Superconvexity of the evolution operator and parabolic eigenvalue problems on $\mathbb{R}^n$

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1. Introduction

The purpose of this paper is to investigate the stability of the zero solution of the equation

$$\partial_t u - k(t)\Delta u = \lambda m(x,t)u \qquad \text{in } \mathbb{R}^N \times (0,\infty) \tag{1.1}$$

as the parameter λ varies over $\mathbb{R}^+$. Here we assume that the *diffusion coefficient* $k\colon \mathbb{R} \to \mathbb{R}$ is a smooth and strictly positive T-periodic function ($T > 0$ a fixed number) and the *weight function* $m\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is smooth and T-periodic in the second argument (for the precise smoothness conditions consult Section 6). Furthermore, we shall assume that m changes sign and that

$$m(x,t) \le -c < 0 \qquad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R} \text{ with } |x| \ge R_0 \tag{1.2}$$

holds with some suitable constants $c, R_0 > 0$. We remark that by suitable rescaling of time we could assume without loss of generality that $k \equiv 1$.

Stability shall be understood as stability with respect to the L_∞-norm and initial values in $C_0(\mathbb{R}^N)$, the space of continuous functions vanishing at infinity. More precisely, we shall interpret (1.1) as an abstract evolution equation in the Banach space $X_0 := (C_0(\mathbb{R}^N), \|\cdot\|_\infty)$. This is accomplished by setting

$$X_1 := D(A) := \{u \in X_0; \Delta u \in X_0\},$$

$$Au := -\Delta u \qquad \text{for } u \in X_1 \text{ and}$$

$$M(t)u := m(\cdot\,,t)u(\cdot) \qquad \text{for } u \in X_0 \text{ and } T \in \mathbb{R}.$$

It is a well-known fact that $-A$ is the infinitesimal generator of a strongly continuous analytic semigroup on X_0 (see e.g. [**10**]). Moreover, $M\colon \mathbb{R} \to \mathcal{L}(X_0)$ is smooth and T-periodic. We may rewrite (1.1) as

$$\dot u + k(t)Au = \lambda M(t)u \qquad \text{for } 0 < t \le T, \tag{1.3}$$

which is a linear evolution equation in X_0. The assumptions we have made are such that for each $u_0 \in X_0$, (1.3) admits a unique solution $t \mapsto u(t;u_0)$ satisfying the initial condition

$$u(0) = u_0. \tag{1.4}$$

Furthermore, for each $t \ge 0$ the mapping $u_0 \mapsto u(t;u_0)$ is a bounded linear operator on X_0 and will be denoted by $U(t)$. Hence, $U(t)u_0 = u(t;u_0)$. We recall the following notions from stability theory. The zero solution of (1.3) is said to be *stable* if there exists a constant $M > 0$ such that $\|U(t)\| \leq M$ for all $t \geq 0$. It is said to be *exponentially stable* if there exist constants $M, \omega > 0$ such that $\|U(t)\| \leq Me^{-t\omega}$ for all $t \geq 0$. Finally, it is said to be *unstable* if it is not stable.

We can now state one of our main results:

1.1 Theorem
Suppose that

$$\mathcal{P}(m) := \int_0^T \max_{x \in \mathbb{R}^N} m(x, \tau) d\tau > 0 \tag{1.5}$$

holds. Then there exists a number $\lambda_1(m) > 0$*, such that the zero solution of* (1.3) *is exponentially stable if* $0 < \lambda < \lambda_1(m)$*, stable (but not exponentially stable) if* $\lambda = \lambda_1$*, and unstable if* $\lambda > \lambda_1$*.*

It turns out that $\lambda_1(m)$ may be interpreted in another way. Consider the periodic-parabolic eigenvalue problem with respect to the indefinite weight function m:

$$\begin{cases} \partial_t \phi - k(t)\Delta\phi = \lambda m(x,t)\phi & \text{in } \mathbb{R}^N \times \mathbb{R} \\ \phi(\cdot, t) \in C_0(\mathbb{R}^N) & \text{for each } t > 0 \\ \phi \text{ is } T\text{-periodic and positive} \end{cases} \tag{1.6}$$

A number $\lambda > 0$ such that there exists a smooth positive $\phi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ solving (1.6) is called a *principal eigenvalue* for (1.6). A corresponding ϕ is then called a *principal eigenfunction.* We have

1.2 Proposition
The number $\lambda_1(m)$ *of Theorem* 1.1 *is a principal eigenvalue for* (1.6) *and is the only one. Moreover, it has, up to linear dependence, a unique principal eigenfunction.*

Results similar to ours have been obtained by A. Beltramo and P. Hess (cf. [**5**], [**6**] and [**13**]) in the case of time-periodic parabolic equations subject to Dirichlet, Neumann or Robin boundary conditions on bounded domains. Moreover, this type of results has been succesfully applied to the study of the qualitative behaviour of solutions of time-periodic semilinear parabolic initial-boundary value problems (for an extensive treatment of these applications consult [**13**]). In fact, our research has been motivated by the desire to understand nonlinear parabolic equations on (the unbounded domain) $\mathbb{R}^N$, and the monograph [**13**] has served as a guiding line for the kind of results one should expect. There are two substantial differences between our problem on $\mathbb{R}^N$ and the bounded domain case. First of all, we do not have the strong positivity of the *period-map* $U(T) \in \mathcal{L}(X_0)$ since the positive cone of X_0 has empty interior. This is not the main issue for the linear problem studied here since the Phragmen-Lindelöf principle (i.e. the parabolic maximum principle on unbounded domains) still gives the irreducibility of $U(T)$. But for nonlinear stability considerations it is indeed a major obstacle (cf. [**18**]). The second difference is the lack of compactness of the period-map. As a consequence we are not able to apply the Krein-Rutman theory for positive irreducible operators which would ensure the existence of a principal eigenvalue. Thus we are forced to use another approach based on estimates for the essential spectral radius of $U(T)$.

We should also mention the paper [**8**] of K.J. Brown, C. Cosner and J. Fleckinger where they prove the existence of a principal eigenvalue for an elliptic eigenvalue problem with indefinite weight function on $\mathbb{R}^N$. However, they do not obtain uniqueness and their method is completely different from ours since they are able to exploit the variational structure of the elliptic problem. But we remark that we do use their result at a certain point, so that our theory extends but does not replace theirs.

On our way to obtaining Theorem 1.1 we shall prove some abstract results which are also of independent interest. All of them are based in the fact that one may approximate the solutions of an abstract linear parabolic equation by the solutions of the equations corresponding to the Yosida approximations of the principal part of the original equation. This was first proved by Kato in [**14**]. From this result we derive in Sections 3 and 4 a series of interesting properties for parabolic evolution operators depending on a parameter. In particular we give conditions under which the spectral radius of say the period-map corresponding to a parameter dependent abstract linear periodic parabolic evolution equation is a logarithmically convex function of the parameter. This proves to be crucial in the proof of Theorem 1.1. Furthermore, we prove in Section 6 an invariance theorem for the essential spectral radius of the evolution operator of an abstract linear evolution equation subject to a 'relatively compact' perturbation. This seems to be new even in the autonomous case.

Notation

If E and F are arbitrary Banach spaces, we denote by $\mathcal{L}(E,F)$ the space of bounded linear operators from E to F equipped with the uniform operator topology. If it is equipped with the strong topology (topology of pointwise convergence), we denote it by $\mathcal{L}_s(E,F)$. We write $T = \text{s-lim}_{n\to\infty} T_n$ if the sequence of bounded linear operators $(T_n)_{n\in\mathbb{N}}$ is strongly convergent to T, that is convergent in $\mathcal{L}_s(E,F)$. The symbol $E \hookrightarrow F$ means that $E \subset F$ and that the natural injection $i\colon E \to F$ is continuous. If, additionally, E lies dense in F we write $E \overset{d}{\hookrightarrow} F$.

If $T\colon D(T) \subset E \to E$ is a closed linear operator with domain of definition $D(T)$, we denote by $\sigma(T)$ its spectrum and by $\varrho(T)$ its resolvent set. If $T \in \mathcal{L}(E)$ we write $\operatorname{spr}(T)$ for its spectral radius.

