies the requirements of (22.7). This proves the theorem. □

**C. Semilinear initial-boundary problems:** We consider here Example 20.5(a) again, and consider the initial-boundary value problem (20.4) and its abstract formulation (20.5). In order that the differentiability requirements needed to apply the results of this section be met, we assume that condition (15.17) be satisfied.

Under this condition we have that the Nemitskii-operator $g : \mathbb{R} \times X_\alpha \to X_0$ is differentiable with respect to the second variable, uniformly on bounded sets. Moreover, if $(t_0, u_0) \in \mathbb{R} \times X_\alpha$ we have that

$$D_2 g(t_0, u_0) = \overline{M}(t_0) \in \mathcal{L}(X_\alpha, X_0)$$

where $\overline{M}(t_0)$ acts on a function $v \in X_\alpha$ in the following way

$$\big[\overline{M}(t_0)v\big](x) := \partial_\xi f\big(x, t_0, u_0(x), \nabla u_0(x)\big)v(x) + \sum_{j=1}^{n} \partial_{\zeta_j} f\big(x, t_0, u_0(x), \nabla u_0(x)\big)\partial_j v(x).$$

Of course if $f$ does not depend on $\nabla u$ $\overline{M}(t_0)$ is just the multiplication operator induced by the function $\partial_\xi f\big(\cdot\,, t_0, u_0(\cdot), \nabla u_0(\cdot)\big)$. Let now $u_0 \in C^1(\mathbb{R}, X_0) \cap C(\mathbb{R}, X_\alpha)$ be a $T$-periodic solution of (20.5). In Section 24 we shall see that the function $\big[(x,t) \mapsto u_0(t)(x)\big]$ is in fact a classical solution of (20.4). In particular, this implies that $\big[(x,t) \mapsto (u_0(t)(x), \nabla u_0(t)(x))\big]$ is Hölder continuous. This together with the regularity conditions we have imposed on $f$ implies that if for each $t \in \mathbb{R}$ we set

$$M(t) := D_2 g\big(t, u_0(t)\big) \in \mathcal{L}(X_\alpha, X_0)$$

we have that $[t \mapsto M(t)]$ is Hölder continuous. Hence, condition $(G)$ of Subsection A is met.

We choose $\frac{1}{2} + \frac{n}{2p} < \alpha \leq 1$ so that $X_\alpha \hookrightarrow C^{1+\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$. In particular the positive cone of $X_\alpha$ has nonempty interior. This will be important in a moment.

Considering the linearized equation

$$\partial_t v + A(t)v = M(t)v \qquad \text{for } t > 0$$

and the corresponding evolution operator $\overline{U}\colon \Delta_\infty \to \mathcal{L}(X_0)$, and setting

$$\overline{K} := \overline{U}(T, 0) \in \mathcal{L}(X_\alpha),$$

as in Subsection A, we note that, by the results of Section 13, $\overline{K}$ is a strongly positive compact operator, in the sense of Section 12. Hence, applying Krein-Rutman (Theorem 12.3), we see that the spectral radius, $r_0 := \mathrm{r}(\overline{K})$, of $\overline{K}$ is an eigenvalue of $\overline{K}$. Moreover, it is the only eigenvalue of modulus $r_0$ and it is algebraically simpleand the corresponding eigenfunction lies in the interior of the positive cone of $X_\alpha$. This means that the linear stability and instability of $u_0$ are completly determined by whether $r_0 < 1$ or $r_0 > 1$ holds, respectively. Or equivalently by whether $\mu_0 > 0$ or $\mu_0 < 0$ holds, where $\mu_0$ is defined as

$$\mu_0 := -\frac{1}{T}\log(r_0).$$

The number $\mu_0$ is called the principal eigenvalue of the *periodic parabolic eigenvalue problem*

$$\text{(22.13)} \qquad \begin{cases} \partial_t \varphi + \overline{\mathcal{A}}(x,t,D)\varphi = \mu\varphi & \text{in } \Omega \times \mathbb{R} \\ \mathcal{B}(x,D)\varphi = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ \varphi(\,\cdot\,,0) = \varphi(\,\cdot\,,T) & \text{in } \Omega, \end{cases}$$

and is the only eigenvalue of (22.13) with a corresponding positive eigenfunction.

**22.7 Remarks**

(a) The study of periodic-parabolic eigenvalue problems in connection with stability of periodic solutions of semilinear parabolic equations has been the subject of a recent book by P. Hess [**67**]. There, the reader will find a variety of nice applications. Compare also Section 14.

(b) Of course one can do the same as above for semilinear initial value problems on the whole of $\mathbb{R}^n$. We refrain from carrying this out here. We remark though that there are some technical complications arising from the fact that the operator $\overline{K}$ is neither strongly positive nor compact for the unbounded domain case. Actually, it is the lack of compactness which makes things hard, since $\overline{K}$ is still irreducible a property which would suffice for the application of the theorem of Krein-Rutman. For more details consult [**38**] and [**80**]. □

**Notes and references:** The proof of the principles of linearized stability and instability for fixed-points of differentiable maps was taken from [**72**]. Stability via linearization has been one of the major achievements in the field of differential equations and builds the corner stone in a variety of applications. Other references for parabolic problems are for example Henry [**66**] and Lunardi [**94**]. See also [**67**] for a proof in the special case of parabolic initial-boundary value problems using arguments involving positivity.

## 23. Boundedness and stability in weaker norms

In this section we continue to study the semilinear time-periodic initial value problem (20.2) under the same assumptions as in Section 20. We shall deal with two different questions which are linked by the use of the same technique, based on Gronwall's inequality, to solve them. In both of them the goal is to obtain a certain property of the solution which is measured in terms of the $X_\alpha$-norm from the same property but in a weaker norm. The first one is when does an a priori bound in a weaker norm imply one in the $X_\alpha$-norm, and the second is when does stability in a weaker norm force stability in the $X_\alpha$-norm.

**A. Boundedness:** We shall make the following assumption on the nonlinearity, which is essentially the same as the one in Section 17. Assume that $(G3')$ of Section 20 holds and that

$(G5)$ $Z$ is a Banach space satisfying $X_\alpha \hookrightarrow Z \hookrightarrow X_0$, and there exist an increasing function $\lambda\colon \mathbb{R}_+ \to \mathbb{R}_+$, and a constant $N \geq 0$, such that for any $\rho > 0$

$$\|g(t,y)\|_0 \leq \lambda(\rho)\big(N + \|y\|_\alpha\big)$$

for all $t \in [0,T]$, and $y \in X_\alpha$, satisfying $\|y\|_Z \leq \rho$.

is met.

**23.1 Remark**

Observe that if $g$ satisfies $(G5)$ then for each $\mu \in \mathbb{R}$ the function $g_\mu := g + \mu$ also satisfies $(G5)$ (possibly with other constants). This will be important later on. □

Before stating and proving the result on bounded solutions we shall need the following easy technical lemma.

**23.2 Lemma**

*Let $\omega > 0$, $\beta, \in [0,1)$ AND $\nu \in (0,1)$ be given. Assume that $w\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying*

$$w(t) \leq c_1 + c_2 \int_0^t e^{-(t-\tau)\omega}(t-\tau)^{-\beta} w(\tau)^\nu \, d\tau$$

*for all $t \geq 0$. Then, $w$ is bounded by a constant depending only on $w(0)$.*

**Proof**

It is easy to see that for $N > 0$ large enough we have that

$$N > c_1 + c_2 \int_0^t e^{-(t-\tau)\beta}(t-\tau)^{-\beta} N^\nu \, d\tau$$

holds. Choose such an $N > 0$ so that $w(0) < N$. Then, by continuity, $w(t) < N$ for all small enough $t > 0$. Assume that there exists a $t_0 >$ such that $w(t) < N$ for all $t \in [0, t_0)$ but $w(t_0) = N$. Then

$$\begin{aligned} N = w(t_0) &\leq c_1 + c_2 \int_0^{t_0} e^{-(t_0-\tau)\omega}(t_0 - \tau)^{-\beta} w(\tau)^\nu \, d\tau \\ &\leq c_1 + c_2 \int_0^{t_0} e^{-(t_0-\tau)\omega}(t_0 - \tau)^{-\beta} N^\nu \, d\tau \\ &< N \end{aligned}$$

which is impossible. Hence, $w(t) < N$ for all $t \geq 0$, proving the lemma. □

**23.3 Theorem**

*Let $R > 0$ and $\alpha \leq \beta < 1$ be given. Assume that $x \in X_\beta$ is such that*

$$\|u(t; x)\|_Z \leq R$$

*holds for all $t \in J(0, x)$.Then,*

$$\|u(t; x)\|_\beta \leq c(R, \beta)$$

*holds for all $t \in J(0, x)$. In particular choosing $\beta = \alpha$ we see that $J(0, x) = \mathbb{R}^+$.*

**Proof**

Let $\omega_0$ be such that

$$\mathrm{r}\big(U(T, 0)\big) = e^{-T\omega_0}.$$

Without loss of generality we may assume that $\omega_0 > 0$ since, otherwise, we write (20.1) as

$$\partial_t u + \big(A(t) - \mu\big)u = g_\mu\big(t, u(t)\big)$$

with $g_\mu$ defined as in Remark 23.1 and $\mu$ large enough.

$(i)$ We first deal with the case $\alpha < \beta$. Fix an $0 < \omega < \omega_0$ and choose $\nu \in (0, 1)$ such that $\alpha = \nu\beta$. By the reiteration theorem (3.3) and since $Z \hookrightarrow X_0$ there exists a constant $\hat{c} > 0$ such that

$$\|y\|_Z \leq \hat{c}\|y\|_0^{1-\nu}\|y\|_\beta^\nu$$

holds for all $x \in X_\beta$.

Hence, using this, the variation-of-constants formula, $(G5)$ and Lemma 6.6 and 6.8 we have

$$\begin{aligned} \|u(t; x)\|_\beta &\leq \|U(t, 0)\|_{\beta,\beta}\|x\|_\beta + \int_0^t \|U(t, \tau)\|_{0,\beta}\|g\big(\tau, u(\tau; x)\big)\|_0 \, d\tau \\ &\leq M(\beta, \omega)\|x\|_\beta + N(0, \beta, \omega)\lambda(\rho)N \int_0^t e^{-(t-\tau)\omega}(t - \tau)^{-\beta} \, d\tau \\ &+ N(0, \beta, \omega)\lambda(\rho) \int_0^t e^{-(t-\tau)\omega}(t - \tau)^{-\beta}\|u(\tau; x)\|_Z \, d\tau \\ &\leq c_1 + c_2 \int_0^t e^{-(t-\tau)\omega}(t - \tau)^{-\beta}\|u(\tau; x)\|_\beta^\nu \, d\tau \end{aligned}$$

where we have set $c_1 := M(\beta,\omega)\|x\|_\beta + N(0,\beta,\omega)\lambda(\rho)N\int_0^t e^{-(t-\tau)\omega}(t-\tau)^{-\beta}\,d\tau$ and $c_2 := N(0,\beta,\omega)\lambda(\rho)\hat{c}R^{1-\nu}$. The assertion follows now from the previous lemma.

$(ii)$ To prove boundedness in case that $x \in X_\alpha$ we choose any nonzero $t_0 \in J(o,x)$ and $\alpha < \beta < 1$. Since $[o,t_0]$ is compact the set $\{u(t;x); t \in [0,t_0]\}$ is bounded in $X_\alpha$. By the smoothing property $u(t_0;x) \in X_1 \subset X_\beta$ so that we may apply part $(i)$ to obtain boundedness in $X_\beta$ and thus in $X_\alpha$. □

**23.4 Remark**

The applications of this result parallel those in Section 17.B. The reader should encounter no dificculties in giving precise formulations. □

**B. Stability:** We now turn to the question of stability in weaker norms. We assume that $(G3')$ of Section 20 holds and that

$(G6)$ $Z$ is a Banach space satisfying $X_\alpha \hookrightarrow Z \hookrightarrow X_0$, and there exist an increasing function $\lambda\colon \mathbb{R}_+ \to \mathbb{R}_+$, and a constant $N \geq 0$, such that for any $\rho > 0$

$$\|g(t,x) - g(t,y)\|_0 \leq \lambda(\rho)\|x-y\|_\alpha$$

for all $t \in [0,T]$, and $x, y \in X_\alpha$, satisfying $\|x\|_Z, \|y\|_Z \leq \rho$,

is met.

**23.5 Remark**

Observe that from $(G6)$ it follows that for each $\rho > 0$

$$\|g(t,y)\|_0 \leq \|g(t,0)\|_0 + \lambda(\rho)\|y\|_\alpha$$

holds for all $t \in [0,T]$ and $y \in X_\alpha$ with $\|y\|_Z \leq \rho$. Hence, by the results of the previous subsection we immediately obtain that a solution of (20.2) which is bounded in the $Z$-norm must be a global solution. □

In a series of lemmas we shall prove the following result:

**23.6 Theorem**

*Let $R > 0$ and $\rho > 0$ be given and suppose that $x, y \in X_\alpha$ are such that*

$$\|u(t;x)\|_Z, \|u(t;y)\|_Z \leq R$$

*for all $t \geq 0$. Furthermore assume that*

$$\|u(t;x) - u(t;y)\|_Z \leq \rho$$

*holds for all* $t \geq 0$. *Then,*

$$\|u(t;x) - u(t;y)\|_\alpha \leq c(\rho, R)$$

*where* $c(\rho, R) > 0$ *is a constant satisfying for all* $t \geq 0$

$$\lim_{\rho \to 0} c(\rho, R) = 0.$$

We start with

**23.7 Lemma**
*Let* $R > 0$ *be given and suppose that* $x, y \in X_\alpha$ *are such that* $\|u(t;x)\|_Z$, $\|u(t;y)\|_Z \leq R$ *for all* $t \geq 0$. *Then,*

$$\|u(t;x) - u(t;y)\|_\alpha \leq c_1(\alpha, R)\|x - y\|_\alpha$$

*holds for all* $t \in [0, T]$, *where* $c_1(\alpha, R) > 0$ *is a suitable constant.*

**Proof**
By Lemma 5.2 we have that

$$\|U(t,s)\|_{\alpha,\alpha} \leq c(\alpha, T) \qquad \text{and} \qquad \|U(t,s)\|_{0,\alpha} \leq c(\alpha, T)(t-s)^{-\alpha}$$

holds for all $(t, s) \in \Delta_T$, for a suitable constant $c(\alpha, T) > 0$. Hence, we obtain by the variation-of-constant formula and assumption $(G6)$

$$\begin{aligned}
&\|u(t;x) - u(t;y)\|_\alpha \\
&\qquad \leq \|U(t,0)\|_{\alpha,\alpha}\|x - y\|_\alpha + \int_0^t \|U(t,\tau)\|_{0,\alpha}\|g\big(\tau, u(\tau;x)\big) - g\big(\tau, u(\tau, y)\big)\|_0 \, d\tau \\
&\qquad \leq c(\alpha, T)\|x - y\|_\alpha + c(\alpha, T)\lambda(\rho) \int_0^t (t-\tau)^{-\alpha}\|u(\tau;x) - u(\tau;y)\|_\alpha \, d\tau.
\end{aligned}$$

Applying Gronwall's inequality (Corollary 16.6) we immediately obtain the assertion. □

**23.8 Lemma**
*Let* $R > 0$ *be given and suppose that* $x, y \in X_\alpha$ *are such that* $\|u(t;x)\|_Z, \|u(t;y)\|_Z \leq R$ *for all* $t \geq 0$. *Then,*

$$\|u(t;x) - u(t;y)\|_\alpha \leq c_2(\alpha, R)t^{-\alpha}\|x - y\|_Z$$

*holds for all* $t \in (0, T]$, *where* $c_2(\alpha, R) > 0$ *is a suitable constant.*

**Proof**
The proof is almost the same as that of the previous lemma. By the variation-of-constants formula and Lemma 5.2 we get

$$\begin{aligned}
&\|u(t;x)-u(t;y)\|_\alpha \\
&\qquad \le \|U(t,0)\|_{0,\alpha}\|x-y\|_0 + \int_0^t \|U(t,\tau)\|_{0,\alpha}\|g\big(\tau,u(\tau;x)\big)-g\big(\tau,u(\tau,y)\big)\|_0\,d\tau \\
&\qquad \le c(\alpha,\alpha)\|x-y\|_\alpha + c(0,\alpha)\lambda(\rho)\int_0^t (t-\tau)^{-\alpha}\|u(\tau;x)-u(\tau;y)\|_\alpha\,d\tau.
\end{aligned}$$

Again the assertion follows by an immediate application of Gronwall's inequality. □

W have the following trivial consequence.

**23.9 Corollary**
*Let $R>0$ be given and suppose that $x,y\in X_\alpha$ are such that $\|u(t;x)\|_Z,\|u(t;y)\|_Z\le R$ for all $t\ge 0$. Then,*

$$\|S(x)-S(y)\|_\alpha \le c_2(\alpha,R)T^{-\alpha}\|x-y\|_Z$$

*holds.*

**Proof of Theorem 23.6**
From the estimate in the $Z$-norm it follows that

$$\|S^{m-1}(x)-S^{m-1}(y)\|_Z\le\rho \qquad \text{and} \qquad \|S^{m-1}(x)\|_Z,\|S^{m-1}(y)\|_Z\le R$$

holds for all $t\ge 0$ and $m\in\mathbb{N}$. Hence, the above corollary yields

$$\|S^m(x)-S^m(y)\|_\alpha\le c_2(\alpha,R)T^{-\alpha}\rho.$$

With Lemma 23.7 we immediately obtain for all $t\in[0,T]$ and $m\in\mathbb{N}$ that

$$\begin{aligned}
\|u(t+mT;x)-u(t+mT;y)\|_\alpha &= \|u(t;S^m(x))-u(t;S^m(y))\|_\alpha \\
&\le c_1(\alpha,R)\|S^m(x)-S^m(y)\|_\alpha \le c_2(\alpha,R)T^{-\alpha}\rho
\end{aligned}$$

which proves the assertion. □

**23.10 Remarks**
(a) It is an easy exercise to see how Theorem 23.6 implies that stability with respect to the $Z$-norm forces stability with respect to the $X_\alpha$-norm.

(b) By arguing in a slightly more careful way we could prove that exponential stability in the $Z$-norm implies exponential stability in the $X_\alpha$-norm.

(c) It is clear how to apply these results to the examples treated in Subsection 17.B. In this way we can give criteria for knowing when stability with respect to the $L_\infty$-norm implies stability with respect to the $X_\alpha$-norm. □

**Notes and references:** The method of proof of all the above results is of course standard. Although the results seem to be in use we could not find a reference. For some more results on boundedness of solutions which are outside the scope of this book consult Redlinger [**103**], [**104**]. Compare also the remarks at the end of Section 17.

# VI. Applications

It is the aim of this chapter to show how to apply the abstract theory developed in the previous chapters to concrete reaction-diffusion equations as far as this was not done in the various examples. We first deal with initial-boundary value problems and establish among other things the relationship between solutions in the 'classical sense' and in the sense of the corresponding abstract equation. In a second section we treat initial value problems on the whole of $\mathbb{R}^n$. In the final section, we show that the abstract results can also be applied to a nonstandard epidemics model.

