

**EXPONENTIAL STABILITY, CHANGE OF STABILITY
AND EIGENVALUE PROBLEMS FOR LINEAR
TIME-PERIODIC PARABOLIC EQUATIONS ON \mathbb{R}^N**

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In memory of Peter Hess

1. Introduction. The main purpose of this article is to give conditions on a non-negative weight function m which are necessary and sufficient for the zero solution of the linear time-periodic parabolic equation

$$\partial_t u - \Delta u = -m(x, t)u \quad \text{in } \mathbb{R}^N \times (0, \infty) \quad (1.1)$$

to be exponentially stable and to apply these results to the study of change of stability in parameter dependent time-periodic parabolic problems. Here, Δ denotes the Laplacian in \mathbb{R}^N and the weight-function lies in $C_T^\nu(\mathbb{R}, BUC(\mathbb{R}^N))$, the space of ν -Hölder continuous and T -periodic functions taking values in the space of bounded uniformly continuous functions on \mathbb{R}^N ($\nu \in (0, 1)$ and $T > 0$ fixed). Stability is to be understood either with respect to the L_∞ - or the L_1 -norm and initial conditions are taken in the spaces $C_0(\mathbb{R}^N)$ and $BUC(\mathbb{R}^N)$ or $L_1(\mathbb{R}^N)$, respectively. As it turns out exponential stability will be a property which does not depend on the underlying space.

Our first characterization of exponential stability will be in terms of a quality that m may have or not as an inhomogeneity in the initial value problem

$$\begin{cases} \partial_t u - \Delta u = m(x, t) & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = 0 & \text{in } \mathbb{R}^N. \end{cases} \quad (1.2)$$

We can actually prove the following result, generalizing the equivalence of statements (1) and (6) of Proposition 4.19 in the paper of C.J.K. Batty [3] which treats the time-independent case, i.e., the case of Schrödinger semigroups.

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Theorem 1.1. *The zero solution of (1.1) is exponentially stable (with respect to either the L_∞ - or the L_1 -norm) if and only if the (mild) solution u of (1.2) in $BUC(\mathbb{R}^N)$ is bounded away from zero at time T , that is, if there exists a constant $\delta > 0$ such that $u(x, T) > \delta$ for all $x \in \mathbb{R}^N$.*

In other words, the source represented by m must be strong enough to push the solution away from zero within time T uniformly on all of \mathbb{R}^N .

The idea for our second characterization goes back to W. Arendt and C.J.K. Batty in the case of time-independent problems (Theorem 1.2 in [1] and Proposition 4.19 in [3]). They prove that a Schrödinger semigroup is exponentially stable if and only if the potential is non-integrable on all sets containing arbitrarily large balls.

We shall prove here the following generalization:

Theorem 1.2. *The zero solution of (1.1) is exponentially stable if and only if the temporal mean, $\frac{1}{T} \int_0^T m(\cdot, \tau) d\tau$, of m is non-integrable on all domains of \mathbb{R}^N which contain arbitrarily large balls.*

Roughly speaking, this is a condition saying that in order to have exponential stability of the zero solution of (1.1) the temporal mean over one period of m , as a function on \mathbb{R}^N , must be positive enough at infinity. Our proof of this fact is based on the first characterization.

It appears that, though similar, our method is simpler than the ones used in [1] or [3]. In [3] the approach is highly probabilistic in nature and while its author remarks that it is possible to prove our Theorem 1.1 in the autonomous case reducing the amount of probabilistic tools involved, he stills needs the Feynman-Kac formula. As in our own approach, in [1] only analytical arguments are used but at some point (proof of the implication (i) \Rightarrow (ii) of Theorem 1.2 of that paper) Hilbert space methods are needed. This kind of argument cannot be generalized to time-periodic equations or to equations which are not in self-adjoint form; ours may be easily extended to cover the case where more general parabolic operators of second order are considered. Thus, in order to give a self-contained analytical treatment we have included complete proofs of Theorems 1.1 and 1.2.

The above results enable us to investigate change of stability of the zero solution of the time-periodic equation

$$\partial_t u - \Delta u = \lambda m(x, t)u \quad \text{in } \mathbb{R}^N \times (0, \infty) \quad (1.3)$$

if the weight m is indefinite, i.e., changes sign, and the parameter λ varies over the positive real axis. We shall generalize some results in [7] where conditions for a change from exponential stability to instability are given. In fact we shall prove the following theorem.

Theorem 1.3. *Assume that the support of the positive part of m is contained in a compact subset of $\mathbb{R}^N \times [0, T]$. Then, there exists a number $\lambda_0 > 0$ such that the zero solution of (1.3) is exponentially stable for $0 < \lambda < \lambda_0$ and unstable for $\lambda > \lambda_0$, if*

and only if the zero solution of (1.1) with m replaced by the negative part, m^- , of m is exponentially stable and such that

$$\mathcal{P}(m) := \int_0^T \sup_{x \in \mathbb{R}^N} m(x, \tau) d\tau > 0.$$

We shall apply this theorem to eigenvalue problems with respect to an indefinite weight function of the form

$$\begin{cases} \partial_t \phi - \Delta \phi = \lambda m(x, t) \phi & \text{in } \mathbb{R}^N \times \mathbb{R} \\ \phi(\cdot, t) \in C_0(\mathbb{R}^N) & \text{for each } t > 0 \\ \phi \text{ is } T\text{-periodic and positive} \end{cases} \quad (1.4)$$

which have also been considered in [7] and [8] in the time-periodic case and in [4] and [5] in the autonomous case.

Theorem 1.3 may be also used to establish bifurcation of positive time-periodic solutions for nonlinear problems as is done in [14] and [12]. In this context see also [17] for a treatment of a class of nonlinear systems.

The remainder of the paper is organized as follows. We start by giving in Section 2 some general facts on linear time-periodic parabolic equations on \mathbb{R}^N which shall be needed later on. We proceed then to introduce in Section 3 the class of functions on \mathbb{R}^N to which weights leading to exponential stability shall have to belong to. In the next two sections we attack first the inhomogeneous equation (1.2) and second the question of exponential stability of (1.1). In Section 6 we collect some simple facts on time-periodic eigenvalue problems on \mathbb{R}^N which are needed in Section 7, our last section, where we turn our attention to the study of change of stability of the zero solution of (1.3) and eigenvalue problems with indefinite weight functions.

2. Linear time-periodic equations. We collect here some results on linear time-periodic parabolic equations of the form

$$\partial_t u - \Delta u + m(x, t)u = 0, \quad \text{in } \mathbb{R}^N \times (s, \infty), \quad (2.1)$$

where $s \in \mathbb{R}$ and m is an arbitrary weight-function lying in $C_T^\nu(\mathbb{R}, BUC(\mathbb{R}^N))$, where $\nu \in (0, 1)$ and $T > 0$ are to remain fixed for the rest of this paper.

A. The period map. It is a well-known fact that Δ is the infinitesimal generator of strongly continuous semigroup of positive operators on each of the Banach spaces $BUC(\mathbb{R}^N)$, $C_0(\mathbb{R}^N)$ and $L_p(\mathbb{R}^N)$ for $1 \leq p < \infty$. Let X_0 be one of the above spaces and define for each $t \in \mathbb{R}$ the closed operator $A_m(t)$ in X_0 by setting

$$D(A_m(t)) := D(\Delta) := \{u_0 \in X_0 : \Delta u_0 \in X_0\},$$

and

$$A_m(t)u_0 := -\Delta u_0 + m(\cdot, t)u_0$$

for each $u_0 \in D(\Delta)$.