If E is an ordered Banach space we shall denote its positive cone either by E_+ or

E^+, as notational convenience dictates. For $x, y \in E$ we write $x \leq y$ if $y - x \in E_+$, and $x < y$ if $x \leq y$ but $x \neq y$. The *dual cone* is defined by $E'_+ := \{x' \in E'; <x', x> \geq 0 \text{ for all } x \in E_+\}$. A point $x \in E$ is called a *quasi-interior point*, denoted by $x \gg 0$, if $<x', x> > 0$ whenever $x' \in E'_+ \setminus \{0\}$. In particular $x > 0$ for any quasi-interior point x of E_+.

2. Yosida approximations of the evolution operator

Let X_0 and X_1 be Banach spaces with $X_1 \overset{d}{\hookrightarrow} X_0$ and $T < \infty$ fixed. In this section we consider the following linear evolution equation

$$\begin{cases} \dot{u} + A(t)u = 0 & \text{for } s < t \leq T \\ u(s) = x \end{cases} \tag{2.1}$$

in X_0, where $s \in [0, T)$. Before stating the precise assumptions on the operators $A(t)$, we introduce some notation. If $\gamma \in (0, \pi)$ is given, we put

$$S_\gamma := \{z \in \mathbb{C} : \ |\arg z| \leq \gamma\} \cup \{0\}. \tag{2.2}$$

Furthermore, we set

$$\Delta_T := \{(t, s) : \ 0 \leq s \leq t \leq T\} \qquad \text{and} \qquad \dot{\Delta}_T := \{(t, s) : \ 0 \leq s < t \leq T\}.$$

On the family $\big(A(t)\big)_{0 \leq t \leq T}$ of closed linear operators in X_0 we impose the following conditions:

- $(A1)$ $D(A(t)) \doteq X_1$ for all $t \in [0, T]$.
- $(A2)$ There is a $\gamma \in (0, \frac{\pi}{2})$ such that $S_{\frac{\pi}{2}+\gamma} \subset \varrho\big(-A(t)\big)$ for all $t \in [0, T]$. Furthermore, there exists an $M > 0$ such that the estimate
  $$\big\|\big(\mu + A(t)\big)^{-1}\big\| \leq \frac{M}{|\mu|}$$
  holds for all $t \in [0, T]$ and $\mu \in S_{\frac{\pi}{2}+\gamma} \setminus \{0\}$.
- $(A3)$ $A(\cdot) \in C^\sigma\big([0, T], \mathcal{L}(X_1, X_0)\big)$ for some $\sigma \in (0, 1)$.

We remark that $(A1)$ and $(A2)$ imply that $-A(t)$ is for each $t \in [0, T]$ the generator of a strongly continuous analytic semigroup in X_0 (see e.g. [**21**], Theorem 2.5.2). Moreover, $(A1)$–$(A3)$ ensure the existence of an evolution operator for the family $\big(A(t)\big)_{0 \leq t \leq T}$ (cf. [**24**], [**21**] and [**3**]). Recall that an *evolution operator* for $\big(A(t)\big)_{0 \leq t \leq T}$ is a function

$$U : \Delta_T \to \mathcal{L}(X_0)$$

satisfying the four subsequent conditions:

(U1) $U \in C\big(\Delta_T, \mathcal{L}_s(X_0)\big) \cap C\big(\Delta_T, \mathcal{L}_s(X_1)\big) \cap C\big(\dot{\Delta}_T, \mathcal{L}(X_0, X_1)\big)$.

(U2) $U(t,t) = \mathbb{1}_{X_0}$, $U(t,r) = U(t,s)U(s,r)$ for all $0 \le r \le s \le t \le T$.

(U3) $[(t,s) \mapsto A(t)U(t,s)] \in C\big(\dot{\Delta}_T, \mathcal{L}(X_0)\big)$ and

$$\sup_{(t,s)\in\Delta_T} (t-s)\|A(t)U(t,s)\| < \infty.$$

(U4) $U(\cdot, s) \in C^1\big((s,T], \mathcal{L}(X_0)\big)$ for each $s \in [0,T)$, and for all $t \in (s,T]$:

$$\partial_1 U(t,s) = -A(t)U(t,s)$$

$U(t,\cdot) \in C^1\big([0,t), \mathcal{L}_s(X_1, X_0)\big)$ for each $t \in (0,T]$, and for all $s \in [0,t)$:

$$\partial_2 U(t,s) \supset U(t,s)A(s).$$

For every $t \in [0,T]$ and $n \in \mathbb{N}$ we may define the *n-th Yosida approximation* of $A(t)$ by

$$A_n(t) := nA(t)\big(n + A(t)\big)^{-1} = A(t)\big(1 + \tfrac{1}{n}A(t)\big)^{-1} = n - n^2\big(n + A(t)\big)^{-1}. \tag{2.3}$$

We remark that

$$\lim_{n\to\infty} n\big(n + A(t)\big)^{-1}x = \lim_{n\to\infty}\big(1 + \tfrac{1}{n}A(t)\big)^{-1}x = x \tag{2.4}$$

for all $x \in X_0$, uniformly in $t \in [0,T]$, and

$$\lim_{n\to\infty} A_n(t)x = A(t)x \tag{2.5}$$

for all $x \in X_1$. Consider for each $n \in \mathbb{N}$ the equation

$$\begin{cases} \dot{u} + A_n(t)u = 0 & \text{for } s < t \le T \\ u(s) = x \end{cases} \tag{2.6}$$

in X_0 with $s \in [0,T)$. Since the Yosida approximation is essentially given by the resolvent of $-A(t)$, and

$$[B \mapsto B^{-1}] \in C^\omega\big(\mathrm{Isom}(X_1, X_0), \mathcal{L}(X_0, X_1)\big), \tag{2.7}$$

we have that $A_n(\cdot) \in C\big([0,T], \mathcal{L}(X_0)\big)$. For this reason there exists an evolution operator U_n for the family $\big(A_n(t)\big)_{0\le t\le T}$ of bounded operators on X_0 (see e.g. [**21**], Chap. 5, Theorem 5.1). As immediately seen, U_n is the solution of the integral equation

$$U_n(t,s)x = x - \int_s^t A_n(\tau)U_n(\tau,s)x\,d\tau. \tag{2.8}$$

We can now state the approximation theorem on which our results rely. The result was proved by T. Kato in [**14**], where he actually constructs the evolution operator in this way.

2.1 Theorem
Suppose $\big(A(t)\big)_{0\le t\le T}$ satisfies assumptions (A1)–(A3), and let $A_n(t)$ be the n-th Yosida approximation of $A(t)$ for $t \in [0,T]$ and $n \in \mathbb{N}$. Then for the corresponding evolution operators U and U_n we have that

$$\lim_{n\to\infty} U_n(t,s)x = U(t,s)x \tag{2.9}$$

holds for any $(t,s) \in \Delta_T$ and $x \in X_0$ uniformly in $t \in [s,T]$.

As an application of this theorem we can prove positivity of the evolution operator, whenever all the $-A(t)$ generate positive semigroups:

2.2 Corollary
Let $\big(A(t)\big)_{0\le t\le T}$ satisfy assumptions (A1)–(A3). Suppose in addition, that X_0 is an ordered Banach space and that $-A(t)$ is for each $t \in [0,T]$ the generator of a positive semigroup. Then the evolution operator for the family $\big(A(t)\big)_{0\le t\le T}$ is positive, that is

$$U(t,s)x \ge 0$$

for all $(t,s) \in \Delta_T$ whenever $x \ge 0$.

Proof
It is well known that the positivity of the semigroups $e^{-tA(s)}$ is equivalent to the fact that the resolvent $\big(n + A(s)\big)^{-1}$ is positive for all sufficiently large $n \in \mathbb{N}$. Therefore, the solution $V_n(t,s)x$ of the integral equation

$$V_n(t,s)x = x + n^2 \int_s^t \big(n + A(\tau)\big)^{-1} V_n(\tau,s)x\,d\tau \tag{2.10}$$

is positive for all $x \ge 0$. Since

$$U_n(t,s)x = e^{-n(t-s)} V_n(t,s)x \tag{2.11}$$

for all $x \in X_0$, $U_n(t,s)$ is positive and the assertion follows by Theorem 2.1. □

We point out that it still seems to be an interesting open problem to find conditions on the family $(A(t))_{0\le t\le T}$ ensuring that the evolution operator is strongly positive, irreducible or even strictly positive. This kind of results are well-known for the autonomous case, i.e. for C_0-semigroups (cf. [**9**] or [**19**]).