## 24. Reaction-diffusion equations in bounded domains

In this section we shall be concerned with initial-boundary value problems of the form

$$(24.1)\qquad \begin{cases} \partial_t u(x,t) + \mathcal{A}(x,t,D)u(x,t) = f\big(x,t,u(t,x),\nabla u(x,t)\big) & (x,t) \in \Omega \times (0,T] \\ \mathcal{B}(x,D)u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

As usual we assume that $\Omega$ is a bounded domain of class $C^\infty$, and that $\mathcal{A}(x,t,D)$ and $\mathcal{B}(x,t)$ satisfy the same hypotheses as in Example 2.9(d). Moreover, we assume that $f$ satisfies either (15.12) or (15.17).

By a (*classical*) *solution* of problem (24.1), we mean a function

$$u \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}\big(\overline{\Omega} \times (0,T]\big).$$

satisfying (24.1). As we have seen in Subsection 15.D we can formulate the above problem in an $L_p$-setting which fits in the theory of abstract parabolic evolution equations of the form

$$(24.2)\qquad \begin{cases} \dot{u} + A(t)u = g\big(t,u(t)\big) & 0 < t \le T \\ u(0) = u_0 \end{cases}$$

in $X_0 := L_p(\Omega)$, where $A(t)$ is for any $t \in [0,T]$ a closed operator on $X_0$ with domain $D(A(t)) \doteq X_1 := W^2_{p,\mathcal{B}}(\Omega)$, and $g$ is the Nemitskii operator induced by $f$.

By means of the theory developed in Section 16, we conclude that equation (24.2) has a unique solution

$$u \in C\big([0,T], X_\alpha\big) \cap C^1\big((0,T], X_0\big)$$

for all initial conditions $u_0$ lying in a suitable interpolation space $X_\alpha$. In the case of the $L_p$-realization of an initial-boundary value problem we have defined $X_\alpha$ for $\alpha \in (0,1)$ by

$$X_\alpha = [X_0, X_1]_\alpha \qquad \text{or} \qquad X_\alpha = (X_0, X_1)_{\alpha,p}\,.$$

For these spaces we have given exact characterizations in Theorem 4.16. In the sequel we shall always choose one of these interpolation scales without further comment.

In a first subsection we investigate regularity properties of solutions of (24.2). Then, we prove a comparison result for sub- and supersolutions of (24.1) and in the last subsection we apply these results to show global existence of solutions of the logistic equation with diffusion term.

**A. Regularity of solutions:** It is rather natural to ask whether the solution of the abstract problem (24.2) is a solution of (24.1) when correctly interpreted and vice versa. It turns out that the answer is positive provided the initial value is smooth enough. We shall actually show that every solution with $u_0 \in X_1$ of the abstract problem is a *regular solution* of (24.1), that is

$$u \in C\big(\overline{\Omega} \times [0,T]\big) \cap C^{2+\eta,1+\frac{\eta}{2}}\big(\overline{\Omega} \times (0,T]\big).$$

To show this we need the following special case of the Schauder theory for parabolic initial-boundary value problems (cf. [**87**], Section IV.5).

**24.1 Theorem**
*Let the above assumptions be satisfied. Then the inhomogeneous linear problem*

$$\begin{cases} \partial_t u(x,t) + \mathcal{A}(x,t,D)u(x,t) = h(x,t) & (x,t) \in \Omega \times (0,T] \\ \mathcal{B}(x,D)u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

*has for all $u_0$ in $C^{2+\eta}(\overline{\Omega})$ and $h$ in $C^{\eta,\frac{\eta}{2}}\big(\overline{\Omega} \times [0,T]\big)$ a unique classical solution $u$ lying in $C^{2+\eta,1+\frac{\eta}{2}}\big(\overline{\Omega} \times [0,T]\big)$.*

We are now ready to prove our regularity result.

**24.2 Theorem**
*Let $u_0 \in X_1 = W_{p,\mathcal{B}}(\Omega)$ with $p > n$. If $f$ does not depend explicitly on $\nabla u$ we assume that $p > n/2$. Then,*

(*a*) *Each classical solution of* (24.1) *is a solution of the abstract equation* (24.2).

(*b*) *Each global solution of* (24.2) *is a regular solution of* (24.1).

**Proof**

(*a*) Let $u$ be a solution of (24.1) with $u_0 \in X_1$. Then, obviously $u \in C\big([0,T],X_0\big) \cap C^1\big((0,T],X_0\big)$ and $u(t) := u(\cdot\,,t) \in X_1$ for all $t \in [0,T]$. Moreover, $u$ satisfies (24.2). To see that $u \in C\big([0,T],X_\alpha\big)$ for each $\alpha \in (0,1)$ observe that

$$\big[t \mapsto f\big(\cdot\,,t,u(\cdot\,,t),\nabla u(\cdot\,,t)\big)\big] \in C\big([0,T],X_0\big).$$

The assertion follows then from (15.2) and Corollary 5.6.

(*b*) Let $u$ be a solution of (24.2) with $u_0 \in X_1$. Fix $\alpha \in \big(\frac{1}{2}+\frac{n}{2p},1\big)$ and observe that the solution $u$ lies in $C\big([0,T],X_\alpha\big)$. Since by Corollary 4.17,

$$X_\alpha \hookrightarrow C^{1+\nu}(\overline{\Omega}) \tag{24.3}$$

for some $\nu \in (0,1)$, we see that $g\big(\cdot\,,u(\cdot)\big) \in C\big([0,T],X_0\big)$. From (15.2), Corollary 5.6 and the fact that $x \in X_1$ we deduce now that $u \in C^{\beta-\alpha}\big([0,T],X_\alpha\big)$ for all $\beta \in (\alpha,1)$. Together with (24.3) this implies that

$$u \in C^{\beta-\alpha}\big([0,T],C^{1+\nu}(\overline{\Omega})\big) \tag{24.4}$$

for $\beta \in (\alpha,1)$. In particular, we see that $u \in C\big(\overline{\Omega}\times[0,T]\big)$ which was one of the assertions.

Put for all $(x,t) \in \overline{\Omega}\times[0,T]$

$$h(x,t) := f\big(x,t,u(t,x),\nabla u(x,t)\big).$$

Then, (24.4) and the assumptions on $f$ make sure that

$$h \in C^{\sigma,\frac{\sigma}{2}}\big(\overline{\Omega}\times[0,T]\big) \tag{24.5}$$

for a suitable $\sigma \in (0,\eta)$. Let $\varepsilon > 0$ be arbitrary but fixed. Moreover, let $\varphi \in C^\infty(\mathbb{R}_+)$ be a nonnegative function satisfying $\varphi(t) = 0$ for $0 \le t \le \varepsilon/4$ and $\varphi(t) = 1$ for $t \ge \varepsilon/2$. Consider now the initial-boundary value problem

$$\begin{cases} \partial_t v + \mathcal{A}(x,t,D)v = \varphi(t)h(x,t) + \dot{\varphi}(t)u(x,t) =: m(x,t) & (x,t) \in \Omega\times(0,T] \\ \mathcal{B}(x,D)v = 0 & (x,t) \in \partial\Omega\times(0,T] \\ v(x,0) = 0 & x \in \Omega. \end{cases} \tag{24.6}$$

By (24.5) and the choice of $\varphi$, $m \in C^{\sigma,\frac{\sigma}{2}}\big(\overline{\Omega}\times[0,T]\big)$. From Theorem 24.1 we conclude now that $v \in C^{2+\sigma,1+\frac{\sigma}{2}}\big(\overline{\Omega}\times[0,T]\big)$. Part (*a*) of this theorem asserts now that $v$ is the unique solution of the abstract problem

$$\begin{cases} \dot{v} + A(t)v = m(t) & 0 < t \le T \\ v(0) = 0 \end{cases} \tag{24.7}$$

in $X_0$. On the other hand, $\varphi u$ is also a solution of (24.7). Hence, $v = \varphi u$ and by definition of $\varphi$ we have that $u \in C^{2+\sigma,1+\frac{\sigma}{2}}\big(\overline{\Omega} \times [\frac{\varepsilon}{2}, T]\big)$. In particular, this implies that $h \in C^{\eta,\frac{\eta}{2}}\big(\overline{\Omega} \times [0,T]\big)$.

Choose now a nonnegative function $\psi \in C^\infty(\mathbb{R}_+)$ such that $\psi(t) = 0$ for $0 \le t \le \varepsilon/2$ and $\psi(t) = 1$ for $t \ge \varepsilon$ and consider the equation

$$\begin{cases} \partial_t v + \mathcal{A}(x,t,D)v = \psi(t)h(x,t) + \dot{\psi}(t)u(x,t) =: m_1(x,t) & (x,t) \in \Omega \times (0,T] \\ \mathcal{B}(x,D)v = 0 & (x,t) \in \partial\Omega \times (0,T] \\ v(x,0) = 0 & x \in \Omega. \end{cases}$$

Observe that $m_1 \in C^{\eta,\frac{\eta}{2}}\big(\overline{\Omega} \times [0,T]\big)$. Thus, a similar argument as the one above shows that $u \in C^{2+\eta,1+\frac{\eta}{2}}\big(\overline{\Omega} \times [\varepsilon,T]\big)$ and that $u$ satisfies (24.1) on $\overline{\Omega} \times [0,T]$. Since $\varepsilon > 0$ was arbitrarily chosen, the assertion of the theorem follows.

If $f$ is independent of $\nabla u$, an analysis of the above proof shows that all the arguments hold whenever $p > n/2$ and $\alpha \in \big(\frac{n}{2p}, 1\big]$ □

**24.3 Remark**

(a) The above theorem remains true if we only require, that $u_0 \in X_\beta$ for some $\frac{1}{2} + \frac{n}{2p} < \beta \le 1$. In case $f$ is independent of $\nabla u$, we need only $p > \frac{n}{2}$ and $u_0 \in X_\beta$ for some $\beta \in \big(\frac{n}{2p}, 1\big]$. This is easily seen from the proof.

(b) Let $u$ be the (global) solution of (24.2) with arbitrary initial condition. Since $u(\varepsilon) \in X_1$ for all $\varepsilon > 0$ for which the solution exists, we conclude from the above theorem, that

$$u \in C^{2+\eta,1+\frac{\eta}{2}}\big(\overline{\Omega} \times (0,T]\big)$$

holds. □

As a corollary we would like to give a theorem on the classical solvability of the initial-boundary value problem (24.1).

**24.4 Corollary**

*Let $p > n$ and $u_0 \in X_\alpha$ for some $\alpha \in \big(\frac{1}{2} + \frac{n}{2p}, 1\big]$. Then, (24.1) has a unique maximal regular solution. Moreover, if $f$ is independent of $\nabla u$, it suffices to assume that $p > \frac{n}{2}$ and $u_0 \in X_\alpha$ for some $\alpha \in \big(\frac{n}{2p}, 1\big]$.*

**Proof**

Consider the $L_p$-realization (24.2) of (24.1). By Theorem 16.2, the abstract equation has a unique maximal solution. Now the assertion follows from Theorem 24.2 and Remark 24.3(a). □

**B. Comparison theorems:** We have seen in Section 17 that, under certain conditions on the nonlinearity, it is possible to get global existence of a solution, provided we have an a priori bound for the solution in an intermediate space $Z$ between $X_1$ and $X_0$. One possibility to obtain such an a priori bound is a comparison principle for sub- and supersolutions for equation (24.1). This shall be carried out in Corollary 24.8.

Comparison principles have proved to be very powerful tools to investigate the asymptotic behaviour of solutions of reaction-diffusion equations of the form (24.1) (see e.g. [**67**]).

First of all a few definitions are in order.

**24.5 Definition**
Let $v \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}\big(\overline{\Omega} \times [0,T]\big)$ satisfy the differential inequalities

$$(24.8) \quad \begin{cases} \partial_t u(x,t) + \mathcal{A}(x,t,D)u(x,t) \geq f\big(x,t,u(t,x),\nabla u(x,t)\big) & (x,t) \in \Omega \times (0,T] \\ \mathcal{B}(x,D)u(x,t) \geq 0 & (x,t) \in \partial\Omega \times (0,T]. \end{cases}$$

Then, $v$ is called a *supersolution* of (24.1). If the reverse inequalities hold in (24.8), $v$ is said to be a *subsolution* of (24.1). If $v$ is not a solution, it is called a *strict super-* or a *strict subsolution* of (24.1) respectively. □

In the sequel we use the notion of ordered Banach spaces introduced in Sections 12 and 13.

**24.6 Theorem**
*Let $u$ and $v$ be sub- and supersolutions of* (24.1) *respectively such that $u(\cdot\,,0) \leq v(\cdot\,,0)$ on $\overline{\Omega}$. Then,*

$$u(\cdot\,,t) \leq v(\cdot\,,t)$$

*holds on $\overline{\Omega}$ for all $t \in [0,T]$. Moreover, if $u(\cdot\,,0) < v(\cdot\,,0)$, then*

$$(24.9) \qquad u(\cdot\,,t) \ll v(\cdot\,,t)$$

*in $C^1_{\mathcal{B}}(\overline{\Omega})$ for all $t \in (0,T]$. If $\mathcal{B}$ is not the Dirichlet boundary operator,* (24.9) *holds in $C(\overline{\Omega})$ for all $t \in (0,T]$.*

**Proof**
Put $w := v - u$ and subtract the inequalities for $u$ and $v$. Then we obtain

$$(24.10) \quad \begin{cases} \partial_t w + \mathcal{A}(t)w \geq f\big(x,t,v,\nabla v)\big) - f\big(x,t,u,\nabla u)\big) & (x,t) \in \Omega \times (0,T] \\ \mathcal{B}w \geq 0 & (x,t) \in \partial\Omega \times (0,T] \\ w(x,0) \geq 0 & x \in \Omega. \end{cases}$$

The difference on the right hand side of the first inequality may be written in the form

$$(24.11) \qquad f\big(x,t,v,\nabla v)\big) - f\big(x,t,u,\nabla u)\big) = \sum_{i=1}^{n} d_i(x,t)\partial_i w + d_0(x,t)w,$$

where we have set for any $j = 0, \dots, n$

$$d_j(x,t) := \int_0^1 \partial_{\xi_j} f\big(x, t, v(x,t) + \tau w(x,t), \nabla v(x,t) + \tau \nabla w(x,t)\big)\, d\tau$$

for all $(x,t) \in \overline{\Omega} \times [0,T]$. Here we have denoted a generic point of $\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^n$ by $(x, t, \xi_0, \xi_1, \dots, \xi_n)$.

Setting $\tilde{\mathcal{A}}(t) := \mathcal{A}(t) - \sum_{i=1}^n d_i \partial_i - d_0$, we see that $w$ satisfies the differential inequality

$$\begin{cases} \partial_t w + \tilde{\mathcal{A}}(t) w \geq 0 & (x,t) \in \Omega \times (0,T] \\ \mathcal{B}(x,D) w \geq 0 & (x,t) \in \partial\Omega \times (0,T] \\ w(x,0) \geq 0 & x \in \Omega. \end{cases}$$

The assertions are now an easy consequence of Theorem 13.5. □

In particular, the above theorem hold for solutions with order related initial data.

**24.7 Corollary**

*Let $u, v$ be solutions of problem* (24.2) *with $u(0) \leq v(0)$. Then, $u(t) \leq v(t)$ for all $t \in [0,T]$. Moreover, if $u(0) < v(0)$, then*

$$u(t) \ll v(t) \tag{24.13}$$

*holds in $C^1_{\mathcal{B}}(\overline{\Omega})$ for all $t \in (0,T]$. If $\mathcal{B}$ is not the Dirichlet boundary operator,* (24.13) *holds in $C(\overline{\Omega})$ for all $t \in (0,T]$.*

**Proof**

By Theorem 24.2 and 24.5, all the assertions of the theorem hold whenever $u(0), v(0) \in X_1$. Since the solution depends continuously on the initial value by Theorem 16.8, and since $X_1 \overset{d}{\hookrightarrow} X_\alpha$ for all $\alpha \in [0,1)$, the first part of the theorem is proved.

Suppose that $u(0) < v(0)$ holds. By continuity of the solution, we conclude from the first part that $u(\varepsilon) < v(\varepsilon)$ whenever $\varepsilon > 0$ is sufficiently small. Since $u(\varepsilon), v(\varepsilon) \in X_1$, the assertion follows. □

**24.8 Corollary**

*Suppose that $\underline{u}$ and $\overline{u}$ are sub- and supersolutions of* (24.1) *respectively, and that $f$ satisfies growth condition* (17.3)*. Assume in addition, that $\big[\underline{u}(0), \overline{u}(0)\big]_{L_p(\Omega)} \cap X_\alpha \neq \emptyset$, where $\alpha$ is chosen as in Corollary* 24.4*. Then, there exists a unique global regular solution of* (24.1) *for all $u_0 \in \big[\underline{u}(0), \overline{u}(0)\big]_{L_p(\Omega)} \cap X_\alpha$.*

**Proof**

By Corollary 24.4 there exists a unique maximal regular solution $u$ of (24.1). From Theorem 24.6 we have that $\underline{u}(\cdot\,,t) \leq u(\cdot\,,t) \leq \overline{u}(\cdot\,,t)$ holds as long as $u$ exists. This implies that

$$\|u(t)\|_\infty \leq \max\big\{\|\underline{u}(t)\|_\infty, \|\overline{u}(t)\|_\infty\big\}$$

for all $t > 0$ for which $u(t)$ exists. In particular, $\|u(t)\|_\infty$ remains bounded on the existence interval uniformly in $t$. We are thus in a situation to apply Corollary 17.2 and hence, the assertion follows. ☐

**C. The logistic equation:** We illustrate the results obtained in the previous subsection by showing some simple properties of the logistic equation. The equation under consideration is

$$
\begin{cases} \partial_t u + \mathcal{A}(t)u = mu - bu^2 & (x,t) \in \Omega \times (0,\infty) \\ \mathcal{B}u = 0 & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{24.12}
$$

where $\Omega$, $\mathcal{A}(t)$ and $\mathcal{B}$ are as in Subsection A with $a_0 \equiv 0$ and $b_0 \geq 0$. Moreover, assume that $m, b \in C^{\eta,\frac{\eta}{2}}\big(\overline{\Omega} \times [0,T]\big)$ with $b \gg 0$.

The equation describes the evolution of the population density of a species living in a domain $\Omega$, called the habitat. The second and first order term in $\mathcal{A}(t)$ takes into account the geographical spread of the population caused by diffusion and a drift, respectively. The time dependence of the coefficients reflects the fact that the diffusion matrix and the drift vector are subject, for example, to seasonal variations for example. The space dependence describes the spacial inhomogeneity of the habitat. The term $m - bu$ is a growth rate, which depends on the population density. This growth rate becomes negative if the population density is large, which describes the limiting effects of crowding. Dirichlet boundary conditions express the inhospitality of the boundary. Finally the equation is subject to an initial density $u_0$.

This equation arises for instance when investigating semisimple solutions of Volterra-Lotka systems (see Chapters IV and V in [**67**]).

Equation (24.12) is of the form (24.1) with

$$f(x,t,\xi) = m(x,t)\xi - b(x,t)\xi^2$$

for all $(x,t,\xi) \in \overline{\Omega} \times [0,T] \times \mathbb{R}$. Obviously, $f$ satisfies both of the standard assumptions (15.12) and (15.17).

