Uniform Continuity of Continuous Functions on Compact Metric Spaces

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A basic theorem asserts that a continuous function on a compact metric space with values in another metric space is uniformly continuous. The usual proofs based on a contradiction argument involving sequences or on the covering property of compact sets are quite sophisticated for students taking a first course on real analysis. We present a direct proof only using results that are established anyway in such an introductory course.

Let \( f : X \to Y \) be a continuous function from the compact metric space \((X,d_X)\) into the metric space \((Y,d_Y)\). The function \( F : X \times X \to \mathbb{R} \) given by

\[
F(x,y) : = d_Y(f(x),f(y))
\]

is continuous with respect to the product metric on \( X \times X \). Fix \( \varepsilon > 0 \) and consider the inverse image

\[ A_\varepsilon := F^{-1}([\varepsilon, \infty]) := \{(x,y) \in X \times X : F(x,y) \geq \varepsilon\}. \]

As \( F \) is continuous and \([\varepsilon, \infty)\) is closed, \( A_\varepsilon \) is a closed subset of the compact metric space \( X \times X \). Hence, \( A_\varepsilon \) is compact. Assume that \( A_\varepsilon \neq \emptyset \). The real valued function \((x,y) \mapsto d_X(x,y)\) is continuous on \( X \times X \) and hence has a minimum on the compact set \( A_\varepsilon \). Thus, there exists \((x_0,y_0) \in A_\varepsilon\) such that

\[
\delta := d(x_0,y_0) \leq d(x,y)
\]

for all \((x,y) \in A_\varepsilon\). As \((x_0,y_0) \in A_\varepsilon\) we have \( \delta > 0 \) as otherwise \( x_0 = y_0 \) and hence \( 0 = F(x_0,y_0) \geq \varepsilon > 0 \). Moreover, if \( d_X(x,y) < \delta \), then \((x,y)\) is in the complement of \( A_\varepsilon \), and therefore

\[
d_X(x,y) < \delta \implies d_Y(f(x),f(y)) = F(x,y) < \varepsilon.
\]

This is exactly what is required for uniform continuity. If \( A_\varepsilon = \emptyset \), then (1) holds for every \( \delta > 0 \). As the arguments work for every choice of \( \varepsilon > 0 \) this proves the uniform continuity of \( f \).

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