2.3 Remark

Condition $(A2)$ may be replaced by

$(A2)'$ There is a $\gamma \in (0, \frac{\pi}{2})$ and a $\mu_0 \in \mathbb{R}$ such that $\mu_0 + S_{\frac{\pi}{2}+\gamma} \subset \varrho\big(-A(t)\big)$ for all $t \in [0,T]$. Furthermore, there exists an $M > 0$ such that the estimate

$$\big\|\big(\mu + A(t)\big)^{-1}\big\| \leq \frac{M}{|\mu - \mu_0|}$$

holds for all $t \in [0,T]$ and $\mu \in \mu_0 + S_{\frac{\pi}{2}+\gamma}\backslash\{0\}$.

In this case we replace the family $\big(A(t)\big)_{0\leq t\leq T}$ by the family $\big(\mu_0 + A(t)\big)_{0\leq t\leq T}$. If U is the evolution operator to the former and U_{μ_0} the evolution operator to the latter family, we have that

$$U(t,s) = e^{(t-s)\mu_0} U_{\mu_0}(t,s) \tag{2.12}$$

holds for all $(t,s) \in \Delta_T$. □

3. Parameter dependent evolution equations

Let X_0 and X_1 be Banach spaces with $X_1 \overset{d}{\hookrightarrow} X_0$ and $T > 0$. Furthermore, we assume that Λ is a bounded open set in $\mathbb{R}$ or $\mathbb{C}$. We consider the following linear parameter dependent evolution equation

$$\begin{cases} \dot{u} + A(\lambda,t)u = 0 & \text{for } s < t \leq T \\ u(s) = x & \end{cases} \tag{3.1}$$

for $\lambda \in \overline{\Lambda}$ and $s \in [0,T)$. Throughout this section we assume that the family

$$\big(A(\lambda,t)\big)_{0\leq t\leq T}$$

satisfies for each $\lambda \in \overline{\Lambda}$ assumption $(A1)$ and $(A2)$ of the previous section uniformly in $\lambda \in \overline{\Lambda}$. We replace $(A3)$ by

$(A3)_0$ $A(\cdot,\cdot) \in C^{0,\sigma}\big(\overline{\Lambda} \times [0,T], \mathcal{L}(X_1,X_0)\big)$ for some $\sigma \in (0,1)$

or

$(A3)_\omega$ $A(\cdot,\cdot) \in C^{\omega,\sigma}\big(\overline{\Lambda} \times [0,T], \mathcal{L}(X_1,X_0)\big)$ for some $\sigma \in (0,1)$.

Under these conditions we can prove the following approximation theorem:

3.1 Theorem
Let $U_\lambda(t,s)$ be the evolution operator for the family $\big(A(\lambda,t)\big)_{0\le t\le T}$ and $U_{\lambda,n}(t,s)$ the evolution operator to the family $\big(A_n(\lambda,t)\big)_{0\le t\le T}$ of the n-th Yosida approximations of $A(\lambda,t)$. Then

$$\lim_{n\to\infty} U_{\lambda,n}(t,s)x = U_\lambda(t,s)x \tag{3.2}$$

holds for all $x \in X_0$ uniformly in $(\lambda,t) \in \overline{\Lambda}\times[s,T]$.

Proof
An inspection of the proofs in [**14**] gives the assertion. In particular, this is shown in Lemma 3.2 and formula (3.19) of [**14**] observing that the convergence depends only on the constants in $(A1)$–$(A3)$. □

We shall now prove the continuous or analytic dependence of $U_\lambda(t,s)$ on the parameter $\lambda \in \overline{\Lambda}$.

3.2 Theorem
If $(A_3)_0$ holds, then

$$\big[\lambda \mapsto U_\lambda(t,s)x\big] \in C\big(\overline{\Lambda}, \mathcal{L}_s(X_0)\big)\,. \tag{3.3}$$

If $(A_3)_\omega$ holds, then

$$\big[\lambda \mapsto U_\lambda(t,s)\big] \in C^\omega\big(\overline{\Lambda}, \mathcal{L}(X_0)\big)\,. \tag{3.4}$$

Proof
Let us first consider the n-th Yosida approximation $U_{\lambda,n}$ of U_λ. Then $U_{\lambda,n}(t,s)$ is the solution of (2.8) for each fixed $\lambda \in \overline{\Lambda}$. The solution of (2.8) is obtained by the Banach contraction mapping theorem. On the other hand, by (2.3) and (2.7), the dependence of $A_{\lambda,n}(t)$ on λ is the same as that of $A_\lambda(t)$. Now, the theorem on continuous or analytic parameter dependence of fixed points (c.f. [**12**], Section 1.2.6) gives the desired regularity of $U_{\lambda,n}(t,s)$ in $\lambda \in \overline{\Lambda}$. By Theorem 3.1, (3.3) follows immediately.

It can be shown, that $(A1)$, $(A2)$ and $(A3)_\omega$ hold also uniformly with respect to λ taken in a complex neighbourhood of the real axis. Hence, the map $\lambda \to U_{\lambda,n}(t,s)$ may be extended to a complex analytic map in this complex neighbourhood. Moreover, approximation (3.2) holds. Note that the uniform limit of (complex) analytic functions is again analytic. Thus, in the case of $(A3)_\omega$, $\lambda \to U_\lambda(t,s)$ is stongly analytic. By [**15**], Theorem III.3.12, this implies analyticity in the uniform operator topology. For details we refer to [**10**], Section 11. □

3.3 Remark
With some extra effort one could prove strong C^k-dependence of $U_\lambda(t,s)$ on $\lambda \in \Lambda$ for every $k \in \mathbb{N}$ if suitable conditions are fulfilled. For details see [**10**], Section 11. □

4. Superconvexity of the spectral radius

Let Λ be an open interval in $\mathbb{R}$ and X an ordered Banach space with normal and generating positive cone. The positive cone in this space will be denoted by X_+.

A real valued function $\varphi\colon \Lambda \to \mathbb{R}$ is said to be *superconvex*, if $\varphi \equiv 0$ or φ is positive and $\log \varphi$ is convex. A vector valued function $x\colon \Lambda \to X$ is said to be *superconvex*, if for any $\varepsilon > 0$ and any triple $\lambda_0 < \lambda_2 < \lambda_1$ in Λ there exists a finite number of $x_1, \dots, x_m \in X_+$ and real valued superconvex functions $\varphi_1, \dots, \varphi_m$ on Λ such that

$$\|x(\lambda_k) - \sum_{j=1}^{m} \varphi_j(\lambda_k) x_j\| \leq \varepsilon \qquad (k = 0, 1, 2). \tag{4.1}$$

Finally, an operator valued function $U\colon \Lambda \to \mathcal{L}(X)$ is called *superconvex*, if for each $x \in X_+$, $U(\cdot)x\colon \Lambda \to X$ is superconvex. These definitions are due to Kato [**17**]. Just as real valued log-convex functions, superconvex functions exhibit a variety of pleasant properties. In the next remark, we list some of them which are relevant for our purposes. For the proofs we refer to [**17**].

4.1 Remark
Let $x_n\colon \Lambda \to X$ and $U_n(\cdot)\colon \Lambda \to \mathcal{L}(X)$ be superconvex functions for all $n \in \mathbb{N}$. Then the following holds:

(a) Superconvex functions are positive.

(b) $\mu_1 x_1 + \mu_2 x_2$ and $\mu_1 U_1 + \mu_2 U_2$ are superconvex whenever μ_1, μ_2 are positive.

(c) Suppose that $x(\lambda) := \lim_{n\to\infty} x_n(\lambda)$ and $U(\lambda) := \text{s-lim}_{n\to\infty} U_n(\lambda)$ exist for all $\lambda \in \Lambda$. Then x and U are again superconvex.

(d) Suppose that $x\colon \Lambda \to X$ and $U\colon \Lambda \to \mathcal{L}(X)$ are superconvex. Then, the function $\lambda \mapsto U(\lambda)x(\lambda)$ is superconvex.