It is evident that $u \equiv 0$ is a solution of (24.12). On the other hand, if $R > 0$ is a sufficiently large constant, it holds that

$$0 = \partial_t R + \mathcal{A}(t)R \geq mR - bR^2 \qquad \text{and} \qquad \mathcal{B}R \geq 0.$$

Hence, if $v \equiv R$ is sufficiently large, $v$ is a supersolution of (24.12). We thus obtain the following existence theorem.

**24.9 Proposition**

*Let $p > \frac{n}{2}$ and $\alpha \in \left(\frac{n}{2p}, 1\right)$. Then, for any nonnegative initial value $u_0 \in X_\alpha$, the logistic equation* (24.12) *has a unique global regular solution.*

**Proof**

Observe that under our hypotheses $X_\alpha \hookrightarrow C(\overline{\Omega})$. Hence, for any $u_0 \in X_\alpha$ there exists a constant $R > 0$ such that $u_0 \le R$. We can thus apply Corollary 24.8 with $\underline{u} \equiv 0$ and $\overline{u} \equiv R$ for some $R$ sufficiently large and the proof of the Proposition is complete. □

Assume for the rest of this section that all coefficients in (24.12) depend $T$-periodically on $t$ for some period $T > 0$. We shall now investigate stability properties of the zero-solution. This may be done by using the principle of linearized stability and instability established in Section 22.

To do this consider the linarization of (24.12) at the zero. Using Theorem 15.8 we see that

$$\partial_2 g_f(t,0)v(x) = m(x,t)v(x).$$

Hence, the linearized equation is

$$(24.13) \qquad \begin{cases} \partial_t v + \mathcal{A}(t)v = mv & (x,t) \in \Omega \times (0,\infty) \\ \mathcal{B}v = 0 & (x,t) \in \partial\Omega \times (0,\infty) \\ v(x,0) = v_0(x) & x \in \Omega, \end{cases}$$

When considering periodic-parabolic eigenvalue problems in Section 22.C we have seen that there exists a unique eigenvalue $\mu \in \mathbb{R}$ with positive eigenfunction $\varphi$ such that

$$(24.15) \qquad \begin{cases} \partial_t \varphi + \mathcal{A}(t)\varphi = \mu\varphi & (x,t) \in \Omega \times (0,\infty) \\ \mathcal{B}\varphi = 0 & (x,t) \in \partial\Omega \times (0,\infty) \\ \varphi(x,0) = \varphi(x,T) & x \in \Omega. \end{cases}$$

Denote by $\bar{K}$ the period-map associated to the linearized equation. By (14.3), the spectral radius of $\bar{K}$ is obtained from $\mu$ via the formula $\mathrm{r}(\bar{K}) = e^{-\mu T}$. Hence, $\mathrm{r}(\bar{K}) < 1$ if $\mu > 0$ and $\mathrm{r}(\bar{K}) > 1$ if $\mu < 0$. Recall that the zero solution is linearly stable iand only if $\mathrm{r}(\bar{K}) < 1$, neutrally stable if and only if $\mathrm{r}(\bar{K}) = 1$, and linearly unstable if and only if $\mathrm{r}(\bar{K}) > 1$. Of course this may be reformulated in terms of the principal eigenvalue, $\mu$, of (24.15): the zero-solution of $u$ is linearly stable if and only if $\mu > 0$, neutrally stable if and only if $\mu = 0$, and linearly unstable if and only if $\mu < 0$. The results of Section 22 give local Ljapunov stability or instability for the zero solution of (24.12). Here, stability and instability are to be understood with respect to the topology of $X_\alpha$.

The time-periodic diffusive logistic equation belongs to a class of problems for which the existence of a unique globally attracting periodic solution is intimately linked to the stability properties of the zero solution (see [**45**] for an abstract approach to this kind of equations). In [**67**] one can find the following result

**24.10 Theorem**
*(i) Assume that the trivial solution of* (24.12) *is linearly or neutrally stable. Then, it is globally asymptotically stable with respect to nonnegative initial data in* $X_\alpha^+$*), i.e. if* $u_0 \in X_\alpha^+$ *then we have that*

$$\|u(t; u_0)\|_\alpha \to 0$$

*as* $t \to \infty$.

*(ii) Assume that the trivial solution of* (24.12) *is linearly unstable. Then, there exists a unique nontrivial positive* $T$*-periodic solution* $u^*$ *of* (24.12). *Furthermore,* $u^*$ *is everywhere positive and is globally asymptotically stable with respect to positive initial data in* $X_\alpha$*, i.e. if* $u_0 \in X_\alpha^+ \setminus \{0\}$ *then*

$$\|u(t; u_0) - u^*(t)\|_\alpha \to 0$$

*as* $t \to \infty$.

This theorem gives a complete description of the long-time behaviour of positive solutions of (24.12).

**24.11 Remark**
One may consider problem (24.12) as a special case of the parameter dependent problem

$$(24.16) \qquad \begin{cases} \partial_t u + \mathcal{A}(t)u = \lambda(mu - bu^2) & (x,t) \in \Omega \times (0,\infty) \\ \mathcal{B}u = 0 & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where $\lambda$ varies over the interval $(0,\infty)$. In virtue of the above theorem we have to solve a periodic-parabolic eigenvalue problem of the form (14.1) and determine the sign of the principal eigenvalue $\mu(\lambda)$ in order to obtain information on the stability properties of the zero solution of (24.16). One can prove that there exists a unique number $\lambda_0$ such thatWe conclude now from Corollary 14.13 that there are at most two values of the parameter where stability or instability is lost. □

**Notes and references:** The proofs of the regularity result and of the comparison principle are taken from Amann [**7**]. The results on the logistic equation can be found in Hess [**67**]. There one finds a diversity of interesting results on the long-time behaviour of time-periodic reaction-diffusion equations on bounded domains.

## 25. Reaction-diffusion equations on $\mathbb{R}^n$

We devote this section to semilinear initial value problems of the form

$$
\begin{cases} \partial_t u(x,t) - k(t)\Delta u(x,t) = f\big(x,t,u(t,x),\nabla u(x,t)\big) & (x,t) \in \mathbb{R}^n \times (0,T] \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n, \end{cases} \tag{25.1}
$$

where we make the same assumptions as in Section 15.E. In particular the nonlinearity $f$ satisfies either (15.23) or (15.24). Since most of the arguments are essentially repetitions of the arguments in the previous section we shall be brief and take the liberty of omitting the proofs.

By a (*classical*) *solution* of problem (25.1), we mean a function

$$
u \in BUC(\mathbb{R}^n \times [0,T]) \cap BUC^{2,1}\big(\mathbb{R}^n \times [\varepsilon, T]\big),
$$

for all $\varepsilon > 0$, satisfying (25.1). It is clear how to define *local* and *maximal* clasical solutions of (25.1). In Subsection 15.E we reformulated the above problem as an abstract semilinear evolution equation in the Banach space $BUC(\mathbb{R}^n)$ or in its closed subspace $C_0(\mathbb{R}^n)$. In other words we wrote (25.1) in the form

$$
\begin{cases} \dot u + A(t)u = g\big(t,u(t)\big) & 0 < t \le T \\ u(0) = u_0 \end{cases} \tag{25.2}
$$

in $X_0$, where

$$
X_0 := BUC(\mathbb{R}^n) \text{ or } C_0(\mathbb{R}^n).
$$

Here, for any $t \in [0,T]$ the operator $A(t)$ is the $X_0$-realization of $k(t)\Delta$, and $g$ is the Nemitskii operator induced by $f$. We remark here that if $X_0 = C_0(\mathbb{R}^n)$ we must additionally assume that (15.27) holds, i.e. $f(x,t,0,\zeta) = 0$ for every $(x,t,\zeta) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$.

The results contained in Section 16 imply that equation (25.2) has a unique solution

$$
u \in C\big([0,T], X_\alpha\big) \cap C^1\big((0,T], X_0\big)
$$

for all initial values $u_0$ lying in the continuous interpolation space $X_\alpha = (X_0, X_1)^0_{\alpha,\infty}$, for any $\alpha \in (\frac{1}{2}, 1)$, or $\alpha \in [0,T]$ if $f$ does not depend on $\nabla u$. For these spaces we have given exact characterizations in Theorem 4.18 .

As already mentioned the structure of of this section shall be more or less the same as that of the previous one. We start by asking which is the connection between the solutions of the abstract equation (25.2) and those of the 'classical' equation (25.1). We then proceed to give a comparison theorem for sub- and supersolutions of (25.1). As an application we consider the unbounded version of the diffusive logistic equation.

**A. Regularity of solutions:** As in the bounded domain case one may show that every solution of (25.2) with initial value $u_0 \in X_1$ is a *regular solution*, that is

$$u \in BUC\big(\mathbb{R}^n \times [0,T]\big) \cap BUC^{2+\eta,1+\frac{\eta}{2}}\big(\mathbb{R}^n \times [\varepsilon,T]\big)$$

for all $\varepsilon > 0$. The proof of this result is the same as the proof of Theorem 24.2. The only difference is that instead of invoking Theorem 24.1 we need to resort to the following unbounded version of the Schauder theory for parabolic initial value problems (cf. [**87**], Section IV.5).

**25.1 Theorem**
*Let the above assumptions be satisfied. Then the inhomogeneous linear problem*

$$\begin{cases} \partial_t u(x,t) - k(t)\Delta u(x,t) = h(x,t) & (x,t) \in \mathbb{R}^n \times (0,T] \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

*has for all $u_0$ in $BUC^{2+\eta}(\mathbb{R}^n)$ and $h$ in $BUC^{\eta,\frac{\eta}{2}}\big(\mathbb{R}^n \times [0,T]\big)$ a unique classical solution $u$ lying in $BUC^{2+\eta,1+\frac{\eta}{2}}\big(\mathbb{R}^n \times [0,T]\big)$.*

We now state our regularity result.

**25.2 Theorem**
*Let $u_0 \in X_1$. Then,*

(*a*) *Each classical solution of* (25.1) *is a solution of the abstract equation* (25.2)*.*

(*b*) *Each global solution of* (25.2) *is a regular solution of* (25.1)*.*

**25.3 Remark**

(a) The above theorem remains true if we only require, that $u_0 \in X_\beta$ for some $\frac{1}{2} < \beta \leq 1$. In case that $f$ is independent of $\nabla u$, we need only $u_0 \in X_0$.

(b) Let $u$ be a (global) solution of (25.2) with initial condition $u_0$. Since $u(\varepsilon) \in X_1$ for all $\varepsilon \in (0,T]$ we conclude from the above theorem, that

$$u \in BUC^{2+\eta,1+\frac{\eta}{2}}\big(\mathbb{R}^n \times [\varepsilon,T]\big)$$

holds for all $\varepsilon > 0$.

(c) It is clear how to formulate and prove local versions of the above results. □

The following corollary on the classical solvability of the initial value problem (25.1) is the companion result to Corollary 24.4.

**25.4 Corollary**
*Let* $u_0 \in X_\alpha$ *for some* $\alpha \in \left(\frac{1}{2}, 1\right]$. *Then,* (25.1) *has a unique maximal regular solution. Moreover, if* $f$ *is independent of* $\nabla u$, *it suffices to assume that* $u_0 \in X_0$.

**B. Comparison theorems:** As in the bounded domain case we state a comparison principle for order related sub- and supersolutions of (25.1). Also in the context of unbounded domains this principle turns out to be an extremely useful instrument when dealing with questions concerning the long-time behaviour of solutions of (25.1) (see [**80**]).

We start by defining what sub- and supersolutions are.

**25.5 Definition**
Let $v \in BUC(\mathbb{R}^n \times [0,T]) \cap BUC^{2,1}\big(\mathbb{R}^n \times [0,T]\big)$ satisfy the differential inequalities

$$\partial_t u(x,t) - k(t)\Delta u(x,t) \geq f\big(x,t,u(t,x),\nabla u(x,t)\big) \qquad (x,t) \in \mathbb{R}^n \times (0,T] \tag{25.3}$$

Then, $v$ is called a *supersolution* of (25.1). If the reverse inequalities hold in (25.3), $v$ is said to be a *subsolution* of (25.1). If $v$ is not a solution, it is called a *strict super*- or a *strict subsolution* of (25.1), respectively. □

The comparison principle, whose proof is identical to that of Theorem 24.6, reads:

**25.6 Theorem**
*Let* $u$ *and* $v$ *be sub- and supersolutions of* (25.1), *respectively such that* $u(\cdot\,,0) \leq v(\cdot\,,0)$ *on* $\mathbb{R}^n$. *Then,*

$$u(\cdot\,,t) \leq v(\cdot\,,t)$$

*holds on* $\mathbb{R}^n$ *for all* $t \in [0,T]$. *Moreover, if* $u(\cdot\,,0) < v(\cdot\,,0)$, *then*

$$u(x\,,t) < v(x\,,t) \tag{25.4}$$

*holds for all* $x \in \mathbb{R}^n$ *and all* $t \in (0,T]$.

Of course the above theorem holds in particular for solutions of (25.2) with order related initial data.

**25.7 Corollary**
*Let* $u, v$ *be solutions of problem* (25.2) *with* $u(0) \leq v(0)$. *Then,* $u(t) \leq v(t)$ *for all* $t \in [0,T]$. *Moreover, if* $u(0) < v(0)$, *then*

$$u(t)(x) < v(t)(x) \tag{25.5}$$

*holds for all* $x \in \mathbb{R}^n$ *and all* $t \in (0,T]$.

The following result follows from Theorem 25.2 and Proposition 17.4.

**25.8 Corollary**
*Suppose that* $\underline{u}$ *and* $\overline{u}$ *are sub- and supersolutions of* (25.1) *respectively, and that* $f$ *satisfies the growth condition* (17.5)*. Then, there exists a unique global regular solution of* (24.1) *for all* $u_0 \in \big[\underline{u}(0), \overline{u}(0)\big]_{X_0} \cap X_\alpha$.

**C. The logistic equation:** As a simple illustration we consider, as in the previous section, the logistic equation, i.e.

$$\begin{cases} \partial_t u - k(t)\Delta u = mu - bu^2 & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n, \end{cases} \tag{25.6}$$

where we assume that $m, b \in BUC^{\mu,\frac{\mu}{2}}(\mathbb{R}^n \times [0,T])$, for some $\mu \in (0,1)$. Moreover, assume that $b(x,t) > 0$ holds for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

The equation describes the population density of a species living in the unbounded habitat $\mathbb{R}^n$. The population dynamical interpretation is, of course, the same as in the bounded domain case so we shall not repeat it.

Equation (25.6) is of the form (25.1) with

$$f(x,t,\xi) = m(x,t)\xi - b(x,t)\xi^2$$

for all $(x,t,\xi) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}$. Moreover, $f$ satisfies either of the standard assumptions (15.23) and (15.24).

Evidently $u \equiv 0$ is a solution of (25.6). On the other hand, if $R > 0$ is a constant sufficiently large, it holds that

$$0 = \partial_t R - k(t)\Delta R \geq mR - bR^2.$$

Hence, if $v \equiv R$ is sufficiently large, $v$ is a supersolution of (25.6). We thus obtain the following existence theorem.

**25.9 Proposition**
*For any nonnegative initial value* $u_0 \in X_0$*, the logistic equation* (25.6) *has a unique global regular solution.*

Assume for the rest of this section that all coefficients in (25.6) are defined for all $t \in \mathbb{R}^n$ and depend $T$-periodically on $t$ for some period $T > 0$. Moreover. we shall deal with the case that

$$X_0 = C_0(\mathbb{R}^n),$$

and make the additional assumption that

$$
(25.7) \qquad \begin{cases} \text{There exist positive constants } \gamma_0' \text{ and } R_0 \text{ such that} \\ \qquad\qquad m(x,t) \leq -\gamma_0' < 0 \\ \text{holds for all } (x,t) \in \mathbb{R}^n \times \mathbb{R} \text{ with } |x| \geq R_0. \end{cases}
$$

Consider the linear equation

$$
(25.8) \qquad \begin{cases} \partial_t v - k(t)\Delta v = mv & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ v(x,0) = v_0(x) & x \in \mathbb{R}^n, \end{cases}
$$

Since by Theorem 15.8

$$
\partial_2 g_f(t,0)v(x) = m(x,t)v(x)
$$

holds, the $X_0$-formulation of this equation is the linearization of (25.2) at zero. By $\bar{K} \in \mathcal{L}(X_0)$ we denote the period-map associated to the linearized equation. Recall that the zero solution is said to be linearly stable if $r(\bar{K}) < 1$, neutrally stable if $r(\bar{K}) = 1$ and linearly unstable if $r(\bar{K}) > 1$.

We remark here that under condition (25.7) one may obtain results on principal eigenvalues of periodic-parabolic problems as in the bounded domain case. The main difficulty in obtaining this results is the fact that although the period-map $\bar{K}$ is irreducible, it fails to be compact. For details consult [**38**]. In [**80**] these results were used to obtain the analogon of Theorem 24.10 for the unbounded domain case. The precise statement reads:

**25.10 Theorem**

(*i*) *Assume that the trivial solution of* (25.6) *is linearly or neutrally stable. Then it is globally asymptotically stable with respect to nonnegative initial data in $C_0(\mathbb{R}^N)$, i.e. if $u_0 \in C_0^+(\mathbb{R}^n)$ then we have that*

$$
\|u(t;u_0)\|_\infty \to 0
$$

*as $t \to \infty$.*

(*ii*) *Assume that the trivial solution of* (25.6) *is linearly unstable. Then there exists a unique nontrivial positive $T$-periodic solution $u^*$ of* (25.6). *Furthermore, $u^*$ is everywhere positive and is globally asymptotically stable with respect to positive initial data in $C_0(\mathbb{R}^N)$, i.e if $u_0 \in C_0 + (\mathbb{R}^n) \setminus \{0\}$ then*

$$
\|u(t;u_0) - u^*(t)\|_\infty \to 0
$$

*as $t \to \infty$.*

Note that this theorem completely settles the question of the asymptotic behaviour of positive solutions of (25.6). One may consider problem (25.6) as a special case of the parameter dependent problem

$$\begin{cases} \partial_t u - k(t)\Delta u = \lambda m u - b u^2 & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n, \end{cases} \tag{25.9}$$

where $\lambda$ varies over the interval $(0,\infty)$. We are interested in the stability or instability of the zero solution in dependence of the parameter $\lambda$. As in the bounded domain case one can prove that there exists a $\lambda_0 > 0$ such that the zero solution of (25.9) is linearly stable iff $\lambda < \lambda_0$, neutrally stable iff $\lambda = \lambda_0$ and linearly unstable iff $\lambda > \lambda_0$.

**Notes and references:** The results of this section were taken from Koch Medina and Schätti [**80**] and Daners and Koch Medina [**38**]. It seems to be the first time that reaction-diffusion equations on $\mathbb{R}^n$ have been studied by the evolution operator approach on the Banach space $C_0(\mathbb{R}^n)$. In [**80**] one can find more qualitative results for this kind of equations.

## 26. A nonstandard example arising in epidemiology

In this last section we would like to present another example, which, at a first sight, does not fit in the theory of abstract parabolic equations treated in this book. It is a model describing the spread of man-environment diseases such as cholera or typhus in a bounded domain bordering on the sea. This model was introduced by Capasso and Kunisch in [**27**] (see also the references given there).

In a first subsection we give a short description of the model and the exact formulation of the problem we shall be concerned with. Then, we present the ideas of how to put the concrete differential equation in the abstract framework. These ideas are carried out in the following subsections. In a final subsection we apply the results of Section 10 to prove a comparison theorem for solutions of the equation.