Under the present assumptions, for each $t \in \mathbb{R}$, the multiplication operator induced by $m(\cdot, t)$ on X_0 is bounded and hence, by standard perturbation theory, $-A_m(t)$ generates a strongly continuous semigroup of operators on X_0 . In fact, the corresponding semigroup is also positive. More is actually true. Our assumptions imply that the family $\{A_m(t) : t \in \mathbb{R}\}$ is such that there exists a unique evolution operator

$$U_m(t, s) \in \mathcal{L}(X_0),$$

where $(t, s) \in \Delta_\infty := \{(t, s) : s \leq t\}$. This means that for each $u_0 \in X_0$ and $s \in \mathbb{R}$ the function $[s, \infty) \rightarrow X_0$ defined by $t \rightarrow U_m(t, s)u_0$ is the unique solution of

$$\dot{u} + A_m(t)u = 0, \quad t > s \tag{2.2}$$

satisfying the initial condition $u(s) = u_0$. Of course, if m does not depend on time we just have $U_m(t, s) = e^{(t-s)(\Delta-m)}$.

A property that is of particular importance is the positivity of the evolution operator. In fact it follows from the Phragmen-Lindelöf principle (cf. [18]) that $U_m(t, s)$ is an irreducible operator on the Banach lattices $C_0(\mathbb{R}^N)$ and $L_p(\mathbb{R}^N)$ for $1 \leq p < \infty$. Note that as an operator on the Banach lattice $BUC(\mathbb{R}^N)$ it fails to be irreducible (since $C_0(\mathbb{R}^N)$ is a $U_m(t, s)$ -invariant closed ideal of $BUC(\mathbb{R}^N)$).

Recall that the zero solution of (2.2) is said to be *stable* if for each $s \in \mathbb{R}$ the set $\{\|U_m(t, s)\| : t \geq s\}$ is bounded. It is called *asymptotically stable* if for each $s \in \mathbb{R}$ and $u_0 \in X_0$ we have that $\|U_m(t, s)u_0\| \rightarrow 0$ as $t \rightarrow \infty$, and *exponentially stable* if there exist constants $M, \omega > 0$ such that

$$\|U_m(t, s)u_0\| \leq M e^{-(t-s)\omega} \|u_0\|$$

for each $u_0 \in X_0$ and $t \geq s$.

In time-periodic problems it is of particular importance to look at the *period-map* S_m defined by

$$S_m := U_m(T, 0) \in \mathcal{L}(X_0).$$

This operator contains much useful information on the asymptotic behavior of solutions of (2.2). Of interest to us will be the fact that the zero solution of (2.2) is exponentially stable if and only if $\text{spr}(S_m) < 1$, where $\text{spr}(S_m)$ denotes the spectral radius of S_m (cf. [6]). Of course, S_m has the same positivity properties as U_m .

B. Norm of the period map. We shall now explore the norm of the period-map in the different spaces. To that end we need to recall that the evolution operator may be represented as the convolution of the initial value with the fundamental solution k_m corresponding to (2.1) (cf. [9]). This is a function defined on $\Gamma := \{(x, t, y, s) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} : t \geq s\}$ with the property that if $u_0 \in X_0$ then

$$[U_m(t, s)u_0](x) = \int_{\mathbb{R}^N} k_m(x, t; y, s)u_0(y) dy \tag{2.3}$$

for all $x \in \mathbb{R}^N$ and $t > s$. Observe that the integral on the righthand side of (2.3) makes sense also for $u_0 \in L_\infty(\mathbb{R}^N)$ and defines a positive bounded operator on this space. Hence, we may consider S_m to be defined as a positive operator on $L_\infty(\mathbb{R}^N)$.

Recall that if T is a positive operator defined on $L_\infty(\mathbb{R}^N)$ or on $BUC(\mathbb{R}^N)$ and taking values in any ordered Banach space E with an order monotone norm (i.e., $x \leq y \Rightarrow \|x\|_E \leq \|y\|_E$), then

$$\|T\| = \|T\mathbf{1}\|_E. \tag{2.4}$$

With this in mind it is very easy to prove the following result which is hardly surprising (it also follows from Lemma 2.2).

Lemma 2.1. *The norm of S_m is the same on $BUC(\mathbb{R}^N)$ and $C_0(\mathbb{R}^N)$.*

To calculate the norm of S_m in $L_1(\mathbb{R}^N)$ we need to study the equation adjoint to (2.1). This is the equation

$$\partial_t v + \Delta v = m(x, t)v, \quad \text{in } \mathbb{R}^N \times [0, s) \tag{2.5}$$

which, because of the change of sign before the Laplacian, fails to be parabolic. While we cannot solve the initial value problem for this equation a fundamental solution k_m^* exists for the *backward problem*. It is given by

$$k_m^*(x, t; y, s) = k_m(y, s; x, t)$$

for $(y, s; x, t) \in \Gamma$ and has the property that if we set

$$v(t, x) := [V_m^*(s, t)u_0](x) := \int_{\mathbb{R}^N} k_m^*(x, t; y, s)u_0(y) dy \tag{2.6}$$

whenever $x \in \mathbb{R}^N$ and $t < s$, then v is the unique solution of equation (2.5) with *terminal value* $v(\cdot, s) = u_0$. Observe that $V_m^*(s, t)$ is a bounded operator on $L_\infty(\mathbb{R}^N)$.

Recall that if $X_0 = L_1(\mathbb{R}^N)$ then $X'_0 = L_\infty(\mathbb{R}^N)$. It is now easy to see that for this choice of X_0 the dual, $U_m(t, s)'$ of $U_m(t, s)$ is given by $V_m^*(t, s)$ for $t > s$. In particular if we set $S_m^* := V_m^*(T, 0)$ we get

$$S'_m = S_m^* \in \mathcal{L}(L_\infty(\mathbb{R}^N)).$$

We would like to identify S_m^* as the period-map corresponding to a forward equation. Let v be the solution of (2.5) with $s = T$ and terminal value $u_0 \in X_0$; then w defined by $w(\cdot, t) := v(\cdot, T - t)$ solves the equation

$$\partial_t w - \Delta w = m^*(x, t)w, \quad \text{in } \mathbb{R}^N \times (0, T] \tag{2.7}$$

with initial value $u(0) = u_0$. Here, we have set

$$m^*(x, t) := m(x, T - t)$$

for $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. Hence, it follows that

$$S'_m = S_m^* = S_{m^*} \in \mathcal{L}(L_\infty(\mathbb{R}^N)). \tag{2.8}$$

Therefore, using (2.4), the following result immediately follows.

Lemma 2.2. *The norm of S_m in $L_1(\mathbb{R}^N)$ is equal to the norm of S_{m^*} in $BUC(\mathbb{R}^N)$.*

Remark 2.3. If m does not depend on time, then $m = m^*$ and hence the norm of S_m is the same on all three spaces $C_0(\mathbb{R}^N)$, $BUC(\mathbb{R}^N)$ and $L_1(\mathbb{R}^N)$. We do not know if this is also true in the case of time-dependent m .

3. The class \mathcal{E} . In this section we introduce, following [1], a class of nonnegative functions defined on \mathbb{R}^N which shall play a central role in the identification of those weights m which induce an exponentially stable evolution operator $U_m(t, s)$.