(e) Let $U\colon \Lambda \times [0,T] \to \mathcal{L}(X)$ be such that $U(\lambda,\cdot) \in C\big([0,T], \mathcal{L}_s(X)\big)$ and superconvex for all $\lambda \in \Lambda$. Then

$$\Lambda \to \mathcal{L}(X),\ \lambda \mapsto \int_s^t U(\lambda,\tau)\,d\tau \tag{4.2}$$

is superconvex for all $(t,s) \in \Delta_T$. Indeed, this follows from (b) and (c) and the fact that the integral is the limit of Riemann sums. □

4.2 Example
Put

$$X := C_0(\mathbb{R}^n) := \{u \in C(\mathbb{R}^n)\colon \lim_{|x|\to\infty} u(x) = 0\} \tag{4.3}$$

and let m be a bounded continuous real valued function on $\mathbb{R}^n$. Denote by M the corresponding multiplication operator $[u \mapsto mu] \in \mathcal{L}(X)$. Then the function

$$\Lambda \to \mathcal{L}(X),\ \lambda \mapsto e^{\lambda M} \tag{4.4}$$

is a superconvex operator valued function.

Since we were not able to find a proof of this result, we include one for completeness.

Proof
Let $\varepsilon > 0$ and $\lambda_0 < \lambda_2 < \lambda_1$ in Λ be given. There exists a constant $c > 0$ such that $|e^{\lambda_k m(x)}| \le c$ holds for all $k = 0, 1, 2$ and $x \in \mathbb{R}^n$. Let $u \in C_0(\mathbb{R}^n)$ fixed and choose $R > 0$ such that $|u(x)| \le \frac{\varepsilon}{c}$ for all $x \in \mathbb{R}^n$ with $|x| \ge R$. Then

$$|e^{\lambda_k m(x)} u(x)| \le \frac{\varepsilon}{2} \tag{4.5}$$

holds for all $x \in \mathbb{R}^n$ with $|x| \ge R$ and $k = 0, 1, 2$. Put $K := \mathbb{B}(0, 2R)$ the ball centered in 0 with radius $2R$ and select a finite open covering $(U_j)_{j=1,\dots,l}$ such that

$$|e^{\lambda_k m(x)} - e^{\lambda_k m(y)}| < \frac{\varepsilon}{2\|u\|_\infty} \tag{4.6}$$

holds for all $x, y \in U_j$ $(j = 1, \dots, l)$ and $k = 0, 1, 2$. Set $U_{l+1} := \overline{\mathbb{B}(0, R)}^c$. Then $(U_j)_{j=1,\dots,l+1}$ is an open covering of $\mathbb{R}^n$. Let $(\psi_j)_{j=1,\dots,l+1}$ a smooth partition of unity on $\mathbb{R}^n$ subordinate to this covering. Choose now $x_j \in U_j$ $(j = 1, \dots, l)$ arbitrary and put

$$\varphi_j(\lambda) := e^{\lambda m(x_j)} \quad (j = 1, \dots, l) \qquad \text{and} \qquad \varphi_{l+1} \equiv 0. \tag{4.7}$$

Furthermore put $u_j := \psi_j u$ $(j = 1, \dots, l+1)$. Then by (4.5) and (4.6)

$$\|\sum_{j=1}^{l+1} \varphi_j(\lambda_k) u_j - e^{\lambda_k M} u\|_\infty \le \varepsilon \tag{4.8}$$

holds for $k = 0, 1, 2$ and the assertion follows. □

Let us now consider a family $\big(A(\lambda)\big)_{\lambda\in\Lambda}$ of closed operators. Such a family is called *resolvent superconvex* if there exists a number $\mu_0 \in \mathbb{R}$ such that $[\mu_0, \infty) \subset \varrho\big(A(\lambda)\big)$ for all $\lambda \in \Lambda$ and $\lambda \mapsto \big(\mu - A(\lambda)\big)^{-1}$ is superconvex for all $\mu > \mu_0$.

The main result of this section is the following:

4.3 Theorem
Suppose that the hypotheses of Theorem 3.1 *hold and that* X_0 *is an ordered Banachspace with normal and generating positive cone. Moreover assume that for each* $t \in [0,T]$ *the family* $\big(A(\lambda,t)\big)_{\lambda\in\Lambda}$ *is resolvent superconvex. Then the map*

$$\Lambda \to \mathcal{L}(X),\ \lambda \to U_\lambda(t,s) \tag{4.9}$$

is superconvex for all $(t,s) \in \Delta_T$.

Proof
Let $V_{\lambda,n}(t,s)$ for each $\lambda \in \Lambda$ be the solution of (2.10). The solution of this equation is obtained with the Banach fixed point theorem (see e.g. [**21**], Chapter 5, proof of Theorem 5.1). Since the family $\big(A(\lambda,t)\big)_{\lambda\in\Lambda}$ is resolvent superconvex, it follows by Remark 4.1 that $\lambda \mapsto V_{\lambda,n}(t,s)$ is superconvex on Λ for all $(t,s) \in \Delta_T$ and $n \in \mathbb{N}$ large enough. Formula (2.11) implies that the same holds for $U_{\lambda,n}(t,s)$. Again by Remark 4.1 and Theorem 3.1 the assertion follows. □

4.4 Remark
(a) Suppose that for every $\lambda \in \Lambda$, the operator $A(\lambda)$ is the infinitesimal generator of a strongly continuous semigroup in X_0. The family $(A(\lambda))_{\lambda\in\Lambda}$ is said to be *semigroup superconvex* if $\lambda \mapsto e^{tA(\lambda)}$ is superconvex for each $t \geq 0$. Theorem 5.2 in [**17**] asserts that $(A(\lambda))_{\lambda\in\Lambda}$ is semigroup superconvex if and only if it is resolvent superconvex. Therefore, the family $(\lambda M)_{\lambda\in\Lambda}$ of Example 4.2 is resolvent superconvex.
(b) Assume that $(A(\lambda))_{\lambda\in\Lambda}$ and $(B(\lambda))_{\lambda\in\Lambda}$ are two semigroup superconvex families of generators as defined in the previous remark, but with $B(\lambda) \in \mathcal{L}(X_0)$ for each $\lambda \in \Lambda$, and with $\|B(\lambda)\|$ uniformly bounded in $\lambda \in \Lambda$. Then the family $(A(\lambda) + B(\lambda))_{\lambda\in\Lambda}$ is also semigroup superconvex. This is Theorem 5.3 in [**17**]. □

We are mainly interested in the following corollary, which is immediately obtained from the above theorem using a generalization of a theorem by Kingman due to Kato ([**17**], Theorem 2.5). This Theorem asserts that the spectral radius of a family of supercovex operators is a superconvex real valued function.

4.5 Corollary
Suppose that the same assumptions as in Theorem 4.3 *are satisfied and let* $\mathrm{r}(\lambda) := \mathrm{spr}(U_\lambda(t,s))$ *be the spectral radius of* $U_\lambda(t,s) \in \mathcal{L}(X)$. *Then* $\mathrm{r}(\cdot)\colon \Lambda \to \mathbb{R}$ *is superconvex.*

4.6 Remark
Corollary 4.5 yields an abstract proof of a result obtained by Beltramo and Hess in [**6**] and Beltramo [**5**] on the log-convexity of the principal eigenvalue of a parameter

dependent periodic parabolic problem. Observe that we do not require that $\operatorname{spr}(U_\lambda(t,s))$ is an eigenvalue of $U_\lambda(t,s)$. $\square$

5. Relatively compact perturbations and the essential spectrum

Before starting with the true subject of this section let us recall the notion of the essential spectrum and make a few pertinent remarks about it.

We assume that X is a Banach space and that $T \in \mathcal{L}(X)$. A complex number λ belongs to the (*Browder*) *essential spectrum* of T, $\sigma_{ess}(T)$, if one of the following conditions is satisfied: (i) λ is a limit point of $\sigma(T)$, (ii) the image of $(\lambda - T)$ is not closed, or (iii) the space $\cup_{k\geq 1} \ker(\lambda - T)^k$ is infinite dimensional. Define now the *essential spectral radius* of T as

$$\operatorname{spr}_{ess}(T) := \sup\{|\lambda|; \lambda \in \sigma_{ess}(T)\}.$$

5.1 Remarks

(a) What will be of interest to us is to know what kind of points are not contained in the essential spectrum. If $\lambda \in \sigma(T) \setminus \sigma_{ess}(T)$ holds, then λ is a pole of the resolvent of T. In particular it is an eigenvalue (cf. [**9**] Theorem A.3.3).

(b) If $S \in \mathcal{L}(X)$ is compact then $\operatorname{spr}_{ess}(T+S) = \operatorname{spr}_{ess}(T)$. This is an easy consequence of Lemma 1 and Theorem 1 in [**20**].