We hope that this section conveys the idea of how to go about placing a concrete equation in the context of our abstract theory.

**A. The problem:** Assume that $\Omega$ is a bounded domain in $\mathbb{R}^2$. We interpret $\Omega$ as the habitat where an infectious agent and a human population live. The spacial density of the infectious agent will be denoted by $u_1 \colon \Omega \to \mathbb{R}$ and the density of the infected human population by $u_2\colon \Omega \to \mathbb{R}$. We consider the following equations:

$$
(26.1) \qquad \begin{cases} \partial_t u_1 - \Delta u_1 - a_0(x,t)u_1 = f_1(x,t,u_2) & \text{in } \Omega \times (0,\infty) \\ \partial_t u_2 - a(x,t)u_2 = f_2(x,t,u_1) & \text{in } \overline{\Omega} \times (0,\infty) \\ \partial_\nu u_1 + b_0(x)u_1 = \displaystyle\int_\Omega k(x,y)u_2(y)\,dy & \text{on } \partial\Omega \times (0,\infty) \\ \big(u_1(\cdot\,,0), u_2(\cdot\,,0)\big) = (v_1, v_2) & \text{on } \Omega\,. \end{cases}
$$

This equation may be interpreted as follows. The diffusion term $-\Delta u_1$ reflects the fact that the infections agent is subject to geographical spread. The coefficient $a_0$ corresponds to a death rate and is thus assumed to be positive. The coefficient $a$ in the second equation has a similar meaning, but now for the human infected population. We shall assume that the infected population is sedentary so that we do not include a diffusion term in the second equation. The coupling terms $f_1$ and $f_2$ are to be understood as source terms.

One of the pecularities of the diseases mentioned above is, that the infectious agent is sent to the sea through the sewage, from where it turns back to the domain via the consumption of sea food. This procedure is modelled by the operator

$$
(26.2) \qquad Ku_2(x) := \int_\Omega k(x,y)u_2(y)\,dy
$$

for all $x \in \partial\Omega$. The kernel $k(x,y)$ describes the transfer mechanism of the infectious agent generated by the human infected population at $y \in \Omega$ to $x \in \partial\Omega$. The term $b_0(x)u_1(x)$ represents that portion of the density in $x \in \partial\Omega$ which leaves the habitat. Furthermore, $(v_1, v_2)$ are the initial densities.

The above model gives now the motivation to consider a more general problem. Let $\Omega$ be a bounded domain of class $C^\infty$ in $\mathbb{R}^n$. Put $u := (u_1, u_2)$ and consider the following system:

$$
(26.3) \qquad \begin{cases} \partial_t u_1 + \mathcal{A}(t)u_1 = f_1(x,t,u) & \text{in } \Omega \times (0,\infty) \\ \partial_t u_2 = f_2(x,t,u) & \text{in } \overline{\Omega} \times (0,\infty) \\ \mathcal{B}u_1 = Ku_2 & \text{on } \partial\Omega \times (0,\infty) \\ u(\cdot\,,0) = v_0 & \text{on } \Omega\,, \end{cases}
$$

where $\mathcal{A}(t) := \mathcal{A}(x,t,D)$ is a strongly elliptic differential operator, $\mathcal{B} := \mathcal{B}(x,D)$ a Neumann or Robin boundary operator and $f := (f_1, f_2)$ a given nonlinearity. Furthermore, $K$ is defined as in (26.2) with

$$
(26.4) \qquad k(\cdot\,,\cdot) \in C^1(\partial\Omega \times \overline{\Omega}, \mathbb{R})
$$

nonnegative. Finally, $u_0$ is an initial condition.

**B. Abstract formulation of the problem:** We start this subsection with some formal considerations which will be justified later. Fix $p \in (1, \infty)$ and set

$$D(\boldsymbol{A}(t)) := \big\{(u_1, u_2) \in W_p^2(\Omega) \times C(\overline{\Omega});\ \mathcal{B}u_1 = Ku_2\big\} \tag{26.5}$$

for all $t \geq 0$. Then we define for any $u := (u_1, u_2) \in D(\boldsymbol{A}(t))$ the operator

$$\boldsymbol{A}(t)u := \begin{bmatrix} \mathcal{A}(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{26.6}$$

Furthermore, we denote by $g$ the substitution operator induced by $f$, that is

$$g(t, u)(x) := \begin{bmatrix} f_1(x, t, u(x)) \\ f_2(x, t, u(x)) \end{bmatrix} \tag{26.7}$$

for all $(x, t) \in \overline{\Omega} \times \mathbb{R}_+$ and all functions $u: \overline{\Omega} \to \mathbb{R}^2$. With this notation we may consider the system (26.4) as an abstract evolution equation of the form

$$\begin{cases} \dot{u} + \boldsymbol{A}(t) = g\big(t, u(t)\big) \\ \qquad u(0) = v \end{cases} \tag{26.8}$$

in the space

$$X_0 := L_p(\Omega) \times C(\overline{\Omega}). \tag{26.9}$$

It is the aim of this subbsection to show that this renders in fact a proper abstract formulation of problem (26.4).

According to the strategy of how to attack this kind of problems we start by investigating the autonomous homogeneous linear problem. Then we turn to the nonautonomous homogeneous problem. After these preparations it is easy to treat the semilinear equation. Finally, under an additional assumptions on the nonlinearity, we shall establish a comparison theorem for sub- and supersolutions of system (26.3).

**C. The linear autonomous equation:** In this subsection we assume that $\mathcal{A}(t) = \mathcal{A}$ is independent of $t \geq 0$ and $f \equiv 0$. Suppose that $D(\boldsymbol{A})$ and $\boldsymbol{A}$ are defined by (24.5) and (24.6), respectively. Observe that by assumption (26.3) we have that $K \in \mathcal{L}\big(C(\overline{\Omega}), C^1(\partial\Omega)\big)$ and thus for any $p \in (1, \infty)$

$$K \in \mathcal{L}\big(C(\overline{\Omega}), W_p^{1-1/p}(\partial\Omega)\big) \tag{26.10}$$

by the imbedding theorem mentioned at the end of Appendix 4.A. Furthermore, by our positivity assumption on the kernel $k(\cdot\,, \cdot)$, the operator $K$ is positive. On the other hand,

$$\mathcal{B} \in \mathcal{L}\big(W_p^2(\Omega), W_p^{1-1/p}(\partial\Omega)\big)$$

for all $p \in (1,\infty)$. Hence, $D(\boldsymbol{A})$ is a well defined linear subspace of $X_0 = L_p(\Omega) \times C(\overline{\Omega})$. The following theorem is the corner stone in our treatment.

**26.1 Theorem**
*For any $p \in (1,\infty)$, the operator $\boldsymbol{A}$ (with domain $D(\boldsymbol{A})$) is a closed densely defined operator on $X_0 = L_p(\Omega) \times C(\overline{\Omega})$. Moreover, there exist constants $\mu_0 \in \mathbb{R}$ and $M \geq 1$ such that*

$$[\operatorname{Re}\mu \geq \mu_0] \subset \varrho(-\boldsymbol{A}) \tag{26.11}$$

*and*

$$\|(\lambda + \boldsymbol{A})^{-1}\| \leq \frac{M}{1+|\lambda|} \tag{26.12}$$

*for all $\lambda \in [\operatorname{Re}\mu \geq \mu_0]$. The constants $M$ and $\mu_0$ depend only on $\Omega$, $n$, $p$, the modulus of continuity of the highest order coefficients of $\mathcal{A}$ and upper bounds for the $C$-norm of the coefficients of $\mathcal{A}$, the $C^1$-norms of the coefficients of $\mathcal{B}$, $\underline{\alpha}^{-1}$ and the norm of $K$. Here, $\underline{\alpha}$ is the ellipticity constant from* (2.9).

In particular, the above theorem asserts that $-\boldsymbol{A}$ is the generator of an analytic $C_0$-semigroup (compare Theorem 1.1). Before we give the proof of this theorem, we would like to establish some easy consequences.

In contrast to the above theorem, the assumption, that $K$ is positive is necessary for the following for the following proposition. It is not hard to see that the assertion is wrong without the positivity assumption.

**26.2 Proposition**
*Let $K \geq 0$ and $p > n$. Then, the semigroup generated by $-\boldsymbol{A}$ on $X_0$ is positive.*

**Proof**
We have simply to show that $(\lambda + \boldsymbol{A})^{-1}$ is positive for all $\lambda$ large enough. If we set $v := (\lambda + \boldsymbol{A})^{-1}u$ for $u \in X_+$ we get $v_2 = \frac{1}{\lambda}u_2 \geq 0$ for $\lambda > 0$ and therefore we have $(\lambda + \mathcal{A})v_1 = u_1 \geq 0$ and $\mathcal{B}v_1 = Kv_2 = \frac{1}{\lambda}Ku_2 \geq 0$. Here, we used the positivity of $K$. Now the assertion follows from the maximum principle Remark 13.7. □

**26.3 Corollary**
*Let $-\boldsymbol{B} \in \mathcal{L}(X_0)$ be an operator generating a positive semigroup $e^{-t\boldsymbol{B}}$, $K$ positive and $p > n$. Then $-(\boldsymbol{A} + \boldsymbol{B})$ generates a positive analytic $C_0$-semigroup.*

**Proof**
By Theorem 1.3($a$) $-(\boldsymbol{A} + \boldsymbol{B})$ generates an analytic $C_0$-semigroup on $X_0$. Positivity follows from the product formula

$$e^{-t(\boldsymbol{A}+\boldsymbol{B})}u = \lim_{n\to\infty} \bigl(e^{-\frac{t}{n}\boldsymbol{A}} e^{-\frac{t}{n}\boldsymbol{B}}\bigr)^n u$$

which we already used in the proof of Theorem 14.7, and the closedness of the positive cone in $X_0$. □

**26.4 Example**

Let $\boldsymbol{B}$ be given by

$$\boldsymbol{B} := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{26.13}$$

with $b_{ij} \in C(\overline{\Omega})$ $(i, j = 1, 2)$ and $b_{12}$, $b_{21} \leq 0$. Then Corollary 26.7 is applicable. Observe that $-\boldsymbol{B}$ is the generator of a positive semigroup if and only if the condition $b_{12}$, $b_{21} \leq 0$ is satisfied (compare [**31**], Section 7.3). □

The rest of this subsection is devoted to the proof of Theorem 26.1. The proof, which is based on the a priori estimates for elliptic boundary value problems, is subdivided in several lemmas. Let us first recall the well known a priori estimates going back to Agmon, Douglis and Nirenberg [**3**],[**4**] (see e.g. [**10**]).

**26.5 Lemma**

*Let $(\Omega, \mathcal{A}, \mathcal{B})$ be an elliptic boundary value problem as described in Section 1.C and $1 < p < \infty$. Then there exists constants $c_0 > 0$ and $\lambda_0 \in \mathbb{R}$ such that*

$$(\lambda + \mathcal{A}, \mathcal{B}) \in \operatorname{Isom}\bigl(W_p^2\Omega), L_p(\Omega) \times W_p^{1-1/p}(\partial\Omega)\bigr) \tag{26.14}$$

*and*

$$|\lambda|\, \|w\|_{0,p} + |\lambda|^{\frac{1}{2}} \|w\|_{1,p} + \|w\|_{2,p} \leq c_0 \Bigl( \|(\lambda + \mathcal{A})w\|_{0,p} + (1 + |\lambda|^{\frac{1}{2}}) \|\mathcal{B}w\|_{W_p^{1-1/p}(\partial\Omega)} \Bigr) \tag{26.15}$$

*holds for all $\lambda \in [\operatorname{Re}\mu \geq \lambda_0]$. The constants $c_0$ and $\lambda_0$ depend only on $\Omega$, $n$, $p$, the modulus of continuity of the highest order coefficients of $\mathcal{A}$ and upper bounds for the $C$-norm of the coefficients of $\mathcal{A}$, the $C^1$-norms of the coefficients of $\mathcal{B}$ and $\underline{\alpha}^{-1}$. Here, $\underline{\alpha}$ is the ellipticity constant from (2.9).*

**26.6 Remark**

Observe that (26.14) implies in particular, that $\mathcal{B} \in \mathcal{L}\bigl(W_p^2(\Omega), W_p^{1-1/p}(\partial\Omega)\bigr)$ is onto for any $1 < p < \infty$. □

**26.7 Lemma**

*The norm $\|u\|_1 := \|u_1\|_{2,p} + \|u_2\|_\infty$ on $D(\boldsymbol{A})$ is equivalent to the graph norm $\|\cdot\|_G$ induced by $\boldsymbol{A}$.*

**Proof**

The inequality $\|u\|_G \leq c_1\|u\|_1$ for some constant $c_1 > 0$ is obvious from the definition of the norm in $\|\cdot\|_{2,p}$ and the definition of $\boldsymbol{A}$. The reverse inequality is an easy consequence of the a priori estimate (26.15) and the boundedness of $K$. $\square$

**26.8 Lemma**

*$\boldsymbol{A}: X_0 \supset D(\boldsymbol{A}) \to X_0$ is a closed densely defined operator.*

**Proof**

(*i*) Lemma 26.7 and the continuity of $\mathcal{B}: W_p^2(\Omega) \to W_p^{1-1/p}(\partial\Omega)$ and $K: C(\overline{\Omega}) \to W_p^{1-1/p}(\partial\Omega)$ imply that $D(\boldsymbol{A})$ equipped with the graph norm induced by $\boldsymbol{A}$ is a Banach space. But this is equivalent to the fact that $\boldsymbol{A}$ is closed.

(*ii*) Let $v := (v_1, v_2) \in X_0$ is arbitrary. By (26.10) and Remark 26.6 there exists a $\overline{v}_1 \in W_p^2(\Omega)$ such that $\mathcal{B}\overline{v}_1 = Kv_2$ holds. Since the space of test functions $\mathcal{D}(\Omega)$ is dense in $L_p(\Omega)$ for every $p \in (1,\infty)$, for every $\varepsilon > 0$ there exists a $\varphi \in \mathcal{D}(\Omega)$ such that $\|(v_1 - \overline{v}_1) - \varphi\| < \varepsilon$. It is now easy to see that $v_\varepsilon := (\overline{v}_1 + \varphi, v_2)$ lies in $D(\boldsymbol{A})$ and that $\|v_\varepsilon - v\|_{X_0} < \varepsilon$. Hence, the proof of the lemma is complete. $\square$

We are now ready to give the proof of Theorem 26.1.

**Proof of Theorem 26.1**

Since $K: C(\overline{\Omega}) \to W_p^{1-1/p}(\partial\Omega)$ is continuous and $1 + |\lambda|^{\frac{1}{2}} \leq 2|\lambda|$ for all $|\lambda| \geq 1$ we have that

$$(1 + |\lambda|^{\frac{1}{2}})\|\mathcal{B}u_1\|_{W_p^{1-1/p}(\partial\Omega)} = (1 + |\lambda|^{\frac{1}{2}})\|Ku_2\|_{W_p^{1-1/p}(\partial\Omega)} \leq 2|\lambda|\,\|K\|\,\|u_2\|_\infty$$

for all $u = (u_1, u_2) \in D(\boldsymbol{A})$ and $\lambda \in \mathbb{C}$ mit $|\lambda| \geq 1$. If we put this in the a priori estimate (26.15) we get that

$$\begin{aligned}
\|u\|_{0,p} + |\lambda|\|u_1\|_{0,p} &\leq c\Big(\|(\lambda + \mathcal{A})u_1\|_{0,p} + (1 + |\lambda|^{\frac{1}{2}})\|\mathcal{B}u_1\|_{W_p^{1-1/p}(\partial\Omega)}\Big) \\
&\leq c\|(\lambda + \mathcal{A})u_1\|_{0,p} + c\|K\|\,\|\lambda u_2\|_\infty \\
&\leq c\max\{1, \|K\|\}\,\|(\lambda + \boldsymbol{A})u\|_{X_0}
\end{aligned}$$

for all $u \in D(\boldsymbol{A})$ and $\lambda \in [\operatorname{Re}\mu \geq \mu_0]$, where $\mu_0 := \max\{1, \lambda_0\}$ with $\lambda_0$ from Lemma 26.5. On the other hand we obviously have that

$$(1 + |\lambda|)\|u_2\|_\infty \leq c\|\lambda u_2\|_\infty$$

for all $u_2 \in C(\overline{\Omega})$ and $\lambda \in [\operatorname{Re}\mu \geq \mu_0]$. Adding both inequalities, we easily get

$$\|u\|_{X_0} + |\lambda|\|u\|_{X_0} \leq c\|(\lambda + \boldsymbol{A})u\|_{X_0} \tag{26.16}$$

for all $u \in D(\boldsymbol{A})$ and $\lambda \in [\operatorname{Re}\mu \geq \mu_0]$.

It remains to show that $\lambda + \boldsymbol{A}\colon X_0 \supset D(\boldsymbol{A}) \to X_0$ is onto whenever $\lambda \in [\operatorname{Re}\mu \geq \mu_0]$. Suppose that $v \in X_0$ and set for any $\lambda \in [\operatorname{Re}\lambda \geq \mu_0]$

$$u_2 := \frac{1}{\lambda} v_2 .$$

By Lemma 26.5 there exists $u_1 \in W_p^2(\Omega)$ such that

$$(\lambda + \mathcal{A})u_1 = v_1 \qquad \text{and} \qquad \mathcal{B}u_1 = K\Big(\frac{1}{\lambda}v_2\Big) = Ku_2 \, .$$

Together with (26.16) it is now clear that $(\lambda + \boldsymbol{A})^{-1}$ exists and satisfies (26.12).

Finally, from Lemma 26.7 and the considerations above it is easy to see that $M := c^{-1}$ and $\mu_0$ depend only on the quantities listed in the theorem.

We have already proved in Lemma 26.8 that $\boldsymbol{A}$ is a closed densely defined operator on $X_0$. Hence, the proof of Theorem 26.1 is complete. □

**D. The nonautonomous linear problem:** We continue now with the time-dependent case under the assumptions on $\mathcal{A}(t)$ of Subsection B. Furthermore, for any $t \geq 0$, let $D(\boldsymbol{A}(t))$ and $\boldsymbol{A}(t)$ be defined by (26.5) and (26.6), respectively. We set

$$X_1 := \big(D(\boldsymbol{A}(t)), \|\cdot\|_1\big), \tag{26.17}$$

where $\|\cdot\|_1$ is the norm defined in Lemma 26.7. Obviously, $X_1$ is well defined since $\mathcal{B}$ and $K$ are independent of $t \geq 0$. Let now $T > 0$ be arbitrary. Then, applying Theorem 26.1 and Lemma 26.7, it is easy to see that the family $\big(\boldsymbol{A}(t)\big)_{0\leq t\leq T}$ satisfies conditions $(A1)$, $(A2')$ and $(A3)$ of Section 2.

**26.9 Proposition**

*There exists a unique evolution operator $U(\cdot\,,\cdot)$ for the family $\big(\boldsymbol{A}(t)\big)_{0\leq t\leq T}$. Moreover, $U(t,s)$ is positive for all $(t,s) \in \Delta_T$.*

**Proof**

The existence of the evolution operator follows from Theorem 2.6 and Remark 2.7(b). The positivity is an easy consequence of Theorem 26.3 and Corollary 10.10. □

As in the autonomous case, the positivity of $K$ is not necessary for the existence of the evolution operator but only for the positivity.

