THE DIRICHLET PROBLEM BY VARIATIONAL METHODS

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be an bounded open set and $\varphi \in C(\partial \Omega)$. Assume that $\varphi$ has an extension $\Phi \in C(\bar{\Omega})$ such that $\Delta \Phi \in H^{-1}(\Omega)$. Then by the Riesz representation theorem there exists a unique $u \in H^1_0(\Omega)$ such that $-\Delta u = \Delta \Phi$ in $\mathcal{D}(\Omega)'$.

We show that $u + \Phi$ coincides with the Perron solution of the Dirichlet problem

$$\Delta h = 0, \quad h|_{\partial \Omega} = \varphi.$$ 


1. The main result and its consequences

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ with boundary $\partial \Omega$. Let $\varphi \in C(\partial \Omega)$. We consider the Dirichlet problem

$$D(\varphi, \Omega) \quad h \in \mathcal{H}(\Omega) \cap C(\bar{\Omega}), \quad h|_{\partial \Omega} = \varphi,$$

where $\mathcal{H}(\Omega) := \{u \in C^2(\Omega) : \Delta u = 0\}$ denotes the space of all harmonic functions. It follows from the maximum principle that $D(\varphi, \Omega)$ has at most one solution. We say that $\Omega$ is Dirichlet regular if for each $\varphi \in C(\partial \Omega)$ there exists a solution $h$ of $D(\varphi, \Omega)$. Such a solution will be called a classical solution in what follows.

If $\Omega$ is not Dirichlet regular, then there always exists a generalised solution of $D(\varphi, \Omega)$ namely the Perron solution $h_\varphi$ (see Section 2 for the definition). Moreover, if $D(\varphi, \Omega)$ has a classical solution $h$, then $h = h_\varphi$. There is an elaborate theory describing the points $z \in \partial \Omega$ for which $\lim_{x \to z} h_\varphi(x) = \varphi(z)$ for all $\varphi \in C(\partial \Omega)$, those are called the regular points (this is equivalent to the existence of a barrier at $z$, see e.g. Kellogg [12, Section XI.17]).

Our aim is to express that $h_\varphi = \varphi$ on $\partial \Omega$ in a weak sense instead by pointwise convergence. We denote by $H^1(\Omega) := \{u \in L^2(\Omega) : D_j u \in L^2(\Omega), j = 1, \ldots, N\}$ the first Sobolev space and by $H^1_0(\Omega)$ the closure of $H^1(\Omega)$ in $L^2(\Omega)$.
of the test functions $D(\Omega)$ in $H^1(\Omega)$. Finally, denote by $D(\Omega)'$ the space of all distributions on $\Omega$. Given $h \in \mathcal{H}(\Omega)$ we say that

$$h = \varphi \quad \text{on} \quad \partial \Omega \quad \text{weakly}$$

if $\varphi$ has an extension $\Phi \in C(\bar{\Omega})$ such that $u := h - \Phi \in H^1_0(\Omega)$. This implies that $-\Delta u = \Delta \Phi$ in the sense of distributions, that is,

$$\langle \Delta \Phi, v \rangle := \int_{\Omega} \Phi \Delta v \, dx = \int_{\Omega} \nabla u \nabla v \, dx$$

for all $v \in D(\Omega)$. As a consequence

$$|\langle \Delta \Phi, v \rangle| \leq c \left( \int_{\Omega} |\nabla v|^2 \, dx \right)^{1/2}$$

for all $v \in D(\Omega)$ where $c = (\int_{\Omega} |\nabla u|^2 \, dx)^{1/2}$. By virtue of Poincaré’s inequality, $(\int_{\Omega} |\nabla v|^2 \, dx)^{1/2}$ defines an equivalent norm on $H^1_0(\Omega)$. Thus (2) means that $\Delta \Phi$ has a continuous extension from $D(\Omega)$ to $H^1_0(\Omega)$. We keep the notation for the extension $\Delta \Phi \in H^{-1}(\Omega) := H^1_0(\Omega)'$. Our main result is the following.