(c) If X is a Banach lattice and T is a positive operator, then it is well-known that $\operatorname{spr}(T)$ lies in $\sigma(T)$ (see [**23**], Proposition V.4.1). So if we can show that $\operatorname{spr}_{ess}(T) < \operatorname{spr}(T)$, we may infer from (a) that $\operatorname{spr}(T)$ is an eigenvalue. Indeed, then any point in the *peripheral spectrum*, $\sigma_{per}(T) := \{\lambda \in \sigma(T); |\lambda| = \operatorname{spr}(T)\}$, is an eigenvalue. In this case if, additionally, T is irreducible we conclude from Theorem 5.2 in [**23**] that:

- (i) $\operatorname{spr}(T)$ is an algebraically simple eigenvalue and has an eigenvector in the quasi-interior of X_+. Moreover, $\operatorname{spr}(T)$ is the only eigenvalue of T having a positive eigenvector.
- (ii) Each point in $\sigma_{per}(T)$ is an algebraically simple eigenvalue of T. $\square$

For the rest of this section we make the same assumptions and use the same notation as in Section 2. We consider here a perturbation of (2.1), namely

$$\begin{cases} \dot{u} + A(t)u + B(t)u = 0 & \text{for } s < t \leq T \\ u(s) = x \end{cases} \tag{5.1}$$

On the perturbation $B(\cdot)$ we impose the following condition

(B) $\quad B(\cdot) \in C^\sigma\big([0,T], \mathcal{L}(X_1, X_0)\big)$ and $B(t)$ is compact for each $t \in [0,T]$, where σ is the constant appearing in $(A3)$.

Now, this assumption together with a simple generalization of a perturbation result originally due to Desch and Schappacher (cf. [**11**], [**9**] and [**4**]) implies that the family $\big(A(t)+B(t)\big)_{0\le t\le T}$ satisfies assumptions $(A1)$-$(A3)$ (where the constants appearing in $(A2)$ may of course be different from those for $\big(A(t)\big)_{0\le t\le T}$). Hence, there exists a unique evolution operator $V:\Delta_T\to\mathcal{L}(X_0)$ corresponding to (5.1). We now set $K(t,s):=V(t,s)-U(t,s)\in\mathcal{L}(X_0)$ for $(t,s)\in\Delta_T$. Thus,

$$V(t,s)=U(t,s)+K(t,s) \tag{5.2}$$

holds. We would now like to study the effect of the perturbation $K(t,s)$ on the essential spectrum of $U(t,s)$. The main result will be

5.2 Theorem
For any $(t,s)\in\Delta_T$ we have that

$$\operatorname{spr}_{ess}(U(t,s))=\operatorname{spr}_{ess}(V(t,s))$$

holds.

Before proving this result we have to recall some facts on the evolution operator on interpolation spaces.

Let $(X_\alpha,\|\cdot\|)$, $\alpha\in(0,1)$, be the Banach space obtained by interpolating between X_1 and X_0 by means of the complex interpolation method, i.e. $X_\alpha=[X_0,X_1]_\alpha$ (consult [**7**] or [**25**] for the relevant facts on interpolation theory). We then have that

$$X_1 \overset{d}{\hookrightarrow} X_\beta \overset{d}{\hookrightarrow} X_\alpha \overset{d}{\hookrightarrow} X_0 \tag{5.3}$$

whenever $0\le\alpha\le\beta\le 1$, and

$$\|x\|_\alpha \le c(\alpha)\|x\|_1^{1-\alpha}\|x\|_0^\alpha \tag{5.4}$$

holds for all $x\in X_1$ and $\alpha\in(0,1)$.

By $(U1)$ we have that $U(t,s)\in\mathcal{L}(X_0,X_1)$ for $(t,s)\in\dot{\Delta}_T$. Therefore, we get that $U(t,s)\in\mathcal{L}(X_\alpha,X_\beta)$ for any $\alpha,\beta\in[0,1]$ and $(t,s)\in\dot{\Delta}_T$. The norm on $\mathcal{L}(X_\alpha,X_\beta)$ shall be denoted by $\|\cdot\|_{\alpha,\beta}$. We shall need the following estimates, a proof of which can be found either in [**3**] or [**10**].

5.3 Lemma
(i) For $0\le\alpha\le 1$ we have $U\in C\big(\Delta_T,\mathcal{L}_s(X_\alpha)\big)$.
(ii) For $0\le\alpha\le\beta\le 1$ and $(t,s)\in\Delta_T$, we have: $\|U(t,s)\|_{\beta,\alpha}\le c(\alpha,\beta)$.

(iii) For $0 \le \alpha < \beta \le 1$ *and* $(t,s) \in \dot{\Delta}_T$*, we have:* $\|U(t,s)\|_{\alpha,\beta} \le c(\alpha,\beta)(t-s)^{\alpha-\beta}$.

The key result in the proof of Theorem 5.2 is the following

5.4 Proposition

(i) For any $\alpha \in (0,1)$ *we have that for each* $(t,s) \in \Delta_T$*,* $K(t,s)$ *is a compact operator from* X_α *into itself.*

(ii) If additionally to (B), we require that $B(\cdot) \in C\big([0,T], \mathcal{L}(X_\alpha, X_0)\big)$ *for some* $\alpha \in [0,1)$*, we also have that* $K(t,s)$ *is a compact operator from* X_0 *into itself.*

Proof

Of course if $t = s$ we have that $K(t,s) = 0$ and everything is clear.

(i) Let $\alpha \in (0,1)$. From the preceding lemma it follows that

$$\|U(t,\tau)B(\tau)V(\tau,s)\|_{\alpha,\alpha} \le c(\alpha)(t-\tau)^{1-\alpha}(\tau-s)^{\alpha-1} \tag{5.5}$$

holds for all $0 \le s \le \tau \le t \le T$. So that the integral

$$\int_s^t U(t,\tau)B(\tau)V(\tau,s)\,d\tau \tag{5.6}$$

exists in $\mathcal{L}(X_\alpha)$. It easily follows from assumption (B) and the smoothing property $(U1)$ that for each $\tau \in (s,t)$ the integrand is actually a compact operator in $\mathcal{L}(X_\alpha)$. Therefore we may conclude that the integral (5.5) defines a compact operator in $\mathcal{L}(X_\alpha)$. From the variations of constants formula we obtain without difficulties that

$$K(t,s) = -\int_s^t U(t,\tau)B(\tau)V(\tau,s)\,d\tau$$

holds, proving (i).

(ii) In this case we may replace (5.5) by

$$\|U(t,\tau)B(\tau)V(\tau,s)\|_{0,0} \le c(\alpha)(\tau-s)^{-\alpha}$$

so that the integral (5.6) actually exists in $\mathcal{L}(X_0)$. The same arguments as in (i) yield the assertion. □

We were not able to prove that $K(t,s)$ is a compact operator from X_0 into itself without the additional assumption of part (ii) of the theorem. Nevertheless, we conjecture that this is true.

To prove Theorem 5.2 we just have to remark that neither the spectrum nor the essential spectrum of $U(t,s)$ or $V(t,s)$ as operators in $\mathcal{L}(X_\alpha)$ depends on the special choice of $\alpha \in [0,1]$. Hence, combining Remark 5.1 (b) with the above proposition we immediately obtain

$$\mathrm{spr}_{ess}(V(t,s)) = \mathrm{spr}_{ess}\big(U(t,s) + K(t,s)\big) = \mathrm{spr}_{ess}(U(t,s)),$$

proving the theorem. □

6. A periodic-parabolic eigenvalue problem

In this section we treat the following *periodic-parabolic eigenvalue problem*:

$$\text{(6.1)} \qquad \begin{cases} \partial_t\varphi - k(t)\Delta\varphi - m(x,t)\varphi = \mu\varphi & \text{in } \mathbb{R}^N \times \mathbb{R} \\ \varphi(\cdot,t) \in C_0(\mathbb{R}^N) & \text{for each } t \in \mathbb{R} \\ \varphi \text{ is } T\text{-periodic and positive} & \end{cases}$$

A number $\mu \in \mathbb{R}$ such that there exists a smooth positive $\varphi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ solving (6.1) is called a *principal eigenvalue* for (6.1). A corresponding φ is then called a *principal eigenfunction*. Note the difference between (6.1) and the eigenvalue problem (1.6) of the introduction. Here m does not have the role of a weight function.