Definition 3.1. A set $G \subset \mathbb{R}^N$ is said to *contain arbitrarily large balls* if for any $r > 0$ there exists an $x \in G$ such that $\mathbb{B}(x, r) \subset G$. The class of all such subsets of \mathbb{R}^N shall be denoted by \mathcal{G} .

Any rotation or translation of a set $E \subset \mathbb{R}^N$ of the form

$$E = \{x \in \mathbb{R}^N : a < x_j < b\},$$

where $-\infty < a < b < \infty$ and $1 \leq j \leq N$, is called a *strip* of width $b - a$.

It is obvious that cutting out a bounded subset of a set $G \in \mathcal{G}$ still leaves us with a set in \mathcal{G} . The following result says that we may even cut out a strip.

Lemma 3.2. *Let E be a strip of width $c > 0$ and $G \in \mathcal{G}$. Then $G \cap E^c \in \mathcal{G}$.*

Proof. If $G \in \mathcal{G}$ and $\mathbb{B}(x_n, r_n)$ is a sequence of balls contained in G with $r_n \nearrow \infty$ we may construct a sequence of balls having radius $r'_n = r_n - |b - a|$ contained in $G \cap E^c$. This proves the assertion.

Definition 3.3. A nonnegative function $f \in L_\infty(\mathbb{R}^N)$ will be said to belong to the class \mathcal{E} whenever

$$\int_G f(y) dy = \infty$$

for all $G \in \mathcal{G}$.

A proof of the following characterization of functions belonging to \mathcal{E} may be found in [1] (Proposition 1.4).

Lemma 3.4. *Let f be a nonnegative function in $L_\infty(\mathbb{R}^N)$. The following statements are equivalent:*

- (i) f belongs to \mathcal{E} .
- (ii) There exist an $r > 0$ and a $c > 0$ such that $\int_{\mathbb{B}(x, r)} f(y) dy \geq c$ for all $x \in \mathbb{R}^N$.
- (iii) There exists a $c > 0$ such that $\int_G f(y) dy \geq c$ for all $G \in \mathcal{G}$.
- (iv) If (x_n) is a sequence in \mathbb{R}^N and $r_n \nearrow \infty$ then $\liminf_{n \rightarrow \infty} \int_{\mathbb{B}(x_n, r_n)} f(y) dy > 0$.

Remark 3.5. Definition 3.3 may of course be given for functions lying in $L^1_{loc}(\mathbb{R}^N)$. Lemma 3.4 is also valid for these functions (cf. [1]).

Our next result is an immediate corollary to Lemma 3.2 and says that altering a function belonging to \mathcal{E} by setting its value to zero on a strip will produce a function which still belongs to \mathcal{E} (see also [1], Theorem 1.10).

Lemma 3.6. *Let $f \in \mathcal{E}$ and suppose that E is a strip of width $c > 0$. Then $g := f \chi_{E^c}$ also belongs to \mathcal{E} .*

The following simple observation shall prove useful when dealing with time-dependent problems.

Lemma 3.7. *\mathcal{E} is relatively open in $L^+_\infty(\mathbb{R}^N)$, i.e., for each $f \in \mathcal{E}$ there exists an $\epsilon > 0$ such that each $g \in L^+_\infty(\mathbb{R}^N)$ satisfying $\|f - g\|_\infty < \epsilon$ lies also in \mathcal{E} .*

Proof. Let f be as above. By Lemma 3.4 there exist constants $c, r > 0$ such that

$$\int_{\mathbb{B}(x,r)} f(y) dy \geq c$$

for all $x \in \mathbb{R}^N$. Take now $\epsilon > 0$ such that

$$c_1 := c - \epsilon \omega(r) > 0.$$

Here $\omega(r)$ denotes the Lebesgue volume of $\mathbb{B}(0, r)$. For any g as above we have then

$$\int_{\mathbb{B}(x,r)} g(y) dy = \int_{\mathbb{B}(x,r)} f(y) dy - \int_{\mathbb{B}(x,r)} (f(y) - g(y)) dy > c - \epsilon \omega(r) = c_1.$$

Hence by Lemma 3.4 the function g belongs to \mathcal{E} proving the assertion.

4. Strongly positive solutions of an inhomogeneous problem. We set for this whole section $X_0 := BUC(\mathbb{R}^N)$ and consider the inhomogeneous abstract Cauchy problem on this space,

$$\dot{u} + A_0 u = f(t), \quad u(0) = 0, \quad 0 < t \leq T, \tag{4.1}$$

where A_0 is the X_0 -realization of $-\Delta$ and $f \in C([0, T], X_0)$ is not necessarily T -periodic but $f(t) \geq 0$ for all $t \in [0, T]$.

Denote by u_f the unique (mild) solution of (4.1). Thus, by the variation of constants formula, u_f is given by

$$u_f(t) := \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau. \tag{4.2}$$

It follows that $u_f(t) \geq 0$ for all $t \geq 0$. We would like to identify those inhomogeneities f for which $u_f(T)$ is bounded away from zero.

Now, any function $f : [0, T] \rightarrow X_0$ may be identified with the function $\mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}$ given by $(x, t) \mapsto f(t)(x)$, and vice-versa. We shall do so without further comment and without any notational distinction whatsoever.

Definition 4.1. Let g be a function $\mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}$ such that for each $x \in \mathbb{R}^N$ the function $g(x, \cdot) : [0, T] \rightarrow \mathbb{R}$ is integrable. The *temporal mean*, \bar{g} , of g is the function

$$\bar{g} : \mathbb{R}^N \rightarrow \mathbb{R}, \quad x \mapsto \int_0^T g(x, \tau) d\tau. \tag{4.3}$$

For notational convenience we shall also define for each $0 \leq \epsilon \leq T$ the function

$$\bar{g}_\epsilon : \mathbb{R}^N \rightarrow \mathbb{R}, \quad x \mapsto \int_0^{T-\epsilon} g(x, \tau) d\tau.$$

Hence, $\frac{1}{T}\bar{g}$ is in fact the temporal mean of g .

We shall need the following simple result.

Lemma 4.2. *Suppose that the temporal mean of f belongs to \mathcal{E} . Then, \bar{f}_ϵ also belongs to \mathcal{E} for $\epsilon > 0$ small enough.*

Proof. Observe that $\bar{f}_\epsilon \rightarrow \bar{f}$ in X_0 for $\epsilon \rightarrow 0$. From this and since \bar{f} belongs to \mathcal{E} , the assertion immediately follows because of Lemma 3.7. \square

Now we are ready to state and prove the main result of this section.

Theorem 4.3. *The function $u_f(T)$ is bounded away from zero if and only if the temporal mean of f belongs to the class \mathcal{E} .*

Proof. We denote by w_t the heat kernel, i.e.,

$$w_t(x) := (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$$

for $x \in \mathbb{R}^N$ and $t \geq 0$. Then we may represent u_f by

$$u_f(x, T) = \int_0^T \int_{\mathbb{R}^N} w_{T-\tau}(x-y) f(y, \tau) dy d\tau. \quad (4.4)$$

Observe that for each $\epsilon > 0$ and $r > 0$ there exists a constant $\underline{\kappa}_{\epsilon,r} > 0$ such that

$$\underline{\kappa}_{\epsilon,r} < w_t(x) \quad (4.5)$$

for all $x \in \mathbb{R}^N$ with $\|x\| \leq r$ and all $t \geq \epsilon$. Analogously, for each $\epsilon > 0$ there exists a constant $\bar{\kappa}_\epsilon > 0$ such that

$$w_t(x) < \bar{\kappa}_\epsilon \quad (4.6)$$

for all $x \in \mathbb{R}^N$ and all $t \geq \epsilon$.