**Theorem 1.1.** Let $\varphi \in C(\partial \Omega)$ and assume that $\varphi$ has an extension $\Phi \in C(\bar{\Omega})$ such that $\Delta \Phi \in H^{-1}(\Omega)$. Let $u \in H^1_0(\Omega)$ be the unique solution of Poisson’s equation

$$- \Delta u = \Delta \Phi \quad \text{in} \quad D(\Omega)' .$$

Then $u + \Phi = h_\varphi$ is the Perron solution of the Dirichlet problem.

As seen before (3) has a unique solution $u \in H^1_0(\Omega)$. It follows that $\Delta (u + \Phi) = 0$ in the sense of distributions and hence $u + \Phi \in \mathcal{H}(\Omega)$, see [7, Chapter II §3, Proposition 1] or [13, Appendix 34, Theorem 14].

Our main point is to prove that $u + \Phi = h_\varphi$, which will be done in Section 2. The Riesz representation theorem also says that $u \in H^1_0(\Omega)$ is the unique minimiser of

$$\frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \langle \Delta \Phi, u \rangle = \min \left\{ \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \langle \Delta \Phi, v \rangle : v \in H^1_0(\Omega) \right\}$$

(see [5, Théorème V.6]). Hence if $\Phi \in H^1(\Omega)$, then $h_\varphi$ is actually the solution of Dirichlet’s principle, which can now be formulated as follows.

**Corollary 1.2.** Assume that $\varphi$ has an extension $\Phi \in C(\bar{\Omega}) \cap H^1(\Omega)$. Then $h_\varphi$ is the unique minimiser of

$$\min \left\{ \int_{\Omega} |\nabla w|^2 \, dx : w \in H^1(\Omega), \ w - \Phi \in H^1_0(\Omega) \right\} .$$

**Proof.** Substitute $v \in H^1_0(\Omega)$ in (4) by $w = v + \Phi$. \qed
Hildebrandt [10, Theorem 1] shows that the minimiser $h$ of (4) satisfies
\[ \lim_{x \to z} h(x) = \varphi(z) \]
for all regular points $z \in \partial \Omega$. Thus, if $\Omega$ is Dirichlet regular, it follows that $h = h_\varphi$, which is also proved by Simader [15, Theorem 1.6] or [7, Proposition II.7.10]. However, even if $\Omega$ is Dirichlet regular, not every $\varphi \in C(\partial \Omega)$ has an extension $\Phi \in C(\overline{\Omega}) \cap H^1(\Omega)$. This follows from Hadamard’s famous example [9] on the unit disc $\mathbb{D}$ of $\mathbb{R}^2$. Let
\[ \varphi(e^{i\theta}) = \sum_{n=1}^{\infty} 2^{-n} \cos(2^{2n} \theta). \]
Then the classical solution of $D(\varphi, \mathbb{D})$ is given by
\[ h_\varphi(re^{i\theta}) = \sum_{n=1}^{\infty} r^{2^{2n}} 2^{-n} \cos(2^{2n} \theta) \]
(see e.g. [6, page 179–180]), and the energy of $h_\varphi$ is
\[ \int_{\mathbb{D}} |\nabla h_\varphi|^2 \, dx = \infty, \]
hence $h_\varphi \notin H^1(\Omega)$. As a consequence of Theorem 1.1, for this $\varphi$ there exists no extension $\Phi \in C(\overline{\Omega}) \cap H^1(\Omega)$ such that $\Phi|_{\partial \Omega} = \varphi$. Indeed, then (3) would imply that also $h_\varphi = \Phi + u \in H^1(\Omega)$. We refer to Maz’ya and Shaposhnikova [14, §123] for the interesting history of Hadamard’s example.

On the other hand, the condition that $\varphi$ has an extension $\Phi \in C(\overline{\Omega})$ such that $\Delta \Phi \in H^{-1}(\Omega)$ is weaker than $\Phi \in H^1(\Omega)$. Indeed, if $D(\varphi, \Omega)$ has a classical solution $h \in H(\Omega) \cap C(\overline{\Omega})$, then $\Delta \Phi = \Delta h \in H^{-1}(\Omega)$ since $\Delta h = 0$.