We may consider perturbations of the generator $\boldsymbol{A}(t)$ of the form

$$\boldsymbol{B}(t) := \begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix} \tag{26.13}$$

with $b_{ij} \in C^{0,\rho}(\overline{\Omega} \times [0,T])$ $(i,j = 1,2)$. As usual we set $b_{i,j}(t) = b_{i,j}(\cdot\,,t)$ for all $t \in [0,T]$. Furthermore, assume that $b_{12}(t)$, $b_{21}(t) \leq 0$, which assures that the semigroup generated by $-\boldsymbol{B}(t)$ is positive for all $t \in [0,T]$. By Theorem 26.3 the perturbation theorem 5.10 as well as Corollary 10.10, we get the following theorem.

**26.10 Theorem**
*Under the above hypotheses, there exists a unique evolution operator $U(\cdot\,,\cdot)$ for the family $\big(\boldsymbol{A}(t) + \boldsymbol{B}(t)\big)_{0 \leq t \leq T}$. Moreover, if $p > n$, $U(t,s)$ is positive for all $(t,s) \in \Delta_T$.*

After these preparations we are ready to deal with the semilinear equation.

**E. The semilinear equation:** In order to deal with the semilinear equation (26.8) we have to know at least some imbedding theorems for the interpolation spaces between $X_1$ and $X_0$. Suppose that $(\cdot\,,\cdot)_\alpha$ is either the real or the comlpex interpolation method described in Section 4. As usual, set

$$X_\alpha := (X_0, X_1)_\alpha$$

for any $\alpha \in (0,1)$. By definition of $X_1$, it is clear that

$$X_1 \hookrightarrow W_p^2(\Omega) \times C(\overline{\Omega}).$$

Hence, Proposition 3.2$(b)$ and 3.4 imply that

$$X_\alpha \hookrightarrow \big(L_p(\Omega) \times C(\overline{\Omega}), W_p^2(\Omega) \times C(\overline{\Omega})\big)_\alpha \doteq \big(L_p(\Omega), W_p^2(\Omega)\big)_\alpha \times C(\overline{\Omega})$$

for all $\alpha \in (0,1)$. In Theorem A3.5, the interpolation spaces $\big(L_p(\Omega), W_p^2(\Omega)\big)_\alpha$ are determined. Applying these results we conclude that

$$X_\alpha \hookrightarrow W_p^{2\alpha}(\Omega) \times C(\overline{\Omega})$$

for $\alpha \in (0,1) \setminus \{\frac{1}{2}\}$ if we take the real interpolation method or

$$X_\alpha \hookrightarrow H_p^{2\alpha}(\Omega) \times C(\overline{\Omega})$$

for $\alpha \in (0,1)$ if we take the complex interpolation method. Applying now the imbedding Theorem A3.7 we obtain

$$X_\alpha \hookrightarrow C(\overline{\Omega}) \times C(\overline{\Omega}) \tag{26.18}$$

whenever $p > \frac{n}{2}$ and $\alpha \in \big(\frac{n}{2p}, 1]\big)$.

In a more general setting it is possible to determine the spaces $X_\alpha$ exactly. This was carried out in the first authors Ph.D. dissertation [**37**].

The knowlegdge of such an imbedding theorem allows us to show continuity properties of a Nemitskii-operator $g_f$ induced by a peticular nonlinearity $f$. On the nonlinearity $f$ we impose the following conditions:

$$f = (f_1, f_2) \in C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R}^2). \tag{26.19}$$

In fact, weaker conditions would suffice. Let $p > \frac{n}{2}$ and $\alpha \in \left(\frac{n}{2p}, 1\right]$. Then, using the imbedding (26.18), we get that

$$g_f \in C^{\mu,1-}\big([0,T] \times X_\alpha, X_0\big) \tag{26.20}$$

for some $\mu \in (0,1)$. We have now established all the assumptions of Theorem 16.2 and get thus existence of solutions of (26.8). We collect these facts in the following theorem.

**26.11 Theorem**
*Let the hypotheses above be satisfied. Furthermore, let $p > \frac{n}{2}$ and $\alpha \in \left(\frac{n}{2p}, 1\right]$. Then, for any $v \in X_\alpha$ the equation* (26.8) *has a unique maximal solution*

$$u \in C\big([0, t^+(v)), X_\alpha\big) \cap C^1\big((0, t^+(v)), X_0\big)\,,$$

*where $t^+(v)$ is the positive escape time.*

**26.12 Remarks**
(a) Observe that it is in general not possible to prove an analogon to Theorem 24.2, since in the second component there is no regularization. This means that if $v_2$ is only a continuous function, so is $u_2(t)$ for $t > 0$. For this reason, the function $f_1\big(\cdot\,, t, u(\cdot\,, t)\big)$ is not Hölder continous for $t > 0$. For that kind of inhomoegneities in a reaction-diffusion equations the solution may not lie in $C^{2,1}\big(\overline{\Omega} \times (0, t^+(v))\big)$. Of course the converse is true, i.e. that solutions in the classical partial differential equations are solutions of the abstract problem, as it is easily seen.

(b) It follows from the above theorem, that the solution $u$ of (26.8) lies in

$$W_p^{2,1}\big(\Omega \times (0,T)\big) \times \Big(C^1\big((0,T], C(\overline{\Omega})\big) \cap C\big([0,T], C(\overline{\Omega})\big)\Big)\,. \tag{26.21}$$

whenever $v \in X_1$. If $p > n+2$, by the imbedding theorem A3.14, the solution lies in

$$C^{1,0}\big(\overline{\Omega} \times [0,T]\big) \times \Big(C^{0,1}\big(\overline{\Omega} \times (0,T]\big) \cap C\big(\overline{\Omega} \times [0,T]\big)\Big)$$

□

**F. Comparison theorems:** In this last subsection we establish a comparison theorem for sub- and supersolutions of (26.8). First we would like to say what we mean by sub- and supersolutions of (26.8).

**26.13 Definition**
Let $w = (w_1, w_2) \in W_p^{2,1}(\Omega \times (0,T)) \times \Big(C^1\big((0,T], C(\overline{\Omega})\big) \cap C\big([0,T], C(\overline{\Omega})\big)\Big)$ satisfy the following inequalities

$$
(26.22) \qquad \begin{cases} \partial_t w_1 + \mathcal{A}(t) w_1 \geq f_1(x,t,w) & \text{in } \Omega \times (0,T] \\ \partial_t w_2 \geq f_2(x,t,w) & \text{in } \partial\Omega \times (0,T] \\ \mathcal{B} w_1 = K w_2 & \text{on } \partial\Omega \times (0,T]\,. \end{cases}
$$

Then, $w$ is called a *supersolution* of (26.8). If the reverse inequalities hold, $w$ is called a *subsolution* of (26.8). □

For general nonlinearities it is not possible to prove comparison Theorems for sub- and supersolutions of (26.8). To do this we need some additional assumption on $f$.

**26.14 Definition**
A function $h = (h_1, h_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called *quasi-monotone*, if $h_1(\xi_1, \cdot)$ and $h_2(\cdot\,, \xi_2)$ are nondecreasing functions for every fixed $(\xi_1, \xi_2) \in \mathbb{R}^2$. If these functions are strictly increasing, we call $h$ *strictly quasi-monotone*. □

**26.15 Remark**
The condition that a function $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ is quasi-monotone is a generalization of the sign condition on the off-diagonal elements of $\boldsymbol{B}$ in Example 26.4. In the linear case it is a necessary condition for the positivity of the semigroup generated by $-\boldsymbol{B}$ as already mentioned in Example 26.4. □

**26.16 Theorem**
*Let $\underline{u}$ and $\overline{u}$ be sub and supersolutions of* (26.8) *satisfying $\underline{u}(0) \leq \overline{u}(0)$. Suppose that $f$ satisfies* (26.19)*, that $f(x,t,\cdot\,,\cdot)$ is quasi-monotone for any $(x,t) \in \overline{\Omega} \times [0,T]$ and that $p > n$. Then $\underline{u}(t) \leq \overline{u}(t)$ holds for all $t \in [0,T]$.*

**Proof**
Set $w = (w_1, w_2) := \overline{u} - \underline{u}$. Subtracting the inequalities for $\overline{u}$ and $\underline{u}$ we get that

$$
(26.23) \qquad \begin{cases} \partial_t w_1 + \mathcal{A}(t) w_1 \geq f_1(x,t,\overline{u}) - f_1(x,t,\underline{u}) & \text{in } \Omega \times (0,T] \\ \partial_t w_2 \geq f_2(x,t,\overline{u}) - f_2(x,t,\underline{u}) & \text{in } \overline{\Omega} \times (0,T] \\ \mathcal{B} w_1 = K w_2 & \text{on } \partial\Omega \times (0,T] \\ w(0) \geq 0 & \text{in } \Omega\,. \end{cases}
$$

Using condition (26.19) we may write

$$\begin{aligned}
&f_1(x,t,\overline{u}) - f_1(x,t,\underline{u}) \\
&\quad = f_1(x,t,\overline{u}_1,\overline{u}_2) - f_1(x,t,\overline{u}_1,\underline{u}_2) + f_1(x,t,\overline{u}_1,\underline{u}_2) - f_1(x,t,\underline{u}_1,\underline{u}_2) \\
&\quad = \int_0^1 \partial_4 f_1(x,t,\overline{u}_1,\underline{u}_2 + \tau(\overline{u}_2 - \underline{u}_2))\,d\tau w_2 + \int_0^1 \partial_3 f_1(x,t,\underline{u}_1 + \tau(\overline{u}_1 - \underline{u}_1),\underline{u}_2)\,d\tau w_1 \\
&\quad =: b_{12}(x,t)w_2 + b_{11}(x,t)w_1.
\end{aligned}$$

In a similar way define $b_{21}(x,t)$ and $b_{22}(x,t)$ by means of the difference on the right hand side of the second equation in (26.23). Since $f(x,t,\cdot\,,\cdot)$ is quasi-monoton, $b_{12}$ and $b_{21}$ are nonnegative functions. Hence,

$$\boldsymbol{B}(t) := \begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix}$$

has nonnegative off-diagonal elements for all $t \in [0,T]$. Observe, that $b_{ij} \in C(\overline{\Omega})$ $(i,j = 1,2)$. Theorem 26.10 asserts now that the evolution operator $U(\cdot\,,\cdot)$ associated to the family $\big(\boldsymbol{A}(t) - \boldsymbol{B}(t)\big)_{0\le t\le T}$ is positive.

Consider now the equation

$$\begin{cases} \dot{u} + \boldsymbol{A}(t)u - \boldsymbol{B}(t)u =: m(t) \geq 0 & \text{for } t \in (0,T]. \\ \qquad u(0) = \overline{u}(0) - \underline{u}(0) = w(0) \end{cases} \tag{26.24}$$

Since $U(t,s)$ is positive for any $(t,s) \in \Delta_T$, the solution

$$u(t) = U(t,0)w(0) + \int_0^t U(t,\tau)m(\tau)\,d\tau$$

of (26.24) is nonnegative. On the other hand, $w$ is also a solution of (26.24). By uniqueness of the solution, we have that $w(t) = u(t) \geq 0$. Hence, the assertion of the theorem follows. □

In the case of one reaction diffusion equation we have seen in Theorem 24.7, that $\underline{u}(t) \ll \overline{u}(t)$ holds whenever $\underline{u}(0) < \overline{u}(0)$. Under the conditions of the above theorem this is no longer true in general as it is easily seen. Nevertheless, under stronger conditions on the nonlinearity and the operator $K$, it is possible to prove a similar theorem for our system.

**26.17 Corollary**

*Let the assumptions of the above theorem be satisfied and $p > n+2$. Assume in addition that either $f(x,t,\cdot\,,\cdot)$ is strictly quasi-monotone for all $(x,t) \in \overline{\Omega} \times [0,T]$ or $f_2(x,t,\cdot\,,\xi_2)$ is strictly increasing $(x,t,\xi_2) \in \overline{\Omega} \times [0,T] \times \mathbb{R}$ and $K > 0$. Then, if $\underline{u}(0) < \overline{u}(0)$ we have that $\underline{u}(t) \ll \overline{u}(t)$ for all $t \in (0,T]$.*

**Proof**
As in the proof of the above theorem, let $w = (w_1, w_2) := \overline{u} - \underline{u}$. As shown in this theorem, it holds that $w(t) \geq 0$ for all $t \in [0, T]$. By our assumption $w(0) > 0$ we find $x_0 \in \overline{\Omega}$ such that $w_1(x_0, 0) > 0$ or $w_2(x_0, 0) > 0$. Suppose that the first case holds. Setting $b_{ij}$ as in the proof of the above theorem for $i, j = 1, 2$ we conclude that

$$
(26.25) \qquad \begin{cases} \partial_t w_1 + \mathcal{A}(t) w_1 - b_{11}(t) w_1 \geq b_{12} w_2(t) \geq 0 & \text{in } \Omega \times (0, T] \\ \mathcal{B} w_1 = K w_2 \geq 0 & \text{on } \partial\Omega \times (0, T] \\ w_1(0) > 0 & \text{on } \Omega \end{cases}
$$

holds. Now by the parabolic maximum principle (Theorem 13.5) we see that $w_1(t) \gg 0$ in $C(\overline{\Omega})$ for all $t \in (0, T]$. Since, by assumption $b_{21}(x_0, t) > 0$ for all $t \in [0, T]$ $w_2$ satisfies the following differential inequality

$$
(26.26) \quad \begin{cases} \partial_t w_2(x, t) - b_{22}(x, t) w_2(x, t) \geq b_{21}(x, t) w_1(x, t) > 0 & \text{for } (x, t) \in \overline{\Omega} \times (0, T] \\ w_2(x, 0) \geq 0 & \text{for } x \in \overline{\Omega} \end{cases}
$$

For the first inequality we used also that $w_1(t) \gg 0$ for all $t \in (0, T]$. Hence, the assertion follows.

Suppose now that $w_2(x_0, 0) > 0$ for some $x_0 \in \overline{\Omega}$. Then, by an inequality similar to (26.26), we have that $w_2(x_0, t) > 0$ holds for all $t \in [0, T]$. If we put this in (26.25) and using our additional assumption on $f_1$ or $K$ we see that $w_1(t) \neq 0$ for $t \in (0, T]$ and thus by the parabolic maximum principle $w_1(t) \gg 0$ for all $t \in (0, T]$. By using (26.26) again, we obtain the assertion. Hence, the proof of the corollary is complete. □

**26.18 Corollary**
*The assertions of Theorem* 24.16 *and Corollary* 24.17 *remain valid for solutions of* (24.8) *whenever* $p > n + 2$.

**Notes and references:** As already mentioned at the beginning of this section, the model described in Subsection A is due to Capasso and Kunisch [**27**]. They prove also existence and uniqueness of solutions for the autonomous semilinear equation with very special nonlinearities. Moreover, they establish a comparison theorem for solutions.

The abstract formulation of the problem as it is given here is taken from the first authors paper [**36**], where a comparison theorem can also be found. Another method to prove that $-\boldsymbol{A}$ is the generator of an analytic semigroup is described in Greiner [**64**].

The proof of the comparison theorem which was based on Corollary 10.10 is much easier than the one given in [**27**] and [**36**].

Capasso and Thieme have given in [**28**] some indication on how to prove threshold theorems for the system. The first author has proved in [**36**] stability and convergence properties for solutions of the nonautonomous problem with time-independent principal part. A complete treatment of the fully nonautonomous semilinear equation with time-dependent boundary conditions and a description of the asymptotic behaviour of its solutions is given in the first authors Ph.D. dissertation [**37**].

# Appendix

It is the intention of this appendix to acquaint with the notation for the function spaces we use in these lecture notes and to state the most useful properties of these spaces. There are many books on function spaces such as Adams [**2**], Bergh and Löfström [**22**], Besov, Il'in and Nikol'skiĭ [**23**], Kufner, John and Fučìc [**60**] or Triebel [**122**] to name just a few. Also in Amann [**18**] a chapter on function spaces is to be found. However, most of these books treat very wide classes of function spaces, which include the spaces we are interested as very special cases. For the non-specialist, this constitutes a major obstacle in finding the assertions in the form we use. Therefore, we have tried to collect the material we felt is necessary for the treatment of initial-boundary value problems in the setting of interpolation spaces. Proofs are mostly ommited and only indicated in those instances where a precise reference is difficult to find.

## A1. Spaces of continuous and differentiable functions

In this first section of the Appendix we shall introduce several functions spaces characterized by the continuity, Hölder-continuity or differentiability properties of their elements. Throughout this section $\mathbb{K}$ shall denote either of the fields $\mathbb{R}$ or $\mathbb{C}$.

**A. Continuous functions:** We fix a Banach space $F$ over $\mathbb{K}$. If $S$ is any set we denote by $B(S, F)$ the vector space of all bounded functions from $S$ into $F$. When equipped with the *supremum norm*,

$$\|u(x)\|_\infty := \sup_{x\in S} \|u(x)\|,$$

$B(S, F)$ becomes a Banach space.

Let $X$ be a topological space and denote by $C(X, F)$ the space of continuous functions from $X$ into $F$. We set

$$BC(X, F) := B(X, F) \cap C(X, F).$$

Then, $BC(X, F)$ is a closed subspace of $B(X, F)$ and as such a Banach space itself. If $X$ is compact then $BC(X, F) = C(X, F)$. If $X$ is $\sigma$-compact then $C(X, F)$ becomes a

Fréchet space if we endow it with the topology induced by the seminorms

$$p_K(u) := \max_{x \in K} \|u(x)\|$$

where $K$ varies over all compact subsets of $X$.

Let $X$ be $\sigma$-compact and $(K_i)$ a sequence of compact sets in $K$ such that $\bigcup_{i=1}^{\infty} = K$ and $K_i \subset K_{i-1}$ holds for all $i \in \mathbb{N}^*$. Then we define the space $C_0(X, F)$ to be the space of all continuous functions $u \colon X \to F$ such that for any $\varepsilon > 0$ there exists a number $i_0 \in \mathbb{N}$ such that $\|u(x)\| \leq \varepsilon$ holds for all $x \in X \setminus K_i$ whenever $i \geq i_0$. The space $C_0(X, F)$ becomes a Banach space when equipped with the norm $\|\cdot\|_\infty$.

Suppose now that $(M, d)$ is a complete metric space. Then, $BUC(M, F)$ denotes the space of bounded uniformly continuous functions from $M$ into $F$. Since $BUC(M, F)$ is a closed subspace of $BC(M, F)$ it is also a Banach space with respect to the supremum norm.

Suppose in addition, that $M$ is $\sigma$-compact. Then, it is obvious that $C_0(M, F)$ is a closed subspace of $BUC(X, F)$.

In all of the above spaces we shall omit the notational reference to $F$ whenever $F = \mathbb{K}$.

**B. Lipschitz and Hölder continuous functions:** Suppose that $(M, d)$ is a metric space and that $F$ is a Banach space over $\mathbb{K}$. Define now for any $\nu \in (0, 1]$, $N \subset M$ and $u \colon M \to F$ the expression

$$[u]_{N,\nu} := \sup_{x,y \in N, x \neq y} \frac{\|u(x) - u(y)\|}{d(x, y)^\nu}.$$

Recall that a function $u \colon M \to F$ is said to be *locally Lipschitz continuous* if to each $x_0 \in M$ there exists a neighbourhood $N$ such that $[u]_{N,1}$ is finite. If $[u]_{M,1}$ is finite we say that $u$ is *(uniformly) Lipschitz continuous.*

By $C^{1-}(M, F)$ we denote the space of all locally Lipschitz continuous functions. The space of all uniformly Lipschitz-continuous bounded functions is then denoted by $BUC^{1-}(M, F)$. On $BUC^{1-}(M, F)$ we may define the norm

$$\|u\|_{1-} := \|u\|_\infty + [u]_{M,1}$$

which makes it into a Banach space.