The exact assumptions we shall make on the coefficients are the following

$$\text{(6.2)} \qquad k \in C^{\frac{\mu}{2}}(\mathbb{R}),\ \mu \in (0,1),\ \text{is strictly positive and } T\text{-periodic,}$$

$$\text{(6.3)} \qquad \begin{aligned} &m \in BUC^{\mu,\frac{\mu}{2}}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \text{ is } T\text{-periodic in the second argument, and} \\ &m(x,t) \le -c \quad \text{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R} \text{ with } |x| \ge R_0, \end{aligned}$$

for suitable positive constants c and R_0.

Problem (6.1) is closely related to the stability properties of the zero solution of the following linear equation:

$$\text{(6.4)} \qquad \partial_t u - k(t)\Delta u = m(x,t)u \qquad \text{in } \mathbb{R}^N \times (0,\infty)\,.$$

As in the introduction, we may reformulate (6.4) as an abstract evolution equation

$$\text{(6.5)} \qquad \dot{u} + k(t)Au = M(t)u \qquad \text{for } t > 0$$

on the space $X_0 = C_0(\mathbb{R}^N)$. We shall exploit the fact that X_0 is a Banach lattice with respect to pointwise ordering. The quasi-interior points of its positive cone are the functions that are everywhere positive on $\mathbb{R}^N$.

Let $U_m\colon \Delta_T \to \mathcal{L}(X_0)$ be the evolution operator corresponding to (6.5). We set

$$S_m := U_m(T,0) \in \mathcal{L}(X_0), \tag{6.6}$$

i.e. S_m is the *period-map* associated wit (6.5). The stability of the zero solution of (6.5) is measured by means of the spectral radius of S_m (see Corollary 6.4). An essential element of our analysis shall be the following easy consequence of the Phragmèn-Lindelöf principle (cf. [**22**] and [**18**]):

$$S_m \text{ is a } \textit{quasi-strongly positive} \text{ operator,}$$

i.e. if $u_0 \in X_+ \setminus \{0\}$ holds, then $S_m u_0$ is a quasi-interior point. This implies in particular that S_m is an irreducible operator on X_0.

We start with a simple observation.

6.1 Proposition

There is a one to one correspondence between the positive real eigenvalues of S_m andthe real eigenvalues of

$$\begin{cases} \partial_t\varphi - k(t)\Delta\varphi - m(x,t)\varphi = \mu\varphi & \text{in } \mathbb{R}^N \times \mathbb{R} \\ \varphi(\cdot\,,t) \in C_0(\mathbb{R}^N) & \text{for each } t > 0 \\ \varphi \text{ is } T\text{-periodic} \end{cases} \tag{6.7}$$

More precisely, we have that λ is an eigenvalue of S_m with eigenfunction u_0, if and only if $\mu := -\frac{1}{T}\log\lambda$ is an eigenvalue of (6.7) *with eigenfunction given by $\varphi(x,t) := e^{\mu t}[U(t,0)u_0](x)$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.*

Proof

The proof is an easy calculation as the proof of Proposition 14.4 in [**13**]. The needed smoothness of φ is obtained by standard regularity theory. □

In order to exploit the contents of Remark 5.2 (c) we shall need a technical result.

6.2 Lemma

The following estimate holds:

$$\operatorname{spr}_{ess}(S_m) \le e^{-cT}, \tag{6.8}$$

where $c > 0$ is the constant appearing in (6.3)*.*

Before proving this lemma let us remark that if in (6.4) we have that $m \equiv 0$ then, the corresponding period-map is given by $S_0 = e^{\int_0^T k(\tau)d\tau A}$ so that

$$\operatorname{spr}(S_0) = \|S_0\| = 1 \tag{6.9}$$

holds. This is easily seen by the fact that $S_0^n = e^{\int_0^{nT} k(\tau)\,d\tau A}$ and that by Example B-III-1.7 in [**19**] $\|S_0^n\| = 1$.

Proof

(i) Write $m = m_c + m_0$ with

$$m_c(x,t) \le -c \qquad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R} \tag{6.10}$$

and with $m_0(\cdot, t)$ having compact support contained in a common compact set $K \subset \mathbb{R}^N$ for all $t \in \mathbb{R}$. Denote by $M_c(t)$ and $M_0(t)$ the corresponding multiplication operators, then $M(t) = M_0(t) + M_c(t)$. It is easy to see (using the pertinent embeddings for function spaces on bounded domains) that $M_0(t)$ is compact from X_1 into X_0 for each $t \in \mathbb{R}$. Therefore, Theorem 5.2 immediately gives

$$\mathrm{spr}_{ess}(S_m) = \mathrm{spr}_{ess}(S_{m_c}) \tag{6.11}$$

where S_{m_c} is the period-map associated with

$$\dot{u} + k(t)Au = M_c(t)u \qquad \text{for } t > 0. \tag{6.12}$$

(ii) Suppose now that $u_0 \in X_0^+$ and let u be the solution of (6.12). Let v is the solution of

$$\begin{cases} \dot{v} + k(t)Av = -cv & \text{for } t > 0, \\ v(0) = u_0 \end{cases} \tag{6.13}$$

Now, v obviously satisfies

$$0 \le \|v(t)\|_\infty \le e^{-ct}\|u_0\|_\infty \tag{6.14}$$

for all $t \ge 0$ (because of (6.9)). Since $-cv \ge m_c(x,t)v$ in $\mathbb{R}^N \times (0,\infty)$ holds, the comparison principle gives

$$0 \le u(t) \le v(t) \le \|v(t)\|_\infty \tag{6.15}$$

for all $t \ge 0$.

Since any initial value $u_0 \in X_0$ may be decomposed in its positive and negative parts, we obtain from (6.13) and (6.14) that

$$\|u(t)\|_\infty \le e^{-ct}\|u_0\|_\infty$$

holds for all $t \ge 0$ and arbitrary $u_0 \in X_0$. This gives

$$\mathrm{spr}_{ess}(S_{m_c}) \le \mathrm{spr}(S_{m_c}) \le \|S_{m_c}\| \le e^{-cT}$$

which together with (6.11) gives the assertion of the lemma. □

From the result above we may easily obtain a simple stability-instability criterion. But we first make the following

6.3 Remark
Suppose that

$$\dot{u} + A(t)u = 0 \qquad t > 0$$

is an abstract evolution equation in any Banach space X_0 of the type studied in Section 2 but now with $A(t)$ being defined for all $t \in \mathbb{R}$ and depending T-periodically on t. Denote the corresponding evolution operator by $U(t,s)$, $0 \le s \le t < \infty$ and set $S := U(T,0)$. Assume that $\omega_0 \in \mathbb{R}$ is such that $\operatorname{spr}(S) = e^{-T\omega_0}$ holds. Then

$$\omega_0 = \sup\{\omega \in \mathbb{R}; \exists M \ge 1 : \|U(t,s)\| \le Me^{-(t-s)\omega}\, \forall 0 \le s \le t\}$$

holds. In other words ω_0 equals the exponential growth bound of the evolution operator. For a proof of these facts we refer to [**10**], Section 6.B. Note that we call ω_0 the exponential growth bound of U and not $-\omega_0$ which would perhaps seem more natural. □

6.4 Corollary
If $\operatorname{spr}(S_m) < 1$ *then the zero solution of* (6.5) *is exponentially stable. If* $\operatorname{spr}(S_m) = 1$ *then it is stable but not exponentially stable. Finally, if* $\operatorname{spr}(S_m) > 1$ *then it is unstable.*

Proof
Set $r := \operatorname{spr}(S_m)$.

That the zero solution is exponentially stable if and only if $r < 1$ is an immediate consequence of the previous remark.

If $r = 1$ we get by Remark 5.1(c) and the above lemma that the peripheral spectrum $\sigma_{per}(S_m)$ consists entirely of isolated simple eigenvalues. Taking the spectral projection with respect to the peripheral spectrum we may write

$$X = X_s \oplus X_{per} \quad \text{and} \quad S_m = S_{m,s} \oplus S_{m,per}$$

in obvious notation.

Now, it is evident that $\|S_{m,s}^n\|$ is uniformly bounded in $n \ge 0$, since $\operatorname{spr}(S_{m,s}) < 1$. Furthermore, $\|S_{m,per}^n\|$ is also uniformly bounded in $n \ge 0$ since $S_{m,per}$ may be represented by a diagonal matrix with entries of modulus 1. These two observations give the stability of the zero solution.