**Remark 1.3.** Let $\Phi \in C(\overline{\Omega})$. The following assertions are equivalent.

1. $\Delta \Phi \in H^{-1}(\Omega)$;
2. $\Phi \in H_{\text{loc}}^1(\Omega)$ and there exists $c > 0$ such that
\[ \left| \int_{\Omega} \nabla \Phi \nabla v \, dx \right| \leq c \left( \int_{\Omega} |\nabla v|^2 \, dx \right)^{1/2} \]
for all $v \in D(\Omega)$.

In fact, if $\Delta \Phi \in H^{-1}(\Omega)$, then $u + \Phi = h_\varphi \in C^\infty(\Omega)$ where $\varphi = \Phi|_{\partial \Omega}$ and $u \in H_0^1(\Omega)$ solves (3). Thus $\Phi \in H_{\text{loc}}^1(\Omega)$. Now (2) implies the estimate. Conversely, (ii) implies (i) since $\left( \int |\nabla v|^2 \, dx \right)^{1/2}$ is an equivalent norm on $H_0^1(\Omega)$. However, as we saw above (ii) does not imply that $\Phi \in H^1(\Omega)$.

We note two further consequences for Poisson’s equation. Let $C_0(\Omega) := \{ u \in C(\Omega) : u|_{\partial \Omega} = 0 \}$.

**Corollary 1.4.** Let $v \in C_0(\Omega)$ such that $\Delta v \in H^{-1}(\Omega)$. Then $v \in H_0^1(\Omega)$. 

Proof. Let \( \Phi := v \) so that \( \phi = \Phi\vert_{\partial \Omega} = 0 \). Let \( u \in H^1_0(\Omega) \) be the solution of (3). Then by Theorem 1.1 we get \( u + v = h_{\phi} = 0 \). \( \square \)

The following result extends [2, Lemma 2.2].

Corollary 1.5. Assume that \( \Omega \) is Dirichlet regular. Let \( f \in L^p(\Omega) \) with \( N/2 < p \leq \infty \) if \( N \geq 2 \), and let \( f \) be a bounded Borel measure on \( \Omega \) if \( N = 1 \). Then there exists a unique solution \( u \in C_0(\Omega) \) of the Poisson equation

\[-\Delta u = f \quad \text{in} \quad \mathcal{D}(\Omega)'.\]

Proof. Since \( H^1_0(\Omega) \subset L^q(\Omega) \) whenever \( q < \frac{N}{N-2} \) if \( N \geq 2 \) and \( H^1_0(\Omega) \subset L^\infty(\Omega) \) in the case \( N = 1 \) we have \( L^p(\Omega) \subset H^{-1}(\Omega) \) if \( p > N/2 \) and \( M(\Omega) \subset H^{-1}(\Omega) \) if \( N = 1 \), where \( M(\Omega) \) denotes the space of all bounded signed Borel measures on \( \Omega \). It follows from the Riesz representation theorem that there exists a unique \( u \in H^1_0(\Omega) \) such that

\[\int_{\Omega} \nabla u \nabla v \, dx = \int_{\Omega} fv \, dx\]

for all \( v \in H^1_0(\Omega) \), that is,

\[-\Delta u = f \quad \text{in} \quad \mathcal{D}(\Omega)'.\]

Let \( \Phi(x) = \int_\Omega f(y)E(x - y) \, dy \) if \( N \geq 2 \) and \( \Phi(x) = \int_\Omega E(x - y) df(y) \) if \( N = 1 \), where \( E \) is the Newtonian potential. Then \( \Phi \in C(\overline{\Omega}) \). This follows from the fact that \( E \in L^q_{\text{loc}}(\mathbb{R}^N) \) if \( N \geq 2 \) and \( E \in C(\mathbb{R}) \) if \( N = 1 \). Moreover \( \Delta \Phi = f \) in \( \mathcal{D}(\Omega)' \). Let \( \varphi = \Phi\vert_{\partial \Omega} \). It follows from Theorem 1.1 that \( h_{\varphi} = \Phi + u \). Since \( \Omega \) is Dirichlet regular, \( h_{\varphi} \in C(\Omega) \) and \( h_{\varphi}\vert_{\partial \Omega} = \varphi \). Thus \( u \in C_0(\Omega) \). In order to prove uniqueness, let \( u \in C_0(\Omega) \) such that \(-\Delta u = f \) in \( \mathcal{D}(\Omega)' \). Then \( h = u + \Phi \in C(\Omega) \) is a classical solution of \( D(\varphi, \Omega) \). So uniqueness follows from the uniqueness of the classical solution of the Dirichlet problem \( D(\varphi, \Omega) \). \( \square \)

We conclude this section commenting on weak solutions of the Dirichlet problem.