To prove the “if” part of the theorem assume that \bar{f} belongs to \mathcal{E} . By Lemma 4.2 we may choose an $\epsilon > 0$ such that \bar{f}_ϵ belongs to \mathcal{E} . Hence, by Lemma 3.4, there exist constants $c, r > 0$ such that

$$\int_0^{T-\epsilon} \int_{\mathbb{B}(x,r)} f(y, \tau) dy d\tau > c \quad (4.7)$$

for all $x \in \mathbb{R}^N$. By (4.4) and (4.5) we get for any $x \in \mathbb{R}^N$ that

$$u_f(x, T) \geq \int_0^{T-\epsilon} \int_{\mathbb{B}(x,r)} w_{T-\tau}(x-y) f(y, \tau) dy d\tau \geq \underline{\kappa}_{\epsilon,r} c > 0,$$

which means that $u_f(\cdot, T)$ is bounded away from zero.

To prove the other implication we assume that $u_f(\cdot, T)$ is bounded away from zero, i.e., there exists a constant $c > 0$ such that

$$u_f(x, T) > c \tag{4.8}$$

for all $x \in \mathbb{R}^N$. We shall prove that \bar{f} belongs to \mathcal{E} arguing by contradiction. If \bar{f} does not belong to \mathcal{E} we infer from Lemma 3.4 that there exist sequences (x_n) in \mathbb{R}^N and (r_n) in \mathbb{R}^N with $r_n \nearrow \infty$ as $n \rightarrow \infty$ such that

$$\int_0^T \int_{\mathbb{B}(x_n, r_n)} f(y, \tau) dy d\tau \rightarrow 0$$

as $n \rightarrow \infty$. Hence, for any $0 < \epsilon \leq T$ we get

$$\begin{aligned} & \int_0^{T-\epsilon} \int_{\mathbb{B}(x_n, r_n)} w_{T-\tau}(x_n - y) f(y, \tau) dy d\tau \\ & \leq \kappa_\epsilon \int_0^{T-\epsilon} \int_{\mathbb{B}(x_n, r_n)} f(y, \tau) dy d\tau \leq \kappa_\epsilon \int_0^T \int_{\mathbb{B}(x_n, r_n)} f(y, \tau) dy d\tau \rightarrow 0. \end{aligned} \tag{4.9}$$

Choose now $0 < \epsilon < T$ such that

$$\int_{T-\epsilon}^T \int_{\mathbb{R}^N} w_{T-\tau}(x - y) f(y, \tau) dy d\tau < \frac{c}{2} \tag{4.10}$$

for all $x \in \mathbb{R}^N$. Observe finally that obviously

$$\begin{aligned} & \int_0^{T-\epsilon} \int_{\mathbb{B}(x_n, r_n)^c} w_{T-\tau}(x_n - y) f(y, \tau) dy d\tau \\ & \leq \|f\|_\infty \int_0^{T-\epsilon} \int_{\mathbb{B}(0, r_n)^c} w_{T-\tau}(y) dy d\tau \rightarrow 0 \end{aligned} \tag{4.11}$$

as n tends to infinity.

We may rewrite $u_f(x_n, T)$ as

$$\begin{aligned} u_f(x_n, T) &= \int_0^T \int_{\mathbb{R}^N} w_{T-\tau}(x - y) f(y, \tau) dy d\tau \\ &= \int_{T-\epsilon}^T \int_{\mathbb{R}^N} w_{T-\tau}(x_n - y) f(y, \tau) dy d\tau \\ &+ \int_0^{T-\epsilon} \int_{\mathbb{B}(x_n, r_n)} w_{T-\tau}(x_n - y) f(y, \tau) dy d\tau \\ &+ \int_0^{T-\epsilon} \int_{\mathbb{B}(x_n, r_n)^c} w_{T-\tau}(x_n - y) f(y, \tau) dy d\tau. \end{aligned}$$

Using (4.9), (4.10) and (4.11) we see that for n large enough the righthand side is smaller than $\frac{\varepsilon}{2}$ contradicting (4.8). Therefore, \bar{f} must belong to \mathcal{E} , concluding the proof of the theorem.

5. Exponential stability. We are now ready to deal with the question of exponential stability of the zero solution of (1.1) or, more precisely, its abstract version

$$\dot{u} + A_m(t)u = 0, \quad t > 0 \quad (5.1)$$

when X_0 is any of the spaces $BUC(\mathbb{R}^N)$, $C_0(\mathbb{R}^N)$ or $L_1(\mathbb{R}^N)$. We shall use the notation of Section 2. Our main result is the following.

Theorem 5.1. *Let X_0 be any of the spaces $BUC(\mathbb{R}^N)$, $C_0(\mathbb{R}^N)$ or $L_1(\mathbb{R}^N)$. Then, the following assertions are equivalent:*

- (i) *The zero solution of (5.1) is exponentially stable.*
- (ii) *The temporal mean \bar{m} of m belongs to \mathcal{E} .*
- (iii) $\|S_m\| < 1$.

Proof. Observe that if we prove the assertion for $X_0 = BUC(\mathbb{R}^N)$, by Lemma 2.1, we immediately get it for $C_0(\mathbb{R}^N)$. Since the temporal mean of m lies in \mathcal{E} if and only if the temporal mean of m^* does, m^* as defined in Section 2, we obtain the assertion for $L_1(\mathbb{R}^N)$ by using Lemma 2.2.

We know that zero is exponentially stable if and only if $\text{spr}(S_m) < 1$ which is evidently the case if and only if

$$\|S_m^n \mathbf{1}\| = \|S_m^n\| < 1 \quad (5.2)$$

for some $n \geq 1$. Now, by the variation of constants formula and since $e^{t\Delta} \mathbf{1} = \mathbf{1}$ for all $t > 0$, we have

$$S_m^n \mathbf{1} = \mathbf{1} + \int_0^{nT} e^{(nT-\tau)\Delta} m(\cdot, \tau) U_m(\tau, 0) \mathbf{1} d\tau.$$

Hence, (5.2) holds if and only if the function of $x \in \mathbb{R}^N$,

$$\int_0^{nT} e^{(nT-\tau)\Delta} m(\cdot, \tau) U_m(\tau, 0) \mathbf{1} d\tau$$

is bounded away from zero. But, since

$$e^{-nT\|m\|_\infty} \leq e^{-s\|m\|_\infty} \leq U_m(s, 0) \mathbf{1} \leq \mathbf{1}$$

for all $s \in [0, nT]$, this happens if and only if

$$\int_0^{nT} e^{(nT-\tau)\Delta} m(\cdot, \tau) d\tau$$

is bounded away from zero. According to Theorem 4.3 this is equivalent to the function $\int_0^{nT} m(\cdot, \tau) d\tau$ belonging to \mathcal{E} . By the T -periodicity of m this will be so if and only if $\int_0^T m(\cdot, \tau) d\tau$ belongs to \mathcal{E} . This proves the first part of the theorem.