Finally, if $r > 1$ we have, again by Remark 5.1(c), that it is a (simple) eigenvalue of S_m. Take a corresponding eigenvector u_0. Then we get

$$\|S_m^n u_0\| = r^n \|u_0\|$$

for all $n \geq 0$. This yields the instability of the zero solution. □

Another consequence of Lemma 6.2 is the following result on the eigenvalue problem (6.1). We set

$$\mu(m) := -\frac{1}{T}\log \operatorname{spr}(S_m). \tag{6.16}$$

6.5 Proposition
If $\operatorname{spr}(S_m) \geq 1$ *holds, there exists a unique principal eigenvalue* $\mu_1(m)$ *of* (6.1)*, which is given by* $\mu_1(m) = \mu(m)$*. Moreover, the corresponding principal eigenfunction is the only eigenfunction (up to linear dependence) and is everywhere positive.*

Proof
The result is an immediate consequence of Lemma 6.2, Remark 5.1(c) and Proposition 6.1. □

In what follows we shall make use of the theory of periodic-parabolic eigenvalue problems on bounded domains as contained in [**13**]. For each $R > 0$ let $\mu(m, R)$ denote the principal eigenvalue and $\varphi_{m,R}$ a corresponding principal eigenfunction of the following periodic-parabolic eigenvalue problem in the bounded domain $\mathbb{B}(0, R)$:

$$\begin{cases} \partial_t\varphi - k(t)\Delta\varphi - m(x,t)\varphi = \mu\varphi & \text{in } \mathbb{B}(0,R)\times\mathbb{R} \\ \varphi = 0 & \text{on } \partial\mathbb{B}(0,R)\times\mathbb{R} \\ \varphi \text{ is } T\text{-periodic and positive} & \end{cases} \tag{6.17}$$

We first establish the following

6.6 Lemma
The function $R \mapsto \mu(m, R)$ *is strictly decreasing and*

$$\mu_\infty(m) := \lim_{R\nearrow\infty} \mu(m, R) \tag{6.18}$$

is finite.

Proof
We first prove the strict monotonicity. Assume to the contrary that there exist $0 < R_1 < R_2$ such that

$$\mu(R_1) \leq \mu(R_2), \tag{6.19}$$

and let φ_1 and φ_2 be the corresponding eigenfunctions, i.e.

$$\begin{cases} \partial_t\varphi_i - k(t)\Delta\varphi_i - m(x,t)\varphi_i = \mu(R_i)\varphi_i & \text{in } \mathbb{B}(0,R_i)\times\mathbb{R} \\ \varphi_i = 0 & \text{on } \partial\mathbb{B}(0,R_i)\times\mathbb{R} \\ \varphi_i \text{ is } T\text{-periodic and positive} & \end{cases} \tag{6.20}$$

holds for $i = 1, 2$.

It is easy to see that we may assume without loss of generality that

$$\varphi_2(x,t) > \varphi_1(x,t)$$

holds for all $(x,t) \in \mathbb{B}(0, R_1) \times \mathbb{R}$. Now, let $\sigma > 1$ be such that

$$\varphi_2(x,0) - \sigma\varphi_1(x,0) \geq 0 \tag{6.21}$$

holds for all $x \in \bar{\mathbb{B}}(0, R_1)$, with equality holding for some $x \in \mathbb{B}(0, R_1)$. Setting

$$\varphi := \varphi_2 - \sigma\varphi_1$$

and using (6.19), (6.20) and (6.21) we obtain

$$\begin{cases} \partial_t\varphi - k(t)\Delta\varphi - m(x,t)\varphi \geq \mu(R_1)\varphi & \text{in } \mathbb{B}(0,R_1) \times \mathbb{R} \\ \varphi > 0 & \text{on } \partial\mathbb{B}(0,R_1) \times \mathbb{R}. \end{cases}$$

But now the maximum principle and the periodicity of φ give

$$\varphi(x,t) > 0$$

for all $(x,t) \in \mathbb{B}(0, R_1) \times \mathbb{R}$, contradicting the choice of σ. Hence, the strict monotonicity is proved.

To prove that the limit in (6.18) is finite we just have to observe that by Lemma 15.6 in [**13**] we have

$$\mu(R) \geq -\frac{1}{T}\mathcal{P}_R(m)$$

where

$$\mathcal{P}_R(m) := \int_0^T \max_{\|x\| \leq R} m(x,\tau)d\tau$$

is bounded from above by assumption (6.3). □

We owe the above proof to G. Schätti. The next result is useful in applications.

6.7 Lemma
We have that

$$\mu(m) \leq \mu_\infty(m) \tag{6.22}$$

holds.

Proof
Let $u_0 \in X_0^+$ and $R > 0$ be arbitrary. In what follows we shall not bother to distinguish (notationally) between a function on $\mathbb{R}^N$ and its restriction to any subset. Let u be the solution of (6.4) with initial data $u(0) = u_0$. It is easy to see that u is a supersolution for

$$\left\{\begin{aligned} \partial_t v - k(t)\Delta v = m(x,t)v \qquad & \text{in } \mathbb{B}(0,R) \times \mathbb{R} \\ v = 0 \qquad & \text{on } \partial\mathbb{B}(0,R) \times \mathbb{R} \\ v(\cdot, 0) = u_0 \qquad & \text{in } \mathbb{B}(0,R) \end{aligned}\right. \tag{6.23}$$

Hence, if v is the solution of (6.23) we obtain from the comparison principle

$$0 \leq v(t) \leq u(t) \qquad \text{in } \mathbb{R}^N$$

for all $t \geq 0$. Therefore, u cannot decay more rapidly then v. Thus, we obtain from Remark 6.3 that $\mu(m) \leq \mu(m, R)$. The assertion follows by the previous lemma. □

Though certainly desirable, it does not seem to be clear whether equality holds in (6.22) or not. The above result gives a possibility to obtain estimates from below for $\operatorname{spr}(S_m)$ whenever one has an estimate from above for $\mu(m, R)$ for some $R > 0$. We may formulate this loosely in the following way: the instability of the zero solution of

$$\left\{\begin{aligned} \partial_t u - k(t)\Delta u = m(x,t)u \qquad & \text{in } \mathbb{B}(0,R) \times \mathbb{R} \\ u = 0 \qquad & \text{on } \partial\mathbb{B}(0,R) \times \mathbb{R} \end{aligned}\right. \tag{6.24}$$

for some $R > 0$, forces the instability of the zero solution of (6.4).

We close this section with two simple technical observations.

6.8 Remark

(a) If m_1 and m_2 are two functions satisfying (6.3) and $m_1 < m_2$, then it is an easy consequence of the Phragmèn-Lindelöf principle that $S_{m_1} u_0 \ll S_{m_2} u_0$ for all $u_0 \in X_0$. From this it follows that

$$\operatorname{spr}(S_{m_1}) \leq \operatorname{spr}(S_{m_2}). \tag{6.25}$$

Moreover, if we know that $\operatorname{spr}(S_{m_1})$ and $\operatorname{spr}(S_{m_2})$ are eigenvalues of S_{m_1} and S_{m_2} respectively, it is possible to show that strict inequality holds in (6.25) (compare Lemma 15.5 in [**13**]). Note that in general, equality may occur.

(b) Suppose now that (m_n) is a sequence of functions satisfying (6.3) and that

$$m_n \to m \qquad \text{in } BUC(\mathbb{R}^N \times \mathbb{R})$$

as $n \to \infty$. Then,

$$\operatorname{spr}(S_{m_n}) \to \operatorname{spr}(S_m)$$

as n tends to infinity. Note that we do not use that the spectral radii are eigenvalues. The proof is quite similar to that of Lemma 15.7 in [**13**].