Remark 1.6. Let \( \varphi \in C(\partial \Omega) \). We call \( h \in H(\Omega) \) a weak solution of \( D(\varphi, \Omega) \) if \( \varphi \) has an extension \( \Phi \in C(\overline{\Omega}) \) such that \( \Delta \Phi \in H^{-1}(\Omega) \) and \( h - \Phi \in H^1_0(\Omega) \).

(a) It is not obvious that weak solutions are unique. Theorem 1.1 gives a positive answer: since \( h = h_{\varphi} \) and since the Perron solution \( h_{\varphi} \) is unique there is at most one solution.

(b) If \( \Omega \) is Dirichlet regular, then \( \varphi \in C(\partial \Omega) \) has an extension \( \Phi \in C(\Omega) \) with \( \Delta \Phi \in H^{-1}(\Omega) \), namely the Perron solution \( h_{\varphi} \). We do not know whether this is true in general. Here is a class of examples where it is true.

(c) Let \( G \subset \mathbb{R}^N \) be a bounded open set which is Dirichlet regular and assume that \( N \geq 2 \). Let \( F \subset G \) be a finite non-empty set and \( \Omega = G \setminus F \). Then \( \Omega \) is not Dirichlet regular. Let \( \varphi \in C(\partial \Omega) \). Let
Let $h \in \mathcal{H}(G) \cap C(\bar{G})$ such that $h(z) = \varphi(z)$ for all $z \in \partial G$. Let $\psi \in C^1(\mathbb{R}^N)$ such that $\psi = 0$ on $\mathbb{R}^N \setminus G$ and $\psi(z) = \varphi(z) - h(z)$ for all $z \in F$. Then $\Delta \Phi = \Delta (h + \psi) \in H^{-1}(\Omega)$ and $\Phi(z) = \varphi(z)$ for all $z \in \partial \Omega = \partial G \cup F$.

2. Proof of Theorem 1.1

We start this section by giving a definition of the Perron solution. A function $u \in C(\Omega)$ is called subharmonic if $\Delta u \geq 0$ in $\mathcal{D}(\Omega)'$ (that is, if $\int_B u \Delta v \, dx \geq 0$ for all $0 \leq v \in \mathcal{D}(\Omega)$) and $u$ is called superharmonic if $-u$ is subharmonic. We write

$$u \leq \varphi \text{ on } \partial \Omega \text{ if } \limsup_{x \to z} u(x) \leq \varphi(z) \text{ for all } z \in \partial \Omega$$

and

$$u \geq \varphi \text{ on } \partial \Omega \text{ if } \liminf_{x \to z} u(x) \geq \varphi(z) \text{ for all } z \in \partial \Omega.$$  

Then by Perron’s method

$$h_\varphi(x) := \sup \{u(x) : u \in C(\Omega) \text{ is subharmonic and } u \leq \varphi \text{ on } \partial \Omega\}$$

$$= \inf \{u(x) : u \in C(\Omega) \text{ is superharmonic and } u \geq \varphi \text{ on } \partial \Omega\}$$

exists for all $x \in \Omega$ and defines a bounded harmonic function $h_\varphi$. The mapping $\varphi \mapsto h_\varphi$ from $C(\partial \Omega)$ into $\mathcal{H}(\Omega) \cap L^\infty(\Omega)$ is linear, positive (that is, $\varphi \geq 0$ implies $h_\varphi \geq 0$) and contractive (that is, $\sup_{x \in \Omega} |h_\varphi(x)| \leq \sup_{x \in \partial \Omega} |\varphi(z)|$). We refer to [7, Chapter II §4] for proofs of these classical results.