Let $\nu \in (0, 1)$. A function $u \colon M \to F$ is said to be *locally $\nu$-Hölder continuous*, if to each $x_0 \in M$ there exists a neighbourhood $N$ such that $[u]_{N,\nu}$ is finite. If $[u]_{M,\nu}$ is finite we say that $u$ is *(uniformly) $\nu$-Hölder continuous.*

The space of all locally $\nu$-Hölder continuous functions will be denoted by $C^\nu(M, F)$ and the space of all uniformly $\nu$-Hölder continuous functions by $BUC^\nu(M, F)$. If we endow $BUC^\nu(M, F)$ with the norm

$$\|u\|_\nu := \|u\|_\infty + [u]_{M,\nu}$$

it becomes a Banach space. Whenever notational convenience dictates we shall write $BUC^0(M,F)$ and $\|\cdot\|_0$ instead of $BUC(M,F)$ and $\|\cdot\|_\infty$, respectively. The norms $\|\cdot\|_\mu$ satisfy an interpolation inequality, which is of foundamental importance when considering Nemitskii-operators in in spaces of Hölder continuous functions in Section 15. The proof is taken from Amann [**18**].

**A1.1 Lemma**

*Suppose that* $0 \le \lambda < \nu < 1$ *or that* $0 \le \lambda < 1$ *and* $\nu = 1-$. *Setting for any* $\theta \in [0,1]$

$$(A1.1) \qquad \mu := \begin{cases} (1-\theta)\lambda + \theta\nu & \text{if } \nu < 1 \\ (1-\theta)\lambda + \theta & \text{if } \nu = 1- \end{cases}$$

*we have the following interpolation inequality*

$$\|u\|_\mu \le \|u\|_\lambda^{1-\theta}\|u\|_\nu^\theta.$$

**Proof**

Let $u \in BUC^\nu(M,F)$. Then it is easy to see that

$$(A1.2) \qquad [u]_{M,\mu} \le [u]_{M,\lambda}^{1-\theta}[u]_{M,\nu}^\theta\,.$$

Let now $x,y,z > 0$ and $\theta \in (0,1)$. Using Young's inequality (cf. [**60**], Corollary 2.2.3), we see that

$$\begin{aligned}\left(\frac{x}{x+y}\right)^{1-\theta}&\left(\frac{x}{x+z}\right)^{\theta} + \left(\frac{y}{x+y}\right)^{1-\theta}\left(\frac{z}{x+z}\right)^{\theta} \\ &\le (1-\theta)\frac{x}{x+y} + \theta\frac{x}{x+z} + (1-\theta)\frac{y}{x+y} + \theta\frac{z}{x+z} = 1\end{aligned}$$

holds. Hence, $x + y^{1-\theta}z^\theta \le (x+y)^{1-\theta}(x+z)^\theta$ holds for all $x,y,z > 0$. By definition of the norm $\|\cdot\|_\mu$ as well as $(A1.2)$ we obtain the assertion. □

The interpolation inequality form above is the basis for the next proposition.

**A1.2 Proposition**

*Suppose that* $0 < \mu < \nu < 1$. *Then, the following imbeddings hold*

$$(A1.3) \qquad BUC^{1-}(M,F) \hookrightarrow BUC^\nu(M,F) \hookrightarrow BUC^\mu(M,F) \hookrightarrow BUC(M,F).$$

*Moreover, these imbeddings are compact whenever* $F$ *is finite dimensional and* $M$ *is compact.*

**Proof**
Let $0 \leq \mu \leq \nu < 1$. Since by definition $\|u\|_0 = \|u\|_\infty \leq \|u\|_\nu$, we get from the interpolation inequality $(A1.1)$ that

$$\|u\|_\mu \leq \|u\|_0^{\mu/\nu} \|u\|_\nu^{1-\mu/\nu} \leq 1$$

holds for all $u \in BUC^\mu(M, F)$ with $\|u\|_\nu \leq 1$. Hence, the imbedding constant, which is given by

$$\sup_{\substack{u \in BUC^\nu(M,F) \\ \|u\|_\nu = 1}} \|u\|_\mu$$

is less than one and the first assertion follows.

The compactness assertion is shown by an argument involving the Arzelà-Ascoli Theorem and the interpolation inequality $(A1.1)$. (For details see e.g. [**18**],[**60**]) $\square$

Again, if $F = \mathbb{K}$ we omit the reference to it in our notation.

**C. Differentiable functions:** Let $E$ and $F$ be Banach spaces over $\mathbb{K}$ and suppose that $U \subset E$ is open. Moreover, fix a positive natural number $k$.

By $C^k(U, F)$ we denote the space of all $k$-times continuously differentiable functions from $U$ into $F$. We write $BC^k(U, F)$ for the space consisting of all those functions in $C^k(U, F)$ with bounded derivatives up to the order $k$. When endowed with the norm

$$\|u\|_k := \|u\|_\infty + \sum_{1 \leq |\alpha| \leq k} \|D^\alpha u\|_\infty ,$$

it becomes a Banach space. The elements of $BUC^k(U, F)$ are the functions in $BC^k(U, F)$ whose derivatives up to the order $k$ are uniformly continuous. As a closed subspace of $BC^k(U, F)$ it is a Banach space.

If $k \geq 2$, we define $C^{k-}(U, F)$ to consist of all functions in $C^{k-1}(U, F)$, such that all their derivatives of the order $k - 1$ are locally Lipschitz continuous functions. We also define $BUC^{k-}$ to be the subspace of $BUC^{k-1}(U, F)$ of functions with uniformly Lipschitz continuous derivatives of the order $k-1$. It becomes a Banach space when equipped with the norm

$$\|u\|_{k-} := \|u\|_{k-1} + \sum_{|\alpha| = k-1} [D^\alpha u]_{U,1}.$$

If $\nu \in (0, 1)$ we denote by $C^{k+\nu}(U, F)$ the space of functions lying in $C^k(U, F)$ whose derivatives of the order $k$ are locally $\nu$-Hölder continuous. Furthermore, we write $BUC^{k+\nu}(U, F)$ for the subspace of $BUC^k(U, F)$ consisting of those functions who have uniformly $\nu$-Hölder continuous derivatives of the order $k$. The norm

$$\|u\|_{k+\nu} := \|u\|_k + \sum_{|\alpha| = k} [D^\alpha u]_{U,\nu}$$

makes it into a Banach space.

Analogously to Proposition 1.1 we have

**A1.3 Proposition**

*Suppose that $U$ is convex. If $0 < \mu < \nu < 1$ and $k$ is a positive natural number, then the following imbeddings hold:*

$$\begin{aligned} BUC^{k+1}(U,F) &\hookrightarrow BUC^{(k+1)-}(U,F) \hookrightarrow BUC^{k+\nu}(U,F) \\ &\hookrightarrow BUC^{k+\mu}(U,F) \hookrightarrow BUC^{k}(U,F). \end{aligned} \tag{1.2}$$

*Moreover, these imbeddings are compact whenever $\overline{U}$ is compact and $F$ is finite dimensional.*

**Proof**

The only thing we have to show is that $BUC^{k+1} \hookrightarrow BUC^{(k+1)-}$. But this is an easy consequence of the Mean Value Theorem and the convexity of $U$. □

The above proposition holds also for more general domains as it is shown in [**60**], Theorem 1.2.14.

We have defined spaces of functions from $U$ into $F$, which have the form $\mathcal{F}(U,F)$. We may also define spaces of functions from $\overline{U}$ into $F$ of the type $\mathcal{F}(\overline{U},F)$ by saying that a function $u$ lies in $\mathcal{F}(\overline{U},F)$ if and only if it belongs to $\mathcal{F}(U,F)$ and every derivative (up to the order characterizing the class $\mathcal{F}(U,F)$) may be extended continuously to the closure of $U$. Thus, for instance, $C^k(\overline{U},F)$ is the space of all functions $u\colon \overline{U} \to F$ such that $D^j u$ may be continuously extended to $\overline{U}$ for each $0 \le j \le k$.

Suppose now that $E$ is finite dimensional. In this case it is well known that one may make the following identifications:

$$\begin{aligned} C^k(\overline{U},F) &= BUC^k(U,F), \\ C^{k-}(\overline{U},F) &= BUC^{k-}(U,F) \text{ and} \\ C^{k+\nu}(\overline{U},F) &= BUC^{k+\nu}(U,F). \end{aligned}$$

**D. Little Hölder spaces:** In this subsection we shall be concerned wit another class of Hölder spaces. These spaces turn out to be the continuous interpolation spaces between the domain of definition of a $BUC(\mathbb{R}^n)$-realization of the Laplacian and $BUC(\mathbb{R}^n)$ (see Theorem 4.18 and Lunardi [**91**]).

Let $k \in \mathbb{N}$ and $\nu \in (0,1)$ be fixed. We denote by

$$buc^{k+\nu}(\mathbb{R}^n)$$

the closed subspace of $BUC^{k+\nu}(\mathbb{R}^n)$ consisting of all functions $u$ satisfying

$$\lim_{r \searrow 0} \sup_{\substack{x \neq y \\ |x-y| \le r}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\nu} = 0$$

for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$. The space $buc^{k+\nu}(\mathbb{R}^n)$ is called *little Hölder space*. We define now a subspace of $buc^{k+\nu}(\mathbb{R}^n)$ by

$$c_0^{k+\nu}(\mathbb{R}^n) := buc^{k+\nu}(\mathbb{R}^n) \cap C_0^k(\mathbb{R}^n).$$

**A1.4 Proposition**
*Let* $0 < \mu < \nu < 1$ *and* $k \in \mathbb{N}$. *Then,*

$$BUC^{k+\nu} \hookrightarrow buc^{k+\mu}(\mathbb{R}^n)$$

**Proof**
Let $u \in BUC^{k+\nu}(\mathbb{R}^n)$. Then,

$$\begin{aligned}\lim_{r \searrow 0} \sup_{\substack{x \neq y \\ |x-y| \leq r}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\mu} &\leq \lim_{r \searrow 0} r^{\nu-\mu} \sup_{\substack{x \neq y \\ |x-y| \leq r}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\nu} \\ &\leq \|\partial^\alpha u\|_\nu \lim_{r \searrow 0} r^{\nu-\mu} = 0\end{aligned}$$

for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$. Hence the assertion follows. $\square$

**E. Function spaces on products:** Let $E_1$, $E_2$ and $F$ be Banach spaces and $U_i$ $(i = 1, 2)$ open subsets of $E_i$. Suppose that $u: U_1 \times U_2 \to F$ is a function. Then we may consider $u$ as a function mapping $x_1 \in U_1$ to the function $u(x_1, \cdot)$. If $u \in BUC(U_1 \times U_2, F)$ it is an easy exercise to verify that

$$\big[x_1 \mapsto u(x_1, \cdot)\big] \in BUC\big(U_1, BUC(U_2, F)\big).$$

On the other hand, every function $v \in BUC\big(U_1, BUC(U_2, F)\big)$ may be considered as a function on $U_1 \times U_2$ mapping $(x_1, x_2)$ to $u(x_1, x_2) := \big[v(x_1)\big](x_2)$. It is now easy to see that

$$u \in BUC(U_1 \times U_2, F).$$

The same holds of course if we interchange the role of $U_1$ and $U_2$. For this reason we make the following identification:

$$(A1.4) \qquad BUC(U_1 \times U_2, F) = BUC\big(U_1, BUC(U_2, F)\big) = BUC\big(U_2, BUC(U_1, F)\big).$$

Furthermore, we do not make any notational difference between their elements.

Let $k_1, k_2 \in \mathbb{N}$ and denote by

$$C^{k_1,k_2}(U_1 \times U_2, F)$$

the space of all functions $u$ having partial derivatives $D_i^{\ell_i} u$ up to the order $k_i$ $(i = 1, 2)$ which are continuous on $U_1 \times U_2$. Similarly, we define the space

$$BUC^{k_1,k_2}(U_1 \times U_2, F)$$

to be the subspace of $C^{k_1,k_2}(U_1 \times U_2, F)$ consisting of all functions $u$ having partial derivatives $D_i^{\ell_i} u$ up to order $k_i$ $(i = 1, 2)$ in $BUC(U_1 \times U_2, F)$.

Let $k \in \mathbb{N}$. Then in an similar way as in $(A1.4)$ we shall make the identification

$$(A1.5) \qquad BUC^{k,0}(U_1 \times U_2, F) = BUC^k\big(U_1, BUC(U_2, F)\big).$$

In the general case $k_1, k_2 \in \mathbb{N}$ arbitrary, the situation is a bit more complicated and we refer to [**18**].

Next we turn to Hölder spaces on products. Let $U_1$, $U_2$ and $F$ be as above and suppose that $M_i \subset U_i$ $(i = 1, 2)$ and $\mu, \nu \in [0, 1) \cup \{1-\}$. For any function $u: U_1 \times U_2 \to F$ we set

$$[u]_{M_1,M_1,\mu,\nu} := \sup_{\substack{(x_1,x_2),(y_1,y_2) \in M_1 \times M_2 \\ (x_1,x_2) \neq (y_1,y_2)}} \frac{\|u(x_1, x_2) - u(y_1, y_2)\|}{\|(x_1 - y_1)\|^\mu + \|(x_2 - y_2)\|^\nu} ,$$

and define

$$C^{\mu,\nu}(U_1 \times U_2, F)$$

to be the subspace of $C(U_1 \times U_2, F)$ of all functions $u$ such that for all $(x_1, x_2) \in U_1 \times U_2$ there exists a neigbourhood $M_1 \times M_2$ such that $[u]_{M_1,M_1,\mu,\nu} < \infty$ holds.

Furthermore, we set

$$BUC^{\mu,\nu}(U_1, U_2, F) := \big\{u \in BUC(U_1 \times U_2);\ [u]_{U_1,U_1,\mu,\nu} < \infty\big\} .$$

Let now $k_1, k_2 \in \mathbb{N}$. Define

$$\begin{aligned} C^{k_1+\mu,k_2+\nu}&(U_1 \times U_2, F) := \\ &\big\{u \in C^{k_1,k_2}(U_1 \times U_2), F);\ \partial_i^{\ell_i} u, \partial_i^{\ell_i} u \in C^{\mu,\nu}(U_1 \times U_2, F) \\ &\qquad \text{whenever } 0 \le \ell_i \le k_i\ (i = 1, 2)\big\} \end{aligned}$$

as well as

$$\begin{aligned} BUC^{k_1+\mu,k_2+\nu}&(U_1 \times U_2, F) := \\ &\big\{u \in BUC^{k_1,k_2}(U_1 \times U_2), F);\ \partial_i^{\ell_i} u, \partial_i^{\ell_i} u \in BUC^{\mu,\nu}(U_1 \times U_2, F) \\ &\qquad \text{whenever } 0 \le \ell_i \le k_i\ (i = 1, 2)\big\} . \end{aligned}$$

Finally, we say that $u \in C^{k_1+\mu,k_2+\nu}(U_1 \times U_2, F)$ *uniformly in* $N_1 \times N_2$, if $u \in BUC^{k_1+\mu,k_2+\nu}(N_1 \times N_2, F)$, where $N_i$ are subsets of $U_i$ $(i = 1, 2)$.

**F. Analytic maps:** Let $E$, $F$ be Banach spaces and $U$ and open subset of $E$. A function $g \in C^\infty(U, F)$ is called *analytic*, if for any $x_0 \in U$ there exists a $\delta > 0$ such that

$$g(x_0 + h) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k g(x) h^k$$

holds for all $h \in E$ such that $\|h\| < \delta$, i.e. the Taylor series converges and represents locally the function $g$. Note that

$$D^k g(x_0) \in \mathcal{L}^k(\underbrace{E \times \cdots \times E}_{k\text{-times}}; F) \qquad \text{and} \qquad D^k g(x_0) h^k := D^k(x_0)(\underbrace{h, \dots, h}_{k\text{-times}}),$$

where $\mathcal{L}^k(E \times \cdots \times E; F)$ denotes the space of all continuous $k$-linear mappings from $E \times \cdots \times E$ to $F$. This space is a Banach space when equipped with the norm

$$|\!|\!| a |\!|\!| := \sup_{u_1, \dots, u_k \in E \setminus \{0\}} \frac{\|a(u_1, \dots, u_n)\|}{\|u_1\| \cdots \|u_k\|} \,.$$

The space of all analytic maps from $U$ to $F$ is denoted by

$$C^\omega(U, F).$$

The following theorem gives a characterizes analytic maps by some growth condition on the derivatives of $g$ (see e.g. Chae [**29**], Theorem 12.6).

**A1.5 Theorem**
*Let $g \in C^\infty(U, F)$. Then, $g$ is analytic if and only if for any $x_0$ in $U$ there exist constants $\delta > 0$ and $M > 0$ (possibly depending on $x_0$) such that*

$$|\!|\!| D^k g(x_0) |\!|\!| \leq M \delta^{-k} k!$$

*holds for all $k \in \mathbb{N}$.*

We end this account by introducing the notation for analytic function spaces on products. Let $E_1$ and $E_2$ be Banach spaces and $U_i$ $(i = 1, 2)$ open subsets of these spaces. Then we write

$$g \in C^{\omega,k}(U_1 \times U_2, F),$$

whenever for any $(x, y) \in U_1 \times U_2$ there exists a neighbourhood such that $D_2^j g(\cdot\,, y)$ $(0 \leq j \leq k)$ has a power series representation of the form

$$D_2^j g(x_0 + h) = \sum_{k=0}^{\infty} \frac{1}{k!} a_k^j(x, y) h^k$$

for all $h$ with norm sufficiently small.

More about analytic maps on Banach spaces may be found in Chae [**29**] or Hille and Philips [**69**].

## A2. Distributions and test functions

We give here the definition of the most widely used spaces of distributions and the corresponding spaces of test functions. For proofs and further properties of these spaces we refer to the vast literature on functional analysis, topological vector spaces and distributions such as [**52**], [**54**], [**71**], [**106**], [**121**], [**127**].

Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. Denote by

$$\mathcal{D}(\Omega) := \{u \in C^\infty(\Omega);\ \operatorname{supp} u \subset \Omega, \overline{\operatorname{supp} u} \text{ compact}\} \tag{A2.1}$$

the *space of test functions*. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges to $\varphi$ in $\mathcal{D}(\Omega)$ if and only if there exists a compact set $K$ in $\Omega$ such that $\operatorname{supp}\varphi_n$ is contained in $K$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\varphi_n = \varphi$ in $C^\infty(K)$. The space $\mathcal{D}(\Omega)$ equipped with this topology becomes a complete locally convex space.

Its (topological) dual is called the *space of distributions* and usually denoted by

$$\mathcal{D}'(\Omega).$$

If $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}'(\Omega)$, it converges to zero in $\mathcal{D}'(\Omega)$ if and only if

$$\lim_{n\to\infty} <u_n, \varphi> = 0 \tag{A2.2}$$

holds for all $\varphi \in \mathcal{D}(\Omega)$.