Proof
By assumption, for any $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$m - \varepsilon < m_n < m + \varepsilon$$

holds for all $n \geq n_0$. Using (6.25) we get that

$$e^{-\varepsilon T} \operatorname{spr}(S_m) = \operatorname{spr}(S_{m-\varepsilon}) \leq \operatorname{spr}(S_{m_n}) \leq \operatorname{spr}(S_{m+\varepsilon}) = e^{\varepsilon T} \operatorname{spr}(S_m)$$

for all $n \geq n_0$. Hence, the assertion follows. □

7. Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1 and Proposition 1.2. So we shall be interested in the stability properties of the parameter dependent equation

$$(7.1)_\lambda \qquad \partial_t u - k(t)\Delta u = \lambda m(x,t)u \qquad \text{in } \mathbb{R}^N \times (0,\infty),$$

as explained in the introduction. On the coefficients we impose assumptions (6.2) and (6.3). Furthermore, we assume that m changes sign. More precisely, we assume that (1.5) holds, i.e.

$$(7.2) \qquad \mathcal{P}(m) := \int_0^T \max_{x \in \mathbb{R}^N} m(x,\tau)\, d\tau > 0$$

The parameter λ shall vary over the nonnegative reals. Recall that we had reformulated (7.1) as an abstract evolution equation

$$(7.3) \qquad \dot{u} + k(t)Au = \lambda M(t)u \qquad \text{for } t > 0$$

on the space $X_0 = C_0(\mathbb{R}^N)$.

For each $\lambda \in \mathbb{R}^+$ let $U_\lambda \colon \Delta_T \to \mathcal{L}(X_0)$ be the evolution operator corresponding to (7.3). We set

$$(7.4) \qquad S_\lambda := U_\lambda(T,0) \in \mathcal{L}(X_0),$$

i.e. S_λ is the *period-map* associated with (7.3).

One of the important ingredients in the proof of Theorem 1.1 is the following consequence of the results of Section 4. We set

$$r(\lambda) := \mathrm{spr}(S_\lambda). \tag{7.5}$$

7.1 Lemma

(i) The function r is log-convex, i.e. $[\lambda \mapsto \log r(\lambda)]$ is convex. In particular, $r(\cdot)$ is convex and, therefore, continuous.

(ii) If r is not identically 1, then there is at most one $\lambda_1 > 0$ such that $r(\lambda_1) = 1$. In this case $r(\lambda) < 1$ if $0 < \lambda < \lambda_1$, and $r(\lambda) > 1$ if $\lambda > \lambda_1$.

Proof

Assertion (i) is an immediate consequence of Example 4.2, Remarks 4.4(a) and Corollary 4.5.

To prove (ii) we start by noting that by Theorem 3.2 the map $[\lambda \mapsto S_\lambda]$ is analytic. Since for any $\lambda_1 > 0$ with $r(\lambda_1) = 1$ we have that $r(\lambda_1)$ is an isolated simple eigenvalue of S_{λ_1}, we find (by analytic perturbation theory, cf. [**15**]) that r is analytic in a neighbourhood of λ_1. This together with the log–convexity of r and the fact that $r(0) = 1$ easily yields the assertion. □

The above lemma allows four qualitatively different possibilities for the graph of r which are depicted in the following figure:

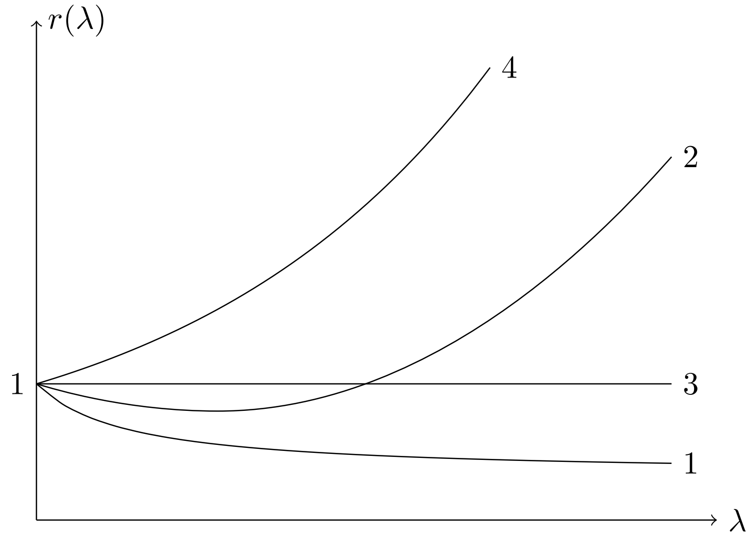

Fig. 1

In the following two lemmas we shall eliminate all possibilities but the second. We start by eliminating possibilities 1 and 3.

7.2 Lemma
We have that $r(\lambda) \to \infty$ *as* $\lambda \nearrow \infty$.

Proof
We use the notation at the end of Section 6. From assumption (7.2) we get that

$$\mathcal{P}_R(m) := \int_0^T \max_{\|x\| \le R} m(x,\tau)\, d\tau \; > 0$$

for some $R > 0$. By Lemma 15.4 in [**13**] we get

$$\mu(\lambda m, R) \to -\infty \qquad \text{as } \lambda \nearrow \infty. \tag{7.6}$$

Since, by Lemmas 6.5 and 6.6,

$$\mu(\lambda m) \le \mu_\infty(\lambda m) \le \mu(\lambda m, R)$$

holds, we get from (7.6) that

$$r(\lambda) = e^{-\mu(\lambda m)T} \to \infty \qquad \text{as } \lambda \nearrow \infty,$$

proving the Lemma. □

We now eliminate possibility 4 in Figure 1.

7.3 Lemma
For small $\lambda > 0$ *we have that* $r(\lambda) < 1$.

Proof
Define

$$\bar{m}(x) := \max_{0 \le t \le T} k(t)^{-1} m(x,t),$$

for $x \in \mathbb{R}^N$. Then, by (6.3), we obviously have $\bar{m}(x) \le -c$ for all $x \in \mathbb{R}^N$ with $|x| \ge R_0$. Here, the c is not necessarily the same than in (6.3). Furthermore, by (7.2), there exists an $x_0 \in \mathbb{R}^N$ such that $\bar{m}(x_0) > 0$. Let u be the solution of (6.4) with $u(0) = u_0 \in X_0^+$ and set

$$v(t) := u\Big(\int_0^t \frac{1}{k(\tau)}\, d\tau\Big), \qquad t \ge 0.$$

A simple calculation gives

$$\dot{v}(t) + Av(t) \le \lambda \bar{M} v(t) \qquad t > 0,$$

where $\bar{M}$ is the multiplication operator on X_0 induced by $\bar{m}$.

Hence, if w is the solution of

$$\dot{w} + Aw = \lambda \bar{M} w, \qquad t > 0, \tag{7.7}$$

with $w(0) = u_0$, the comparison principle gives

$$0 \leq v(t) \leq w(t) \qquad t \geq 0. \tag{7.8}$$

By Theorem 2.1 in [**8**] and standard regularity theory, there exists a $\bar{\lambda}_1 >$ such that $\mathrm{spr}(\bar{S}_{\bar{\lambda}_1}) = 1$, where $\bar{S}_\lambda$ is the period-map corresponding to (7.7). Combining this with the fact that – since (7.7) satisfies the same assumptions as (7.3) (being additionally autonomous) – we may apply Lemmas 7.1 and 7.2, we obtain that for $0 < \lambda < \bar{\lambda}_1$ the zero solution of (7.7) is exponentially stable. But from (7.8) we infer that for $0 < \lambda < \bar{\lambda}_1$,

$$\|v(t)\| \leq \|w(t)\| \leq Me^{-t\omega}\|u_0\|, \qquad t \geq 0 \tag{7.9}$$

with suitable constants $M, \omega > 0$. From (7.9) and the definition of v we obtain a similar estimate for u. This, and the fact that we may decompose any initial value $u_0 \in X_0$ in its positive and negative parts, finally give the exponential stability of the zero solution of (7.3) for all $0 < \lambda < \bar{\lambda}_1$ proving the lemma. □

We were not able to prove Lemma 6.5 without resorting to the result in [**8**]. It would be nice to obtain the result directly not having to make a detour via the elliptic problem.

Putting together Lemmas 7.1 to 7.3 we see that the function $r(\cdot)$ starts at $r(0) = 1$, immediately goes down, below the line '$r = 1$', and then goes up to infinity crossing the line '$r = 1$' only once. In other words, there exists only one $\lambda_1 > 0$ such that $r(\lambda_1) = 1$. Moreover, $r(\lambda) < 1$ whenever $0 < \lambda < \lambda_1$, and $r(\lambda) > 1$ whenever $\lambda > \lambda_1$. This together with Corollary 6.2 proves Theorem 1.1.

Proposition 1.2 is now a simple consequence of (6.4), Lemma 7.1 and Theorem 1.1.

7.4 Remark
Note that the only instance where assumption (7.2) was needed was in the proof on Lemma 7.2. The other two lemmas remain also valid if we only assume (6.2) and (6.3). □

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