Let $\Omega$ be a bounded open set and let $\varphi \in C(\partial \Omega)$. For the proof we use the following alternative description of the Perron solution $h_\varphi$. Let $\Omega_n \subset \Omega$ be open, Dirichlet regular such that $\Omega_n \subset \Omega_{n+1}$ and $\bigcup_{n \in \mathbb{N}} \Omega_n = \Omega$. Such $\Omega_n$ can always be constructed even of class $C^\infty$ (see [7, Chapter II §4 Lemma 1]). Extend $\varphi$ to a function $\Phi \in C(\bar{\Omega})$. Let $h_n \in C(\Omega_n) \cap \mathcal{H}(\Omega_n)$ such that $h_n = \Phi$ on $\partial \Omega_n$. Then

$$h_\varphi(x) = \lim_{n \to \infty} h_n(x)$$

uniformly on compact subsets of $\Omega$. We refer to [11, Theorem II] or [3, Theorem 3.4] for this result. Now we assume in addition that

$$\Omega_n \subset \{x \in \Omega : \text{dist}(x, \partial \Omega) > 1/n\}.$$  

This can always be arranged by re-numerating the $\Omega_n$. Let $\varphi_n$ be a mollifier, that is, $0 \leq \varphi_n \in \mathcal{D}(\mathbb{R}^N)$, supp $\varphi_n \subset B(0, 1/n)$ and $\int \varphi_n \, dx = 1$. We also assume that $\varphi_n(x) = \varphi_n(-x)$ for all $x \in \mathbb{R}^N$. Extend $\Phi$ to a uniformly continuous function on $\mathbb{R}^N$, which we still denote by $\Phi$, and let $\Phi_n := \varphi_n * \Phi$. Then $\Phi_n \to \Phi$ uniformly on $\mathbb{R}^N$. Let $k_n \in \mathcal{H}(\Omega_n) \cap C(\Omega_n)$ such that $k_n = \Phi_n$ on $\partial \Omega_n$, that is, $k_n$ is the solution of $D(\Omega_n, \Phi_n|_{\partial \Omega_n})$. We show that also

$$k_n(x) \to h_\varphi(x)$$
uniformly on compact subsets of \( \Omega \). In fact, let \( K \subset \Omega \) be compact. There exists \( n_0 \) such that \( K \subset \Omega_n \) for all \( n \geq n_0 \). By the maximum principle we have

\[
\|k_n - h_n\|_{C(K)} \leq \|k_n - h_n\|_{C(\Omega_n)} \leq \|k_n - h_n\|_{C(\partial \Omega_n)} = \|\Phi_n - \Phi\|_{C(\partial \Omega_n)} \to 0
\]

as \( n \to \infty \). Now (6) follows from (5).

Consider the function

\[
u_n = k_n - \Phi_n \in C_0(\Omega_n).
\]

Then \(-\Delta u_n = \Delta \Phi_n \in D(\Omega_n)' \). It follows from [2, Lemma 2.2], (see also [3]) that \( u_n \in H^1_0(\Omega_n) \). Now we assume in addition that \( \Delta \Phi \in H^{-1}(\Omega) \), that is, there exists a constant \( c > 0 \) such that

\[
|\int_{\Omega} \Phi \Delta v \, dx| \leq c \left( \int_{\Omega} |\nabla v|^2 \right)^{1/2}
\]

for all \( v \in D(\Omega) \). This will allow us to prove that

\[
\left( \int_{\Omega} |\nabla u_n|^2 \, dx \right)^{1/2} \leq c
\]

for all \( n \in \mathbb{N} \). In order to prove (8) fix \( n \in \mathbb{N} \). Let \( v \in D(\Omega_n) \). Then

\[
\int_{\Omega_n} \nabla u_n \nabla v \, dx = - \int_{\Omega_n} u_n \Delta v \, dx = \int_{\Omega_n} (\Phi_n - k_n) \Delta v \, dx = \int_{\Omega_n} \Phi_n \Delta v \, dx
\]