Remarks 5.2. (a) The above theorem together with Theorem 4.3 proves Theorems 1.1 and 1.2 of the introduction.

(b) From the above and using the Riesz-Thorin interpolation theorem we obtain that if the mean of m belongs to \mathcal{E} , then the zero solution of (5.1) is exponentially stable in $X_0 = L_p(\mathbb{R}^N)$ for any $1 < p < \infty$. In the time-independent case it can be proved (cf. [1]) that this condition is also necessary and, hence, exponential stability is actually independent of the underlying space (in fact, in this case it is true that the spectrum of S_m is the same on all spaces as shown in [10]). We were not able to show this in the T -periodic case but we conjecture that this is true.

(c) As already pointed out in the introduction Theorem 5.1 generalizes a result by W. Arendt and C.J.K. Batty for the autonomous case. In their Theorem 1.2 they actually admit more general weight functions, but the core of their proof is the result on $L_1(\mathbb{R}^N)$ with bounded weights. It is easy to see that our method also works for such weights and hence for the more general weights of the first part of [1]. In fact, the core result Theorem 4.3 may be formulated for $f = f_1 + f_2 \in L_\infty(\mathbb{R}^N) + L_1(\mathbb{R}^N)$, since (4.4) still has a meaning for these potentials. The only modification needed in the proof is the replacement of inequality (4.11) by

$$\begin{aligned} & \int_0^{T-\epsilon} \int_{\mathbb{B}(x_n, r_n)^c} w_{T-\tau}(x_n - y) f(y, \tau) dy d\tau \\ \leq & \|f_1\|_\infty \int_0^{T-\epsilon} \int_{\mathbb{B}(0, r_n)^c} w_{T-\tau}(y) dy d\tau + c(T - \epsilon)^{-\frac{N}{2}} \int_0^{T-\epsilon} \int_{\mathbb{B}(0, r_n)^c} f_2(y) dy d\tau, \end{aligned}$$

whose righthand side also tends to zero as n tends to infinity.

6. Periodic parabolic evolution equations. In this section we shall collect some properties of the evolution operator associated to problem (2.1) which are needed in the next section. Throughout we shall denote by m and m_n arbitrary functions in $C_T^v(\mathbb{R}, BUC(\mathbb{R}^N))$. We shall use freely the notation of Section 2.

As we have already seen, the spectral radius of the period map S_m plays a prominent role in deciding whether the zero solution of (2.1) is exponentially stable or not. Here we give an overview of those properties of the spectral radius of the period map which are relevant for our purposes. The assertions in Remark 6.8 in [7] remain valid in this general context.

Proposition 6.1. *The following assertions are true.*

- (a) *If $m_1 \leq m_2$, then $S_{m_2} \leq S_{m_1}$ and $\text{spr}(S_{m_2}) \leq \text{spr}(S_{m_1})$. If $\text{spr}(S_{m_2})$ and $\text{spr}(S_{m_1})$ are eigenvalues, then the inequality is strict whenever $m_1 \neq m_2$.*
- (b) *If $\lim_{n \rightarrow \infty} m_n = m$ in $BUC(\mathbb{R}^N \times \mathbb{R})$ we have that*

$$\lim_{n \rightarrow \infty} \text{spr}(S_{m_n}) = \text{spr}(S_m).$$

Consider the periodic parabolic eigenvalue problem

$$\partial_t \phi - \Delta \phi + m\phi = \mu \phi \quad \text{on } \mathbb{R} \times \mathbb{R}^N, \quad \phi \in C_T(\mathbb{R}, C_0(\mathbb{R}^N)). \quad (6.1)$$

A number $\mu \in \mathbb{R}$ is said to be an eigenvalue for (6.1) if there exists a nontrivial solution ϕ solving (6.1). The function ϕ is then called an eigenfunction. Of particular interest are eigenvalues having positive eigenfunctions. They are called principal eigenvalues. The following result involves only an easy computation.

Proposition 6.2. *A number $\gamma > 0$ is an eigenvalue for S_m with eigenfunction ϕ_0 if and only if $\mu := -\frac{1}{T} \log \gamma$ is an eigenvalue for (6.1) with eigenfunction $\phi(t) = e^{\mu t} U(t, 0)\phi_0$.*

When studying the existence of principal eigenvalues for (6.1) the following condition is of particular interest:

$$\text{spr}_{\text{ess}}(S_m) < \text{spr}(S_m), \quad (6.2)$$

$\text{spr}_{\text{ess}}(S_m)$ being the radius of the essential spectrum (cf. [16]). Recall that since S_m is irreducible, by Theorem V.5.2 in [19] and its corollary this condition and Proposition 6.2 imply the following result.

Proposition 6.3. *Suppose that (6.2) holds. Then, $\text{spr}(S_m)$ is an algebraically simple eigenvalue and the only one having a positive eigenfunction. The same holds true for the dual operator S'_m . In particular, problem (6.1) has a unique principal eigenvalue.*

Concerning the stability properties of the zero solution, the following holds.

Proposition 6.4. *Let (6.2) hold and assume in addition that $\text{spr}_{\text{ess}}(S_m) < 1$. Then, the zero solution of (2.1) is exponentially stable if $\text{spr}(S_m) < 1$, stable but not exponentially stable if $\text{spr}(S_m) = 1$, and unstable if $\text{spr}(S_m) > 1$.*

The proof of the above proposition is the same as in [7], Corollary 6.4. These results show that it is important to be able to estimate the radius of the essential spectrum. Our next result provides a possibility to do this.

Proposition 6.5. *Suppose that $q \in C([0, T], C_0(\mathbb{R}^N))$. Then,*

$$\text{spr}_{\text{ess}}(S_{m+q}) = \text{spr}_{\text{ess}}(S_m).$$

Proof. First we show the compactness of the multiplication operator induced by $q(t)$, $t \in \mathbb{R}$. Suppose $g \in C_c(\mathbb{R}^N)$. It is well known that $D(\Delta)$ is continuously embedded in $C^1(\mathbb{R}^N)$. On the other hand, $C^1(\mathbb{R}^N)$ is compactly embedded in the Fréchet space $C(\mathbb{R}^N)$. Thus, any bounded sequence (u_n) in $D(\Delta)$ has a subsequence which is uniformly convergent on bounded subsets of \mathbb{R}^N . As g has compact support, we find a subsequence of (gu_n) converging uniformly on \mathbb{R}^N . This shows the compactness of the multiplication operator induced by g as an operator from $D(\Delta)$ to $BUC(\mathbb{R}^N)$ and $C_0(\mathbb{R}^N)$, respectively. If $g \in C_0(\mathbb{R}^N)$ approximate g in $C_0(\mathbb{R}^N)$ by a sequence

(g_n) having compact support. Since $\|M_g - M_{g_n}\| \leq \|g - g_n\|_\infty$, and the space of compact operators is closed, it follows that M_g is also compact.

The results in the previous paragraph show that $M_{g(\cdot)} \in C([0, T], \mathcal{K}(D(\Delta), X))$. Hence Theorem 5.2 in [7] gives the assertion. \square

A consequence of the preceding proposition is the following corollary to Theorem 5.1 which will be instrumental in proving change of stability for parameter dependent equations.