Every function $u \in L_{1,\mathrm{loc}}(\Omega)$, that is $\int_U |u(x)|\,dx < \infty$ for all bounded and closed subsets of $\Omega$, may be considered as an element of $\mathcal{D}'(\Omega)$ via the identification

$$<u, \varphi> := \int_\Omega u(x)\varphi(x)\,dx\,. \tag{A2.3}$$

Next we give a short description of the space of rapidly decreasing functions on $\mathbb{R}^n$. For a function $\varphi \in C^\infty(\Omega)$ and $k \in \mathbb{N}$ we define

$$p_k(\varphi) := \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} \big(1 + |x|^2\big)^k |\partial^\alpha \varphi(x)| \tag{A2.4}$$

if it exists. The *Schwartz space* or the *space of rapidly decreasing functions* is now defined by

$$\mathcal{S} := \mathcal{S}(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n);\ p_k(\varphi) < \infty \text{ for all } k \in \mathbb{N}\}. \tag{A2.5}$$

The family of seminorms $(p_k)_{k\in\mathbb{N}}$ makes $\mathcal{S}$ into a Fréchet space.

The dual $\mathcal{S}'$ of $\mathcal{S}$ is called the space of *tempered distributions*. It can be shown, that $\lim_{n\to\infty} u_n = 0$ in $\mathcal{S}'$ if and only if $(A2.2)$ holds for all $\varphi \in \mathcal{S}$.

The inclusions between the spaces defined above are given by

$$(A2.6) \qquad \mathcal{D}(\mathbb{R}^n) \overset{d}{\hookrightarrow} \mathcal{S} \overset{d}{\hookrightarrow} \mathcal{S}' \overset{d}{\hookrightarrow} \mathcal{D}'(\mathbb{R}^n).$$

Every element of $\mathcal{D}'(\Omega)$ and $\mathcal{S}'$ can be differentiated arbitrarily often, if differentiatiation is properly defined. For any multiindex $\alpha \in \mathbb{N}^n$ and $u \in \mathcal{D}'(\Omega)$, or $\mathcal{S}'$, we define

$$(A2.7) \qquad <\partial^\alpha u, \varphi> := (-1)^{|\alpha|} <u, \partial^\alpha \varphi>$$

for all $\varphi \in \mathcal{D}'(\Omega)$ or $\mathcal{S}'$, respectively. Moreover, we have that

$$(A2.8) \qquad \partial^\alpha \in \mathcal{L}\big(\mathcal{D}'(\Omega)\big) \cap \mathcal{L}(\mathcal{S}').$$

## A3. Sobolev spaces and interpolation

We divide this owerview into several subsections. In the first we define the notion of a domain of class $C^{\ell+\eta}$. The second is concerned with the definition of the classical Sobolev spaces $W_p^k(\Omega)$. In a third, we describe the spaces which are obtained by real and complex interpolation between Sobolev spaces defined on $\mathbb{R}^n$. It turns out that the interpolation spaces are the Sobolev-Slobodeckii and the Bessel-potential spaces, respectively. These results are generalized in the final subsection to spaces on a bounded domain with smooth boundary. For simplicity, we deal only with domains of class $C^\infty$ although weaker regularity conditions would suffice.

**A. Domains of class $C^{\ell+\eta}$:** Let $\Omega$ be a domain in $\mathbb{R}^n$. We say $\Omega$ is *of class* $C^{\ell+\eta}$ ($\ell \in \mathbb{N}^* \cup \{\infty\}$, $\eta \in [0,1) \cup \{1-\}$), if its boundary $\partial\Omega$ is a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. More precisely, for every $x \in \overline{\Omega}$, there exists an open neighbourhood $U_x$ of $x$ in $\mathbb{R}^n$ and a $C^{\ell+\eta}$-diffeomorphism $\varphi_x$ mapping $U_x$ onto the open unit ball $\mathbb{B}^n := \mathbb{B}_{\mathbb{R}^n}(0,1)$ and $U_x \cap \Omega$ onto $\mathbb{B}^n \cap \mathbb{H}^n$. By $\mathbb{H}^n$ we denote the half space $\mathbb{H}^n := \mathbb{R}^{n-1} \times (0,\infty)$.

The family $(U_x, \varphi_x)_{x\in\overline{\Omega}}$ is called $C^{\ell+\eta}$*-atlas* for $\overline{\Omega}$ and $(U_x, \varphi_x)$ a (*coordinate*) *chart*.

If $\Omega$ is a bounded domain of class $C^{\ell+\eta}$, then, by the compactness of $\overline{\Omega}$, there exists a finite atlas $(U_i, \varphi_i)_{i=1,\dots,N}$ for $\overline{\Omega}$, that is $(U_i, \varphi_i)$ $(i = 1, \dots, N)$ are charts and $\bigcup_{i=1}^N U_i \supset \overline{\Omega}$. In this case there exists a $C^\infty$-partition of unity which is subordinate to the covering $(U_i)_{i=1,\dots,N}$ of $\overline{\Omega}$, that is a family $(\pi_i)_{i=1,\dots,N}$ of functions such that

(*i*) $\pi_i \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} \pi_i \subset U_i$ $(i = 1, \dots, N)$,

(*ii*) $0 \le \pi_i \le 1$ $(i = 1, \dots, N)$,

(*iii*) $\sum_{i=1}^N \pi_i(x) = 1$ for all $x \in \overline{\Omega}$.

**B. Sobolev spaces:** Let $n \geq 1$ and $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. If $u$ is any measurable function we put

$$(A3.1) \qquad \|u\|_p := \begin{cases} \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \operatorname*{ess-sup}_{x \in \Omega} |u(x)| & \text{if } p = \infty. \end{cases}$$

As usual, the essential supremum of a measurable function is given as follows:

$$(A3.2) \qquad \operatorname*{ess-sup}_{x \in \Omega} |u(x)| := \inf\bigl\{M > 0;\ \lambda_n\bigl(\{x \in \Omega;\ |u(x)| > M\}\bigr) = 0\bigr\},$$

where $\lambda_n$ is the $n$-dimensional Lebesgue measure. We now define for any $p \in [1, \infty]$ the *Lebesgue spaces*

$$(A3.3) \qquad L_p(\Omega, \mathbb{K}) = \{u: \Omega \to \mathbb{K}^n \text{measurable};\ \|u\|_p < \infty\}.$$

Equipped with the norm $\|\cdot\|_p$ they become Banach spaces. If no confusion seems possible we shall write $L_p(\Omega)$ instead of $L_p(\Omega, \mathbb{K})$.

We are now in a position to give the definition of the Sobolev spaces. For any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ put

$$(A3.4) \qquad W_p^k(\Omega) := W_p^k(\Omega, \mathbb{K}) := \bigl\{u \in L_p(\Omega);\ \partial^\alpha u \in L_p(\Omega) \text{ for all } |\alpha| \leq k\bigr\}.$$

In the definition of $W_p^k(\Omega, \mathbb{K})$, the derivatives of $u$ are taken in the sense of distributions. This is possible since $L_p(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. When equipped with the norm

$$(A3.5) \qquad \|u\|_{k,p} := \|u\|_{W_p^k(\Omega)} := \begin{cases} \Bigl(\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_p^p\Bigr)^{1/p} & \text{if } 1 \leq p < \infty \\ \max_{|\alpha| \leq k} \|\partial^\alpha u\|_\infty & \text{if } p = \infty\,, \end{cases}$$

they become Banach spaces (see e.g. [**2**], Theorem 3.2). Moreover, $W_p^k(\Omega)$ is separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$ (see e.g. [**2**], Theorem 3.5). For convenience we put $W_p^0 = L_p$ and $\|\cdot\|_{0,p} = \|\cdot\|_p$.

Of course there are smooth functions contained in the Sobolev spaces. In fact, it can be shown (see e.g. [**97**], [**131**], Theorem 2.3.2), that

$$(A3.6) \qquad C^\infty(\Omega) \cap W_p^k(\Omega) \quad \text{is dense in } W_p^k(\Omega)$$

whenever $1 \leq p < \infty$. If moreover $\Omega$ is of class $C^1$, it can be shown (see e.g. [**2**], Theorem 3.18), that the set of restrictions of functions in $\mathcal{D}(\mathbb{R}^n)$ to $\Omega$ is dense in $W_p^k(\Omega)$. This implies in particular, that (if $\Omega$ is of class $C^1$)

$$(A3.7) \qquad W_p^{k_1}(\Omega) \overset{d}{\hookrightarrow} W_p^{k_2}(\Omega)$$

whenever $k_1, k_2 \in \mathbb{N}$ with $k_2 \leq k_1$ and $p < \infty$.

Many properties of $W_p^k(\Omega)$ such as imbedding and interpolation properties are obtained by using the corresponding results for spaces in $\mathbb{R}^n$. The main tool which allows us to do this is an extension operator

$$(A3.8) \qquad \mathcal{E} \in \mathcal{L}\big(W_p^k(\Omega), W_p^k(\mathbb{R}^n)\big).$$

By an extension operator we mean an operator satisfying $(A3.8)$ such that $\mathcal{E}u \restriction_\Omega = u$ for all $u \in W_p^k(\Omega)$. The existence of such an operator depends on the geometric properties of the boundary of $\Omega$. We have the following

**A3.1 Proposition**
*Let $\Omega$ be a bounded domain of class $C^\infty$ and $\ell \in \mathbb{N}^*$ fixed. Then there exists an extension operator*

$$\mathcal{E} \in \mathcal{L}\big(W_p^k(\Omega), W_p^k(\mathbb{R}^n)\big)$$

*independent of $1 \leq p \leq \infty$ and $0 \leq k \leq \ell$. It depends only on $\Omega$ and $\ell$.*

A proof can be found in [**2**], Theorem 4.26. We proceed now to give characterizations of the real and complex interpolation spaces between Sobolev spaces on $\mathbb{R}^n$.

**C. Interpolation of the Sobolev spaces in $\mathbb{R}^n$:** For notational simplicity we write throughout $W_p^k$ instead of $W_p^k(\mathbb{R}^n)$. By $(A3.7)$, it holds that

$$W_p^{k+1} \overset{d}{\hookrightarrow} W_p^k \qquad (k \in \mathbb{N},\ 1 \leq p < \infty).$$

Therefore, $(W_p^k, W_p^{k+1})$ is a Banach couple (see Section 3.B) and it is possible to apply an interpolation method to it. We shall describe here the spaces obtained by real and complex interpolation. These interpolation methods – denoted by $(\cdot\,,\cdot)_{\theta,p}$ and $[\cdot\,,\cdot]_\theta$, respectively – were introduced in Section 4. It turns out that $(W_p^k, W_p^{k+1})_{\theta,p}$ are the *Sobolev-Slobodeckii spaces* and $[W_p^k, W_p^{k+1}]_\theta$ the *Bessel-potential spaces.*

Let $1 \leq p < \infty$, $\sigma \in (0,1)$ and $u\colon \mathbb{R}^n \to \mathbb{K}$ a measurable function. Then we define

$$(A3.9) \qquad I_{\sigma,p}(u) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\sigma p}}\, dx\, dy.$$

If now $s > 0$ is a noninteger real number, we denote its integer part by $[s]$, i.e. $[s]$ is the largest integer satisfying $[s] < s$. Then we define

$$(A3.10) \qquad W_p^s := W_p^s(\mathbb{R}^n) := \big\{ u \in W_p^{[s]};\ I_{s-[s],p}(\partial^\alpha u) < \infty \text{ for } |\alpha| = [s] \big\}$$

and equip it with the norm

$$(A3.11) \qquad \|u\|_{W_p^s} := \Big(\|u\|_{[s],p}^p + \sum_{|\alpha|=[s]} I_{s-[s],p}(\partial^\alpha u)\Big)^{1/p},$$

which makes it into a Banach space. Moreover, $W_p^s$ is separable for $1 \le p < \infty$ and reflexive if $1 < p < \infty$ (see e.g. [**60**], Theorem 6.8.4). The spaces $W_p^s$ are called *Sobolev-Slobodeckii spaces.*

**A3.2 Remark**

The Sobolev-Slobodeckii spaces are special cases of the so called *Besov spaces* $B_{p,q}^s := B_{p,q}^s(\mathbb{R}^n)$. It holds that

$$W_p^s \doteq B_{p,p} \qquad \text{if } s \notin \mathbb{N},\ s > 0$$

(see e.g. [**122**], Remark 2.5.1/4). If $s \in \mathbb{N}$ and $p \neq 2$ this is not true (see e.g. [**60**], Section 8.4.6). □

Let us now turn to the definition of the Bessel-potential spaces. Suppose that $s$ is any real number (not necessarily positive) and that $1 < p < \infty$. Then we define the space

$$(A3.12) \qquad H_p^s := H_p^s(\mathbb{R}^n) := \big\{u \in \mathcal{S}';\ \mathcal{F}^{-1}\big((1+|x|^2)^{-s/2}\mathcal{F}u\big) \in L_p(\mathbb{R}^n)\big\}$$

and the norm

$$(A3.13) \qquad \|u\|_{H_p^s} := \|\mathcal{F}^{-1}\big((1+|x|^2)^{-s/2}\mathcal{F}u\big)\|_p\,.$$

Here, $\mathcal{S}'$ is the space of tempered distributions introduced in Appendix 2 and $\mathcal{F}$ the Fourier transform on $\mathcal{S}'$ defined in Section 1 by (1.24) and (1.25). The spaces $H_p^s$ are Banach spaces when equipped with the norm ($A$3.13) (see e.g. [**22**], Section 6.2 or [**2**], Section 7.59). The spaces $H_p^s$ are called *Bessel-potential spaces* or sometimes *generalized Lebesgue spaces* or *Liouville spaces* and denoted by $L_p^k(\mathbb{R}^n)$.

It can be shown (see e.g. [**122**], Theorem 2.3.3) that

$$(A3.14) \qquad H_p^k(\mathbb{R}^n) \doteq W_p^k(\mathbb{R}^n)$$

if $k \in \mathbb{N}$.

**A3.3 Remark**

The Bessel-potential spaces are special cases of the *Triebel spaces* $F_{p,q}^s(\mathbb{R}^n)$. It holds that

$$H_p^s(\mathbb{R}^n) \doteq F_{p,2}(\mathbb{R}^n)$$

These facts can be found for instance in [**122**], Theorem 2.3.3. □

We now have two scales of Banach spaces at our disposal which contain the Sobolev spaces for the natural numbers:

$$W_p^s(\mathbb{R}^n) \qquad (s \geq 0) \qquad \text{the Sobolev-Slobodeckii spaces}$$

$$H_p^s(\mathbb{R}^n) \qquad (s \in \mathbb{R}^n) \qquad \text{the Bessel-potential spaces.}$$

For noninteger $s \geq 0$ they can be recovered as interpolation spaces between the Sobolev spaces as our next theorem shows.

**A3.4 Theorem**
*Let $s \geq 0$ be a noninteger real number and $1 < p < \infty$. Then*

$$H_p^s \doteq [W_p^{[s]}, W_p^{[s]+1}]_{s-[s]}$$

*and*

$$W_p^s \doteq (W_p^{[s]}, W_p^{[s]+1})_{s-[s],p}.$$

*Furthermore, $\mathcal{D}(\mathbb{R}^n)$ is dense in $W_p^s$ and $H_p^s$.*

**Proof**
The interpolation results follow from $(A3.14)$, Remark A3.2 and [**22**], Theorem 6.2.4 and 6.4.5 or [**122**], Remark 2.4.2/2. The density of $\mathcal{D}(\mathbb{R}^n)$ is a consequence of Remark A3.2 and A3.3 and [**122**], Theorem 2.3.2. □

In particular, it follows from the above theorem, that

$$(A3.15) \qquad W_p^{s_1}(\mathbb{R}^n) \overset{d}{\hookrightarrow} W_p^{s_0}(\mathbb{R}^n) \qquad \text{and} \qquad H_p^{s_1}(\mathbb{R}^n) \overset{d}{\hookrightarrow} H_p^{s_0}(\mathbb{R}^n)$$

whenever $0 \leq s_0 < s_1$. Therefore, $(W_p^{s_0}, W_p^{s_1})$ and $(H_p^{s_0}, H_p^{s_1})$ are again Banach couples. They have the following interpolation properties:

**A3.5 Theorem**
*Let $1 < p < \infty$, $0 \leq s_0 < s_1$ and $0 < \theta < 1$ be real numbers and define*

$$s := (1-\theta)s_0 + \theta s_1.$$

*Then we have that*

(i) $[H_p^{s_0}, H_p^{s_1}]_\theta \doteq H_p^s$

(ii) $(H_p^{s_0}, H_p^{s_1})_{\theta,p} \doteq W_p^s$

(iii) $[W_p^{s_0}, W_p^{s_1}]_\theta \doteq W_p^s$ *if either $s_0, s_1, s \notin \mathbb{N}$ or $s_0, s_1, s \in \mathbb{N}$*

(iv) $(W_p^{s_0}, W_p^{s_1})_{\theta,p} \doteq W_p^s$ *if $s \notin \mathbb{N}$.*

**Proof**
The assertion follows from Remark A3.2 and [**122**], Theorem 2.4.2 and Remark 2.4.2/2 or [**22**], Theorem 6.4.5. □

The following imbedding results relate the Sobolev-Slobodeckii spaces and the Bessel-potential spaces:

**A3.6 Proposition**
*Let $s > 0$ and $1 < p < \infty$ be arbitrary. Then*

$$(A3.16) \qquad W_p^s \overset{d}{\hookrightarrow} H_p^s \quad \text{if } 1 < p \leq 2 \qquad \text{and} \qquad H_p^s \overset{d}{\hookrightarrow} W_p^s \quad \text{if } 2 \leq p < \infty$$

*holds. Moreover, we have that*

$$(A3.17) \qquad H_p^{s+\varepsilon} \overset{d}{\hookrightarrow} W_p^s \overset{d}{\hookrightarrow} H_p^{s-\varepsilon}$$

*and*

$$(A3.18) \qquad W_p^{s+\varepsilon} \overset{d}{\hookrightarrow} H_p^s \overset{d}{\hookrightarrow} W_p^{s-\varepsilon}$$

*for all $\varepsilon \in (0, s]$.*

**Proof**
The first part follows from [**22**], Theorem 6.4.4 and the density of $\mathcal{S}$ in $H_p^s$ and $W_p^s$, respectively (Theorem A3.4). The second assertion follows from Theorem A3.4 and the properties of the real and complex interpolation methods described in Remark 4.15. □

Finally, we would like to give some imbedding theorems for the spaces considered above:

**A3.7 Theorem**
*Let $1 < p < \infty$ and $s, t \geq 0$. Then*

$$(A3.19) \qquad W_p^s(\mathbb{R}^n) \hookrightarrow BUC^t(\mathbb{R}^n) \qquad \text{if } s - \frac{n}{p} > t$$

*and*

$$(A3.20) \qquad H_p^s(\mathbb{R}^n) \hookrightarrow BUC^t(\mathbb{R}^n) \qquad \text{if } s - \frac{n}{p} > t$$

*hold. If $t$ is not an integer, the imbeddings hold also if $s - \frac{n}{p} = t$.*

A proof of this theorem can be found in [**122**], Theorem 2.8.1. In the next subsection, these results are generalized to spaces on domains.