where \( v_n = \varrho_n * v \in C^\infty(\mathbb{R}^N) \) with \( \text{supp} \, v_n \subset B(0, 1/n) + \text{supp} \, v \subset \Omega \). Thus \( v_n \in D(\Omega) \) and it follows from (7) that

\[
|\int_{\mathbb{R}^N} \nabla u_n \nabla v \, dx| \leq c \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx \right)^{1/2} \leq c \left( \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \right)^{1/2}
\]

for all \( v \in D(\Omega_n) \) since \( D_j v_n = \varrho_n * D_j v \) and \( \|\varrho_n\|_{L^1} = 1 \). As \( D(\Omega_n) \) is dense in \( H^1_0(\Omega_n) \) and \( \int_{\Omega_n} \nabla u_1 \nabla u_2 \, dx \) defines an equivalent scalar product on \( H^1_0(\Omega) \), the claim (8) follows. Now we define \( \tilde{u}_n(x) = u_n(x) \) if \( x \in \Omega_n \) and \( \tilde{u}_n(x) = 0 \) if \( x \notin \Omega_n \). Then \( \tilde{u}_n \in H^1_0(\Omega) \) and \( \nabla \tilde{u}_n = \nabla u_n \) (see e.g. [5, Proposition IX.18]). We identify \( \tilde{u}_n \) and \( u_n \) to simplify the notation. By (8) the sequence \( (u_n) \) is bounded in \( H^1_0(\Omega) \). Hence there exists a subsequence \( (u_{nm}) \) converging weakly to a function \( u \in H^1_0(\Omega) \) as \( m \to \infty \). Since

\[
u_{nm} + \Phi_{nm} = k_{nm}
\]

we have

\[
- \int_{\Omega_{nm}} \nabla u_{nm} \nabla v + \int_{\Omega_{nm}} \Phi_{nm} \Delta v = \int_{\Omega_{nm}} k_{nm} \Delta v = 0
\]

for all \( v \in D(\Omega_{nm}) \). Letting \( m \to \infty \) we conclude that

\[
- \int_{\Omega} \nabla u \nabla v + \int_{\Omega} \Phi \Delta v = 0
\]
for all \( v \in \mathcal{D}(\Omega) = \bigcup_{m \in \mathbb{N}} \mathcal{D}(\Omega_{nm}) \). Thus \( u \) is the solution of

\[
    u \in H^1_0(\Omega), \quad -\Delta u = \Delta \Phi \quad \text{in} \quad \mathcal{D}(\Omega)^\prime.
\]

On the other hand, it follows from (9) and (6) that

\[
    u + \Phi = h_\varphi.
\]

This completes the proof of Theorem 1.1.

3. Further comments

1. Our proof of Theorem 1.1 is based on exhausting \( \Omega \) by Dirichlet regular sets. It allows us actually to identify the weak solution with Perron’s solution. Hildebrandt [10] and Simader [15], in the case where \( \varphi \) has an extension \( \Phi \in C(\Omega) \cap H^1(\Omega) \), use barriers to show that the weak solution has the same regularity properties as the Perron solution. Simader’s proof [15, Theorem 1.6] depends on the notion of \( H^1 \)-barriers which are introduced in [2, Definition 3.1] (cf. [15, Definition 3.1]). By [2, Lemma 3.1] \( \Omega \) is Dirichlet regular if and only if at every point \( z \in \partial \Omega \) an \( H^1 \)-barrier exists (cf. [15, Theorem 1.7]).

2. A further consequence of Corollary 1.4 is that

\[
    H^1_0(\Omega) \cap C(\bar{\Omega}) \subset C_0(\Omega)
\]

whenever \( \Omega \) is Dirichlet regular [15, Corollary 5.3]. We mention that Biegert and Warma [4] actually showed that (10) holds if and only if \( \text{cap}(B(z,r) \setminus \Omega) > 0 \) for each \( z \in \partial \Omega, r > 0 \), that is, if \( \Omega \) is regular in capacity [1, Definition 3.12]. By Wiener’s criterion [8, (2.37)] regularity in capacity is weaker than Dirichlet regularity.

References

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