Corollary 6.6. *Let $m = m^+ - m^-$ be the decomposition of m in its positive and negative parts. Furthermore, assume that $m^- \in C([0, T], C_0(\mathbb{R}^N))$. Then, $\text{spr}_{\text{ess}}(S_m) < 1$ if and only if the mean \bar{m}^+ lies in \mathcal{E} .*

Proof. The only thing to prove is that $\text{spr}_{\text{ess}}(S_{m^+}) < 1$ implies that $\text{spr}(S_{m^+}) < 1$. As $m^+ \geq 0$ it is clear that $\text{spr}(S_{m^+}) \leq 1$. Suppose now that $\text{spr}(S_{m^+}) = 1$. Then, by Proposition 6.3, 1 is an algebraically simple eigenvalue for S_{m^+} with positive eigenfunction $\phi \in C_0(\mathbb{R}^N)$. On the other hand it is clear that

$$\phi = S_{m^+}^n \phi \leq e^{nT\Delta} \phi$$

for all $n \in \mathbb{N}$. But the right hand side of the above inequality tends to zero as n goes to infinity, which leads to a contradiction. Hence, $\text{spr}(S_{m^+}) < 1$, completing the proof of the corollary. \square

For many applications (for instance for the main result of the next section) it is useful to know a bit more about the integrability properties of the eigenfunction appearing in Proposition 6.3.

Proposition 6.7. *Suppose that*

$$m^- \in C_T^\nu(\mathbb{R}, C_0(\mathbb{R}^N)) \cap C_T^\nu(\mathbb{R}, L_p(\mathbb{R}^N))$$

for some $1 < p < \infty$. Furthermore, assume that $\text{spr}(S_{m^+}) < \text{spr}(S_m)$ and $\text{spr}(S_m) \geq 1$. Then, (6.2) holds and the eigenfunction corresponding to the eigenvalue $\text{spr}(S_m)$ lies in the Sobolev space $H_p^2(\mathbb{R}^N)$. The principal eigenfunction associated to (6.1) then belongs to $C_T(\mathbb{R}, H_p^2(\mathbb{R}^N))$.

Proof. The first assertion follows from Proposition 6.5 and the assumption on the spectral radii. The existence of a positive eigenfunction ϕ_0 is guaranteed by Proposition 6.3.

Consider $U_m(\cdot, \cdot)$ as an evolution operator on $L_p(\mathbb{R}^N)$ and define ϕ and μ as in Proposition 6.2. By the variation of constants formula we obtain that

$$\phi_0 = e^{\mu T} S_m \phi_0 = e^{\mu T} S_{m^+} \phi_0 + \int_0^T e^{\mu\tau} U_{m^+}(t, \tau) m^-(\tau) \phi(\tau) d\tau,$$

and, hence,

$$(1 - e^{\mu T} S_{m^+}) \phi_0 = \int_0^T e^{\mu\tau} U_{m^+}(t, \tau) m^-(\tau) \phi(\tau) d\tau. \tag{6.3}$$

Since $m^- \phi \in C_T(\mathbb{R}, L_p(\mathbb{R}^N))$ this equality holds in $L_p(\mathbb{R}^N)$. By our assumptions it follows that

$$\text{spr}(e^{\mu T} S_{m^+}) = e^{\mu T} \text{spr}(S_{m^+}) < 1$$

in $\mathcal{L}(L_p(\mathbb{R}^N))$. Indeed, in case that $\mu < 0$ (i.e., $\text{spr}(S_m) \geq 1$) the assertion is obvious, and in case that $\mu = 0$ it easily follows using Remark 5.2(b). Therefore, $(\mathbf{1} - e^{\mu T} S_{m^+})^{-1}$ exists in $\mathcal{L}(L_p(\mathbb{R}^N))$ and we get from (6.3) that $\phi_0 \in L_p(\mathbb{R}^N)$. As $\phi(T) = \phi_0$, we have that $\phi_0 \in D(\Delta) = H_p^2(\mathbb{R}^N)$. The remaining assertion is an obvious consequence of this fact.

7. Change of stability. We now proceed to investigate the stability properties of the parameter dependent linear time-periodic parabolic equation

$$\partial_t u - \Delta u = \lambda m(x, t)u, \quad \text{in } \mathbb{R}^N \times (0, \infty), \tag{7.1}$$

where the parameter λ varies over the positive real axis and m is a weight function in $C_T^v(\mathbb{R}, BUC(\mathbb{R}^N))$. Throughout this section we shall always take $X_0 = C_0(\mathbb{R}^N)$ and write $S_\lambda \in \mathcal{L}(X_0)$ for the period map associated to (7.1) and $r(\lambda)$ for its spectral radius, i.e., $r(\lambda) := \text{spr}(S_\lambda)$. Note that in contrast to Sections 2 and 6, m appears here with the opposite sign.

Since the stability properties of the zero solution of (7.1) are closely related to the behavior of $r(\lambda)$ as λ varies over \mathbb{R}^+ we shall first focus our attention on this function. We break down the analysis into three parts. First we look at what happens as λ tends to infinity and then proceed to study r in a neighborhood of zero. In a third part we put this information together to obtain our result on change of stability.

But before doing this let us recall a nice property of r . Since part (i) of Lemma 7.1 in [7] is also valid for arbitrary m (same proof) we have that the logarithm of the function $r : \mathbb{R} \rightarrow \mathbb{R}^+$ is convex. It follows that r itself is also convex. This implies its continuity and the existence of its right derivative at each point $\lambda \geq 0$.

A. Behavior as λ tends to infinity. For the behavior of $r(\lambda)$ as λ tends to infinity it is necessary to consider the number

$$\mathcal{P}(m) := \int_0^T \sup_{x \in \mathbb{R}^N} m(x, \tau) d\tau. \tag{7.2}$$

The following condition is necessary and sufficient to ensure the existence of a principal eigenvalue for the Dirichlet problem in bounded domains (see [11], Theorem 16.1).

Lemma 7.1. *For each $\lambda \geq 0$, $\|S_\lambda\| \leq e^{\lambda \mathcal{P}(m)}$ and, in particular, $r(\lambda) \leq e^{\lambda \mathcal{P}(m)}$.*

Proof. Define the function $\tilde{m} : \mathbb{R} \rightarrow \mathbb{R}$ by setting $\tilde{m}(t) := \sup_{x \in \mathbb{R}^N} m(x, t)$ for each $t \in \mathbb{R}$. Denote by \tilde{S}_λ the period map associated to

$$\partial_t u - \Delta u = \lambda \tilde{m}(t)u, \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{7.3}$$

Since we obviously have $m \leq \tilde{m}$ it follows that $\|S_\lambda\| \leq \|\tilde{S}_\lambda\|$. But, as is easily seen from the variation of constants formula,

$$\tilde{S}_\lambda = e^{T\Delta} e^{\int_0^T \lambda \tilde{m}(\tau) d\tau} = e^{T\Delta} e^{\lambda \mathcal{P}(m)}$$

for all $\lambda \geq 0$. Since $\|e^{T\Delta}\| = 1$, the assertion of the lemma follows. \square

The main result of this subsection is the following.

Theorem 7.2. *If $\mathcal{P}(m) > 0$ then*

$$\lim_{\lambda \rightarrow \infty} r(\lambda) = \infty.$$

If $\mathcal{P}(m) = 0$ then $r(\lambda) \leq 1$ for all $\lambda \geq 0$. Finally, if $\mathcal{P}(m) < 0$ then $r(\lambda) < 1$ for all $\lambda > 0$ and

$$\lim_{\lambda \rightarrow \infty} r(\lambda) = 0.$$

Proof. The first assertion may be proved in the same way as Lemma 7.2 in [7]. The other two assertions follow immediately from Lemma 7.1.