**D. Spaces on domains:** Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. Then the Sobolev-Slobodeckii spaces can be defined in the same way as in the case of $\mathbb{R}^n$: Let $u\colon \Omega \to \mathbb{K}$ be a measurable function and $\sigma \in (0,1)$. Then we put

$$I_{\sigma,p}(u) := \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\sigma p}}\, dx\, dy$$

and

$$W_p^s(\Omega) := \bigl\{u \in W_p^{[s]};\ I_{s-[s],p}(\partial^\alpha u) < \infty \text{ for } |\alpha| = [s]\bigr\}$$

for any noninteger positive real number. When equipped with the norm

$$\|u\|_{W_p^s} := \Bigl(\|u\|_{[s],p}^p + \sum_{|\alpha|=[s]} I_{s-[s],p}(\partial^\alpha u)\Bigr)^{1/p},$$

$W_p^s(\Omega)$ becomes a Banach space as in the case of $\mathbb{R}^n$.

There is another possibility to define the spaces $W_p^s(\Omega)$, namely by restriction: Set

$$\tilde{W}_p^s(\Omega) := \bigl\{u \in L_p(\Omega);\ \text{there exists } v \in W_p^s(\mathbb{R}^n) \text{ with } v\restriction_\Omega = u\bigr\}.$$

On this space we define a norm by

$$\|u\|_{\tilde{W}_p^s(\Omega)} := \inf\bigl\{\|v\|_{W_p^s(\mathbb{R}^n)};\ v \in W_p^s(\mathbb{R}^n) \text{ with } v\restriction_\Omega = u\bigr\},$$

which makes $\tilde{W}_p^s(\Omega)$ into a Banach space.

The question is now whether these two definition lead to the same spaces. The answer turns out to be positive, if the domain is smooth enough. In fact, this is an easy consequence of the following extension theorem (see e.g. [**2**], Lemma 7.45 or [**10**], Lemma 11.2):

**A3.8 Proposition**

*Let $\Omega$ be a bounded domain of class $C^\infty$ and $\ell \in \mathbb{N}^*$. Then there exists an extension operator*

$$\mathcal{E} \in \mathcal{L}\bigl(W_p^s(\Omega), W_p^s(\mathbb{R}^n)\bigr)$$

*independent of $1 \le p < \infty$ and $0 \le s \le \ell$. Moreover, $\mathcal{E}$ depends only on $\Omega$ and $\ell$ and coincides with the operator from Proposition* A3.1.

**A3.9 Corollary**
*Let $1 \leq p < \infty$ and suppose that $\Omega$ is a bounded domain of class $C^\infty$. Then*

$$\tilde{W}_p^s(\Omega) \doteq W_p^s(\Omega)$$

*for all $s \in \mathbb{R}$.*

For this reason, we shall not make a difference between the two definitions when dealing with smooth domains.

Let us now turn to the Bessel-potential spaces. In contrast to the Sobolev-Slobodeckii spaces it is not possible to carry over the construction to spaces on domains, since the Fourier transformation is only defined on $\mathbb{R}^n$. Hence, we define the space $H_p^s(\Omega)$ by restriction. Let $\Omega$ be an arbitrary domain. Then we define for any $s \geq 0$

$$(A3.21) \qquad H_p^s(\Omega) := \big\{u \in L_p(\Omega);\ \text{there exists } v \in H_p^s(\mathbb{R}^n) \text{ with } v\restriction_\Omega = u\big\}.$$

which we endow with the norm

$$(A3.22) \qquad \|u\|_{H_p^s(\Omega)} := \inf\big\{\|v\|_{H_p^s(\mathbb{R}^n)};\ v \in H_p^s(\mathbb{R}^n) \text{ with } v\restriction_\Omega = u\big\}.$$

With this norm, $H_p^s(\Omega)$ becomes a Banach space. As in the case of $\mathbb{R}^n$ we get the following characterization of the Sobolev-Slobodeckii and Bessel-potential spaces.

**A3.10 Theorem**
*Let $\Omega$ be a bounded domain of class $C^\infty$ and $1 < p < \infty$. Then for any noninteger $s \in \mathbb{R}$ it holds that*

$$H_p^s(\Omega) \doteq \big[W_p^{[s]}(\Omega), W_p^{[s]+1}(\Omega)\big]_{s-[s]}$$

*and*

$$W_p^s(\Omega) \doteq \big(W_p^{[s]}(\Omega), W_p^{[s]+1}(\Omega)\big)_{s-[s],p}.$$

*Moreover, for any $k \in \mathbb{N}$, there exists an extension operator*

$$\mathcal{E} \in \mathcal{L}\big(W_p^s(\Omega), W_p^s(\mathbb{R}^n)\big) \cap \mathcal{L}\big(H_p^s(\Omega), H_p^s(\mathbb{R}^n)\big)$$

*independent of $1 < p < \infty$ and $0 \leq s \leq k$ and depending only on $k$ and $\Omega$. Furthermore, $C^\infty(\overline{\Omega})$ is dense in $W_p^s(\Omega)$ and $H_p^s(\Omega)$.*

**Proof**
Let us denote the restriction operator $u \mapsto u\restriction_\Omega$ by $\mathcal{R}_\Omega$. Then, by Corollary A3.9 and the definition of $H_p^s(\Omega)$, it is clear that

$$\mathcal{R}_\Omega \in \mathcal{L}\big(W_p^s(\mathbb{R}^n), W_p^s(\Omega)\big) \cap \mathcal{L}\big(H_p^s(\mathbb{R}^n), H_p^s(\Omega)\big).$$

By Theorem A3.4, the assertions are true for $\Omega = \mathbb{R}^n$. By the properties of the interpolation methods, the following diagram is commutative:

$$\begin{array}{ccccc}
W_p^{[s]+1}(\mathbb{R}^n) & \overset{d}{\hookrightarrow} & W_p^s(\mathbb{R}^n) & \overset{d}{\hookrightarrow} & W_p^{[s]}(\mathbb{R}^n) \\
{\scriptstyle \mathcal{R}_\Omega}\big\downarrow\big\uparrow{\scriptstyle \mathcal{E}} & & {\scriptstyle \mathcal{R}_\Omega}\big\downarrow\big\uparrow{\scriptstyle \mathcal{E}} & & {\scriptstyle \mathcal{R}_\Omega}\big\downarrow\big\uparrow{\scriptstyle \mathcal{E}} \\
W_p^{[s]+1}(\Omega) & \overset{d}{\hookrightarrow} & \big(W_p^{[s]}(\Omega), W_p^{[s]+1}(\Omega)\big)_{s-[s],p} & \overset{d}{\hookrightarrow} & W_p^{[s]}(\Omega)
\end{array}$$

The density of the injections in the upper row is due to the fact that the real interpolation method is admissible (see Section 4.A).

Therefore, we see that $\big(W_p^{[s]}(\Omega), W_p^{[s]+1}(\Omega)\big)_{s-[s],p} = W_p^s(\Omega)$ as sets. Since $\mathcal{R}_\Omega\mathcal{E}$ is a bounded injection, by the open mapping theorem, the spaces have equivalent norms. The analogous assertion for the Bessel-potential spaces follows in the same way replacing the real by the complex interpolation method.

The density of $C^\infty(\overline{\Omega})$ follows from the corresponding result for the Sobolev spaces stated after ($A3.6$). □

In a similar way, one may obtain the following interpolation theorem:

**A3.11 Theorem**
*Let $\Omega$ be a domain of class $C^\infty$. Then the assertions of Theorem* A3.5 *and Proposition* A3.6 *remain valid for the corresponding spaces over $\Omega$.*

Furthermore, the following imbedding theorem hold:

**A3.12 Theorem**
*Let $\Omega$ be a domain of class $C^\infty$. Then the assertions of Theorem* A3.7 *remain valid for the corresponding spaces over $\Omega$. Moreover, these imbeddings are compact.*

**Proof**
For the Sobolev-Slobodeckii spaces the assertion follows from Theorem A3.7 and A3.10 considering the following commutative diagram:

$$\begin{array}{ccc}
W_p^s(\mathbb{R}^n) & \hookrightarrow & BUC^t(\mathbb{R}^n) \\
\big\uparrow{\scriptstyle \mathcal{E}} & & {\scriptstyle \mathcal{R}_\Omega}\big\downarrow \\
W_p^s(\Omega) & \hookrightarrow & C^t(\overline{\Omega})
\end{array}$$

The same diagram may be constructed if $W_p^s$ is replaced by $H_p^s$.

The compactness of the imbeddings follows for example from [**2**], Theorem 6.2 or [**131**], Theorem 2.5.1 and Remark 2.5.2. □

Actually, we also have imbeddings in the opposite direction. In fact, this turns out to be much easier to prove. Since we were not able to find a precise reference we include a proof.

**A3.13 Theorem**
*Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $s \geq 0$. Then for any $\alpha \in \big(s-[s],1\big]$ we have that*

$$C^{[s]+\alpha}(\overline{\Omega}) \hookrightarrow W_p^s(\Omega) \tag{A3.23}$$

*If the hypotheses of Theorem* A3.11 *are satisfied, the assertion holds also for the Bessel-potential spaces.*

**Proof**
We shall only prove the case where $s \in (0,1)$. If $s > 1$, the assertion is obtained applying the results for $s \in (0,1)$ to the derivatives of highest order.

Let $s$ be in $(0,1)$ and $u$ in $C^\alpha(\overline{\Omega})$ with $\alpha \in (s,1)$. Then there exists a constant $c_0 > 0$ such that

$$|u(x)-u(y)| \leq c_0|x-y|^\alpha$$

holds for all $x$ and $y$ in $\overline{\Omega}$. Since $\Omega$ is bounded, $\alpha - s > 0$ and passing over to spherical coordinates we see that

$$I_{s,p}(u) := \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\sigma p}}\,dx\,dy \leq c_0^p \iint_{\Omega\times\Omega} |x-y|^{-n+(\alpha-s)p}\,dx\,dy < \infty$$

This gives the required estimates for the noninteger part of the norm. For the integer part note that $C^\alpha(\overline{\Omega})$ is continuously imbedded in $L_p(\Omega)$. Hence, $(A3.23)$ is proved.

The assertion for the Bessel-potential spaces follows from $(A3.18)$ which is valid by Theorem A3.11. □

**E. Anisotropic Sobolev spaces on a cylinder:** Let $\Omega$ be a domain of class $C^2$ in $\mathbb{R}^n$ and $T > 0$. As usual, we denote a generic point of $\Omega \times (0,T)$ by $(x,t)$ and by $\alpha \in \mathbb{N}^n$ a multiindex. Then we define for any $p \in [1,\infty)$ the anisotropic Sobolev space $W_p^{2,1}\big(\Omega \times [0,T]\big)$ by

$$W_p^{2,1}\big(\Omega\times[0,T]\big) := \big\{u \in L_p\big(\Omega\times[0,T]\big);\ \partial_t u, \partial_x^\alpha u \in L_p\big(\Omega\times[0,T]\big) \text{ for } |\alpha| \leq 2\big\}$$

and equip it with the norm

$$\|u\|_{W_p^{2,1}} := \Big(\sum_{|\alpha|\leq 2} \|\partial_x^\alpha u\|_p^p + \|\partial_t\|_p^p\Big)^{1/p}.$$

The space $W_p^{2,1}\big(\Omega \times [0,T]\big)$ endowed with this norm becomes a Banach space. In the following theorem, we list some imbedding properties of these spaces (see e.g. [**23**], Theorem 10.4).

**A3.14 Theorem**
*If $p > (n+2)/2$ we have that*

$$W_p^{2,1}\big(\Omega \times (0,T)\big) \hookrightarrow C\big(\overline{\Omega} \times [0,T]\big).$$

*Moreover, if $p > n+2$ we have that*

$$W_p^{2,1}\big(\Omega \times (0,T)\big) \hookrightarrow C^{1,0}\big(\overline{\Omega} \times [0,T]\big).$$

## A4. Boundary spaces and the trace operator

We start giving some notation which is necessary to define the Sobolev-Slobodeckiĭ spaces on the boudary of a smooth domain. Then we state the trace theorems which allow to put an initial-boundary value problem in the $L_p$-setting.

**A. The boundary of a smooth domain as a manifold:** Let $\Omega$ be a domain of class $C^\infty$ in $\mathbb{R}^n$ and $(U_x, \varphi_x)_{x\in\overline{\Omega}}$ the atlas for $\overline{\Omega}$ which was introduced in Subsection A of Appendix 3. We define for any $x \in \partial\Omega$

$$V_x := U_x \cap \partial\Omega \qquad \text{and} \qquad \psi_x := \varphi_x \restriction_{V_x} : V_x \to \mathbb{B}^{n-1} \times \{0\}.$$

Then $(V_x, \psi_x)_{x\in\partial\Omega}$ is an atlas for $\partial\Omega$ as a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. If $\partial\Omega$ is compact, it can be covered by a finite number of $V_x$. Hence, there exists a finite atlas $(V_i, \psi_i)_{i=1,\dots,N}$ for $\partial\Omega$.

A function $f : \partial\Omega \to \mathbb{R}$ is called a $C^k$ function, if $f \circ \psi_x \in C^k(\mathbb{B}^{n-1}, \mathbb{R})$ for all $x \in \partial\Omega$. If $\Omega$ has a finite atlas, this is equivalent by saying that $f \circ \psi_i \in C^k(\mathbb{B}^{n-1}, \mathbb{R}^n)$ for all $i = 1, \dots, N$. The totality of $C^k$ functions we denote by

$$C^k(\partial\Omega) := C^k(\partial\Omega, \mathbb{R}).$$

Note that this definition is independent of the special choice of the atlas.

Assume for the rest of this subsection that $(V_i, \psi_i)_{i=1,\dots,N}$ is a $C^\infty$-atlas for $\partial\Omega$. Then, it can be shown that there exists a $C^\infty$-partition of unity subordinate to the covering $(V_i)_{i=1,\dots,N}$ of $\partial\Omega$. By this we mean a family $(\tilde{\pi}_i)_{i=1,\dots,N}$ of functions $\tilde{\pi}_i : \partial\Omega \to \mathbb{R}$ such that

$(i)$ $\tilde{\pi}_i \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} \pi_i \subset U_i$ $(i = 1, \ldots, N)$,

$(ii)$ $0 \le \tilde{\pi}_i \le 1 \qquad (i = 1, \ldots, N)$,

$(iii)$ $\sum_{i=1}^{N} \tilde{\pi}_i(x) = 1$ for all $x \in \partial\Omega$.

Observe that one may obtain $(\tilde{\pi}_i)_{i=1,\ldots,N}$ from a partition of unity $(\pi_i)_{i=1,\ldots,\overline{N}}$ associated to a finite atlas of $\Omega$ as it was described in Appendix 3 by restriction.

**B. The Sobolev-Slobodeckii spaces on $\partial\Omega$:** In this subsection we give a description of the Sobolev-Slobodeckii spaces on $\partial\Omega$ by means oflocal coordinates, if $\Omega$ is a domain of class $C^\infty$. In the same way one may define Bessel-potential spaces on $\partial\Omega$, but it turns out, that this is not necessary for the description of the boundary values of $W_p^s$-and $H_p^s$-functions.

In the sequel, let $\Omega$ be a bounded domain of class $C^\infty$ with atlas $\mathfrak{A} := (V_i, \psi_i)_{i=1,\ldots,N}$. Suppose that $(\pi_i)_{i=1,\ldots,N}$ is a $C^\infty$-partition of unity which is subordinate to the covering $(V_i)_{i=1,\ldots,N}$ of $\partial\Omega$. Then we define for any $s \in \mathbb{R}$ and $1 \le p < \infty$ the space

$$W_p^{s,\mathfrak{A}}(\partial\Omega) := \bigl\{u\colon \partial\Omega \to \mathbb{R};\ (\pi_i u) \circ \psi^{-1} \in W_p^s(\mathbb{B}^{n-1}) \text{ for } i = 1, \ldots, N\bigr\}$$

and equip it with the norm

$$(A4.1) \qquad \|u\|_{W_p^s}^{\mathfrak{A}} := \Bigl(\sum_{i=1}^{N} \|(\pi_i u) \circ \psi_i^{-1}\|_{W_p^s(\mathbb{B}^{n-1})}\Bigr)^{1/p}.$$

If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two finite atlases for $\partial\Omega$, it can be shown that the norms $\|\cdot\|_{W_p^s}^{\mathfrak{A}_1}$ and $\|\cdot\|_{W_p^s}^{\mathfrak{A}_2}$ are equivalent. Hence the spaces $W_p^{s,\mathfrak{A}}(\partial\Omega)$ is independent of the special choice of $\mathfrak{A}$ for $\partial\Omega$. We thus omit the index $\mathfrak{A}$ and write simply

$$W_p^s(\partial\Omega)$$

and equip it with one of the equivalent norms of the form $(A4.1)$, which we denote by $\|\cdot\|_{W_p^s}$. It can be shown that $W_p^s(\partial\Omega)$ is a Banach space. For the proofs of these facts we refer to [**122**], Section 3.6.1.

**C. The trace operator:** Let $\Omega$ be a domain of class $C^\infty$ in $\mathbb{R}^n$. If $u \in C(\overline{\Omega})$ we define

$$\gamma\, u := u \restriction_{\partial\Omega}$$

and call $\gamma\, u$ the *trace of $u$ on* $\partial\Omega$ and $\gamma$ the *trace operator*. Observe that

$$\gamma \in \mathcal{L}\bigl(C^k(\overline{\Omega}), C^k(\partial\Omega)\bigr)$$

holds for all $k \in \mathbb{N}$. When dealing with elliptic boundary value problems in an $L_p$-setting, we are forced to make sense of the 'trace' of a function defined almost everywhere. Of course this is not possible in the classical sense since $\partial\Omega$ is a set of measure zero. Nevertheless, it is possible to extend the trace operator to suitable subspaces of $L_p(\Omega)$. The following theorem gives the precise assertions.

**A4.1 Theorem**

*Let $\Omega$ be a domain of class $C^\infty$ in $\mathbb{R}^n$. Then the trace operator $\gamma \in \mathcal{L}\big(C^k(\overline{\Omega}), C^k(\partial\Omega)\big)$ has a unique extension, which we denote again by $\gamma$, such that*

$$\gamma \in \mathcal{L}\big(W_p^s(\Omega), W_p^{s-1/p}(\partial\Omega)\big) \cap \mathcal{L}\big(H_p^s(\Omega), W_p^{s-1/p}(\partial\Omega)\big)$$

*whenever $1 < p < \infty$ and $1/p < s < \infty$).*

**Proof**

The assertion follows as in [**122**], Theorem 4.7.1. □

**A4.2 Remark**

It is possible to consider unbounded domains with noncompact boundaries. In this case, one has to require more conditions on the atlas of $\partial\Omega$. For precise statements we refer to [**10**] and [**25**]. □

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## ABOUT THIS VOLUME

In this Research Note the authors have endeavoured to give an introduction to the theory of abstract semilinear evolution equations of parabolic type, with special emphasis on periodic problems. Throughout they have made use of the theory of interpolation spaces rather then fractional power spaces. While the latter involves a lot of technicalities, the former allows a more elegant and complete treatment of semilinear problems, and brings conceptual clarity. It is shown how these abstract results can be applied to concrete reaction-diffusion equations.

Many of the results appear for the first time in book form and thus these notes should serve as a useful reference.

*Readership*: Researchers and postgraduate students in linear and nonlinear differential equations and dynamical systems.

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