B. Behavior near zero. We shall give necessary and sufficient conditions for the existence of a $\lambda_0 > 0$ such that $r(\lambda) < 1$ whenever $0 < \lambda < \lambda_0$. This means that in this case the zero solution of (7.1) is exponentially stable for these λ 's. Of course this is equivalent to proving the corresponding result for the function $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by

$$\mu(\lambda) := -\frac{1}{T} \log r(\lambda) \tag{7.4}$$

which is evidently concave.

We shall assume throughout that the support of the positive part of m is contained in a set of the form $K \times \mathbb{R}$ with K compact, i.e.,

$$\text{supp}(m^+) \subset K \times \mathbb{R}. \tag{7.5}$$

This implies in particular that $r_{ess}(\lambda) \leq 1$ for all $\lambda \geq 0$ (see Proposition 6.5).

We shall obtain the above mentioned result as a corollary to a more general one whose proof will be prepared in a series of lemmas.

Lemma 7.3. *Suppose that $r(\lambda) > r_{ess}(\lambda)$ holds for $\lambda \in (0, \lambda_0)$ for some $\lambda_0 > 0$. Then, for this range of λ 's, $\mu(\lambda)$ is a principal eigenvalue for*

$$\partial_t \phi - \Delta \phi - \lambda m(x, t) \phi = \mu(\lambda) \phi \quad \text{in } \mathbb{R}^N \times \mathbb{R} \tag{7.6}$$

and for the adjoint problem

$$-\partial_t \psi - \Delta \psi - \lambda m(x, t) \psi = \mu(\lambda) \psi \quad \text{in } \mathbb{R}^N \times \mathbb{R}. \tag{7.7}$$

Denote by ϕ_λ and ϕ_λ^* the corresponding unique principal eigenfunctions for these problems which we normalize by $\|\phi_\lambda^*\|_\infty, \|\phi_\lambda\|_\infty = 1$.

If we additionally assume that $r(\lambda) \geq 1$ then $\phi_\lambda, \phi_\lambda^* \in C_T(\mathbb{R}, H_2^2(\mathbb{R}^N))$ and also $\phi_\lambda \rightarrow 1$ uniformly on compact subsets of $\mathbb{R}^N \times \mathbb{R}$, as λ tends to zero.

Proof. The assertion on the adjoint problem follows from Proposition 6.3 and (2.8).

By the results in Section 6 it remains only to prove the convergence assertion. We first prove that all ϕ_λ 's take their maximum on $K \times \mathbb{R}$. To see this observe that

$$\partial_t \phi_\lambda - \Delta \phi_\lambda + m^- \phi_\lambda - \mu(\lambda) \phi_\lambda \leq 0 \quad \text{in } K^c \times \mathbb{R}.$$

By the maximum principle and since $\phi_\lambda(t)$ lies in $C_0(\mathbb{R}^N)$ for each t we get that the maximum of ϕ_λ on $\bar{K}^c \times \mathbb{R}$ can be taken only on $K^c \times \{0\}$ or on $\partial K \times [0, T]$. By periodicity we may exclude the first alternative. It now follows that ϕ_λ can take its maximum only in $K \times \mathbb{R}$. Set, for each $t \in \mathbb{R}$,

$$h_\lambda(t) := (\lambda m(\cdot, t) - \mu(\lambda)) \phi_\lambda(t).$$

Then, by the variation of constants formula,

$$\phi_\lambda(t) = e^{t\Delta} \phi_\lambda(0) + \int_0^t e^{(t-\tau)\Delta} h_\lambda(\tau) d\tau.$$

Observing that $\{\phi_\lambda(0) : \lambda \in (0, \lambda_0)\}$ and $\{h_\lambda : \lambda \in (0, \lambda_0)\}$ are bounded sets in X_0 and $L_\infty([0, T], X_0)$ respectively we may proceed as in the proof of Theorem 2.4 in [12] and obtain a function ϕ_0 in $BU C^{2,1}(\mathbb{R}^N \times \mathbb{R})$ which is periodic in the second argument and such that $\phi_\lambda \rightarrow \phi_0$ as $\lambda \rightarrow 0$ uniformly on compact subsets of $\mathbb{R}^N \times \mathbb{R}$ and satisfying

$$\partial_t \phi_0 - \Delta \phi_0 = 0. \tag{7.8}$$

Moreover, since each ϕ_λ takes its maximum in $K \times [0, T]$ this function cannot equal zero. Furthermore, the compactness of K implies that the maximum of ϕ_0 is equal to 1. Since 1 is the only periodic solution of (7.8) satisfying this, the assertion of the lemma follows. \square

Recall that whenever $r_{ess}(\lambda) < r(\lambda)$ for some $\lambda > 0$ we know that $r(\lambda)$ is a simple eigenvalue of S_λ . Since S_λ depends analytically on λ as a map to $\mathcal{L}(H_2^2)$ (cf. [7], Theorem 11.8) it follows by perturbation theory (cf. [13], Section VII.1.3) that $r(\cdot)$ is analytic in a neighborhood of this point. Moreover, the corresponding eigenfunctions ϕ_λ , with $\|\phi_\lambda\|_\infty = 1$, depend analytically on λ in that neighborhood. We shall now give a formula for $\mu'(\lambda)$ in such a neighborhood.

Lemma 7.4. *Suppose that $r(\lambda) > r_{ess}(\lambda)$ and $r(\lambda) \geq 1$ for some $\lambda > 0$. Then,*

$$\mu'(\lambda) = - \frac{\int_0^T \int_{\mathbb{R}^N} m \phi_\lambda \phi_\lambda^* dy d\tau}{\int_0^T \int_{\mathbb{R}^N} \phi_\lambda \phi_\lambda^* dy d\tau}. \tag{7.9}$$

Proof. Let ϕ_λ and ϕ_λ^* be as in Lemma 7.2. Since ϕ_λ satisfies (7.6) we may differentiate (7.6) with respect to λ and obtain that

$$\partial_t \phi'_\lambda - \Delta \phi'_\lambda - \lambda m(x, t) \phi'_\lambda - m(x, t) \phi_\lambda = \mu(\lambda) \phi'_\lambda + \mu'(\lambda) \phi_\lambda \quad \text{in } \mathbb{R}^N \times \mathbb{R}. \quad (7.10)$$

Multiplying (7.10) by ϕ_λ^* and reordering we obtain that

$$(\partial_t - \Delta - \lambda m(x, t) - \mu(\lambda)) \phi'_\lambda \phi_\lambda^* - m(x, t) \phi_\lambda \phi_\lambda^* = \mu'(\lambda) \phi_\lambda \phi_\lambda^* \quad \text{in } \mathbb{R}^N \times \mathbb{R}.$$

Integrating both sides over $\mathbb{R}^N \times [0, T]$ and observing that, by Green's formula and (7.7),

$$\int_0^T \int_{\mathbb{R}^N} (\partial_t - \Delta - \lambda m(y, \tau) - \mu(\lambda)) \phi'_\lambda(y, \tau) \phi_\lambda^*(y, \tau) dy d\tau = 0,$$

we obtain the assertion. \square

We are now ready to prove the main result of this subsection.

Theorem 7.5. *Assume that (7.5) holds. Then there exists a $\lambda_0 > 0$ such that $r(\lambda) < 1$ for all $\lambda \in [0, \lambda_0]$ if and only if the temporal mean of m^- belongs to the class \mathcal{E} .*

Proof. The necessity of the condition follows from Corollary 6.6. To prove the sufficiency part assume that the temporal mean of m^- belongs to \mathcal{E} . If $r_{ess}(\lambda) = r(\lambda)$ for λ near zero there is nothing to prove since $r_{ess}(\lambda) < 1$ for all $\lambda > 0$. If not we may apply Lemmas 7.3 and 7.4 freely. Observe that our assumptions on m imply that

$$\int_0^T \int_{\mathbb{R}^N} m(y, \tau) dy d\tau = -\infty. \quad (7.11)$$

Hence for some $R > 0$ and some $\delta > 0$ we get

$$\int_0^T \int_{\mathbb{R}^N} m^+(y, \tau) dy d\tau < \int_0^T \int_{\mathbb{B}(0, R)} m^-(y, \tau) dy d\tau - \delta. \quad (7.12)$$

Take now $\epsilon > 0$ such that

$$\epsilon \int_0^T \int_{\mathbb{B}(0, R)} m^-(y, \tau) dy d\tau < \delta \quad (7.13)$$

and $\lambda_0 > 0$ such that

$$1 - \epsilon < \phi_\lambda \phi_\lambda^* \leq 1 \quad (7.14)$$

on $\mathbb{B}(0, R) \times \mathbb{R}$ for all $\lambda \in (0, \lambda_0]$. Then, using (7.12)–(7.13) we get

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} m^+ \phi_\lambda \phi_\lambda^* dy d\tau \leq \int_0^T \int_{\mathbb{R}^N} m^+ dy d\tau < \int_0^T \int_{\mathbb{B}(0, R)} m^- dy d\tau - \delta \\ & \leq \int_0^T \int_{\mathbb{B}(0, R)} m^- (1 - \epsilon) dy d\tau + \epsilon \int_0^T \int_{\mathbb{B}(0, R)} m^- dy d\tau - \delta \\ & < \int_0^T \int_{\mathbb{B}(0, R)} m^- \phi_\lambda \phi_\lambda^* dy d\tau < \int_0^T \int_{\mathbb{R}^N} m^- \phi_\lambda \phi_\lambda^* dy d\tau. \end{aligned}$$

Hence,

$$\int_0^T \int_{\mathbb{R}^N} m \phi_\lambda \phi_\lambda^* dy d\tau < 0.$$

By (7.9) this implies that $\mu'(\lambda) > 0$ for all $\lambda \in (0, \lambda_0]$. By the concavity of $\mu(\cdot)$ and since $\mu(0) = 0$ we immediately obtain that $\mu(\lambda) > 0$ for all $\lambda \in (0, \lambda_0]$. By (7.2) this proves the theorem. \square

Using the same method of proof one can actually prove the following result.

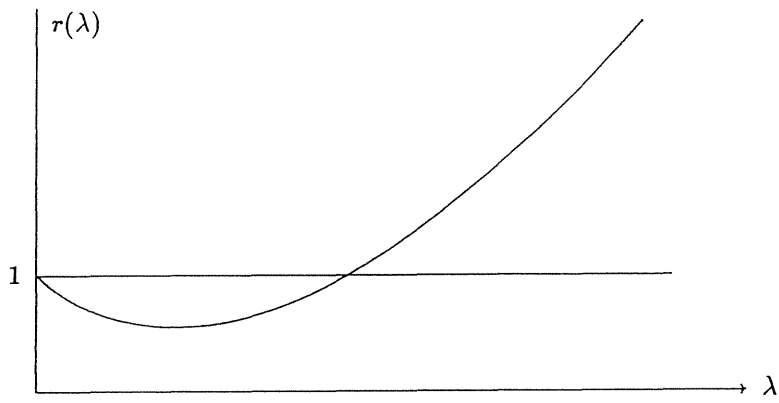
Theorem 7.6. *Suppose that (7.5) holds. Then, if*

$$\int_0^T \int_{\mathbb{R}^N} m(y, \tau) dy d\tau < 0 \tag{7.15}$$

(possibly $-\infty$), there exists a $\lambda_0 > 0$ such that $r(\lambda) \leq 1$ for all $\lambda \in [0, \lambda_0]$.

Proof. Observe that our assumptions are such that we obtain estimates as in (7.12) and (7.13). Now, if the assertion were not true we would have, by concavity of $\mu(\cdot)$, that $\mu(\lambda) > 0$ for all $\lambda > 0$. In particular, we would be able to apply Lemmas 7.3 and 7.4 to obtain an estimate like (7.14). Proceeding as in the proof of Theorem 7.5 it would follow that $\mu(\lambda) < 0$ for all $\lambda \in (0, \lambda_0]$ for some $\lambda_0 > 0$, which is evidently a contradiction.

C. Change of stability. We are interested in giving conditions on m such that the graph of the function $r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ has the form



This means that we have change of stability at a certain point λ_0 : The zero solution of (7.1) starts off being exponentially stable for λ close to zero and becomes unstable as λ crosses λ_0 from the left to the right.

As an immediate consequence of the results in Subsections A and B we obtain the following theorem.

Theorem 7.7. *Suppose that (7.5) holds. Then, there exists a number $\lambda_0 > 0$ such that the zero solution of (7.1) is exponentially stable if $0 < \lambda < \lambda_0$ and unstable if $\lambda > \lambda_0$, if and only if the temporal mean of m^- belongs to the class \mathcal{E} and $\mathcal{P}(m) > 0$.*

Remark 7.8. Of course by the results of Section 2 and Section 5 Theorem 7.7 remains valid if we take X_0 equal to $BUC(\mathbb{R}^N)$ or $L_1(\mathbb{R}^N)$.

D. Principal eigenvalues. The critical value λ_0 from Theorem 7.7 also admits another interpretation. Consider the periodic-parabolic eigenvalue problem with respect to the indefinite weight function m ,

$$\begin{cases} \partial_t \phi - \Delta \phi = \lambda m(x, t) \phi & \text{in } \mathbb{R}^N \times \mathbb{R} \\ \phi(\cdot, t) \in C_0(\mathbb{R}^N) & \text{for each } t > 0 \\ \phi \text{ is } T\text{-periodic and positive.} \end{cases} \quad (7.16)$$

Any number $\lambda > 0$ such that there exists a classical T -periodic solution $\phi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ of (7.16) is called a *principal eigenvalue* for (1.6). The corresponding solution ϕ is then called a *principal eigenfunction*.

From Lemma 7.3 we see that if $r_{ess}(\lambda) < 1$ for all $\lambda > 0$, then λ is a principal eigenvalue for (7.16) if and only if $r(\lambda) = 1$ or, equivalently, if and only if $\mu(\lambda) = 0$. Now, Theorem 7.7 immediately implies the following existence result for principal eigenvalues of (7.16).

Theorem 7.8. *Assume that m^- belongs to the class \mathcal{E} and that (7.5) holds. Then there exists a principal eigenvalue for (7.16) if and only if $\mathcal{P}(m) > 0$. The principal eigenvalue is unique and equals the number λ_0 of Theorem 7.7 and has, up to linear dependence, a unique eigenfunction ϕ . Moreover, $\phi \in C_T(\mathbb{R}, H^2(\mathbb{R}^N))$.*

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