1. Simplified: use the coordinate-system invariant form
   \[ u = \nabla \times (0, 0, \psi(x, y)) = \nabla \times (0, 0, \psi(r, \theta)) \]
   in plane polar coordinates with \( z \)-axis pointing out of paper. This is equivalent to cylindrical coordinates \((r, \theta, z)\) so use the formula for curl in cylindrical polar to give
   \[ u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}. \]

   Alternatively, to do from first principles, convert between coordinate systems and use the chain rule to express derivatives wrt \( x \) and \( y \) in terms of derivatives wrt \( r \) and \( \theta \).

   From the diagram:
   \[ u_r = u_x \cos \theta + u_y \sin \theta, \quad u_\theta = u_y \cos \theta - u_x \sin \theta \]
   \[ u_x = \frac{\partial u_r}{\partial r} \cos \theta - \frac{\partial u_\theta}{\partial \theta} \sin \theta \]
   \[ u_y = \frac{\partial u_r}{\partial \theta} \sin \theta + \frac{\partial u_\theta}{\partial r} \cos \theta \]
   \[ r^2 = x^2 + y^2 \]
   \[ \tan \theta = \frac{y}{x} \]
   \[ \theta = \tan^{-1} \frac{y}{x} \]
   \[ \frac{\partial}{\partial x} = \frac{\partial}{\partial r} \cos \theta - \frac{\partial}{\partial \theta} \sin \theta \]
   \[ \frac{\partial}{\partial y} = \frac{\partial}{\partial r} \sin \theta + \frac{\partial}{\partial \theta} \cos \theta \]

   \[ u_r = \left( \frac{\partial \psi}{\partial x} \cos \theta + \frac{\partial \psi}{\partial y} \sin \theta \right) \cos \theta + \left( \frac{\partial \psi}{\partial r} \frac{\partial \theta}{\partial r} \sin \theta + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial r} \cos \theta \right) \sin \theta \]
   \[ u_\theta = \left( \frac{\partial \psi}{\partial x} \cos \theta + \frac{\partial \psi}{\partial y} \sin \theta \right) \sin \theta - \left( \frac{\partial \psi}{\partial r} \frac{\partial \theta}{\partial r} \cos \theta + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial r} \sin \theta \right) \cos \theta \]

   This gives \( u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \), \( u_\theta = -\frac{\partial \psi}{\partial r} \).
2. The Stokes Stream Function is defined by

\[ u_1 = -\frac{1}{R} \frac{\partial \Psi}{\partial \theta}, \quad u_2 = \frac{1}{R} \frac{\partial \Psi}{\partial \phi} \]

Integrating the second one,

\[ \Psi(R, \phi) = \int_0^R \delta u_2(\delta, \phi) \, d\delta + F(\phi) \]

where \( F(\phi) \) is (arbitrarily) adjustable.

Differentiate this w.r.t \( \phi \),
\[ \frac{\partial \Psi}{\partial \phi} = \int_0^R \frac{\partial}{\partial \phi} \delta u_2(\delta, \phi) \, d\delta + F'(\phi) \]

But from \( \nabla \cdot \mathbf{u} = 0 \) in cylindrical polar coordinates,
\[ \frac{\partial}{\partial \phi} \delta u_2(\delta, \phi) = - \frac{1}{\delta} \frac{\partial}{\partial \delta} (\delta u_2(\delta, \phi)) \]

Putting this into the line above, we can do the resulting integral to give
\[ \frac{\partial \Psi}{\partial \phi} = - \int_0^R \delta u_2(\delta, \phi) \, d\delta + F'(\phi) \]

But \( u_2(R, \phi) = -\frac{1}{R} \frac{\partial \Psi}{\partial \phi} \), so in fact \( F'(\phi) = -u_2(R, \phi) \)

Thus \( F(\phi) = -\int_0^R u_2(\delta, \phi) \, d\delta \)

\[ \Rightarrow \Psi(R, \phi) = \int_0^R \delta u_2(\delta, \phi) \, d\delta - \int_0^b a u_R(a, \eta) \, d\eta \]

is the sought-after answer.

Interpretation: \( 2 \pi \Psi \) is the upward-directed \( z \)-velocity source flux across the shaded area indicated on the diagram. \( 2 \pi \Psi \) (including the -sign) is the inward-directed \( u_2 \) flux through the shaded cylindrical surface.

By continuity, \( 2 \pi \Psi \) thus represents the volume flux through any inscribed surface (eg 2) linking \( (a,b) \) and \( (R, \phi) \).
3. In cylindrical polar coordinates \( u = V_x(0, y(R, \phi), \phi) \).

This is for coordinates \((R, \phi, z)\) in spherical polar coordinates \((r, \theta, \phi)\). The coordinate \( \phi \) plays the same role but is in a different place. Since definitions of vector relations like \( \mathbf{v} = \nabla \times \mathbf{a} \) are true in any coordinate system, we can say that \( u = V_x(0, 0, \frac{\phi(R, \phi)}{R}) \) in spherical polar \((r, \theta, \phi)\). But \( R = \frac{\sin \theta}{r} \), and \( z = \frac{\cos \theta}{r} \), so in spherical polar \( u = V_x(0, 0, \frac{\phi(R, \phi)}{R}) \).

Using the formula for curl in spherical polar \( \nabla \times \mathbf{u} \) gives immediately
\[
u = \left( \frac{1}{r \sin \theta}, \frac{1}{r \sin \theta}, 0 \right) \text{ where now}
\]
\[u = (u_r, u_\theta, u_\phi) \text{ in spherical polar.}
\]
(This can also be done from first principles using the chain rule as in Q.1 ... it's a lengthy calculation.)

4. \( u = U \beta x^{-1/2} \sech^2(\alpha y x^{-1/2}) \). 

\[\frac{\partial u}{\partial y} = \psi(y) = \int u(x, y) \, dy + \text{some function of } x. \]

The jet is symmetric in \( y \) for \( u \) (and, as we shall see, antisymmetric for \( \nu \)). So we can choose the streamline on the axis of symmetry \( y = 0 \) to be \( \psi = 0 \), and then the function of \( x \) is zero.

\[\psi(x, y) = \int U \beta x^{-1/2} \sech^2(\alpha y x^{-1/2}) \, dy \]
But \[\frac{1}{\alpha} \tanh(\alpha y) = A \sech(\alpha y), \]
so this integral just gives
\[\psi(x, y) = x^{1/2} U \beta \tanh(\alpha y x^{-1/2}) \]
The \( y \)-component, \( \nu(x, y) \), is \[-\frac{\partial \psi}{\partial x}, \text{ which equals} \]
\[-\frac{U \beta}{\alpha} x^{-1/2} \tanh(\alpha y x^{-1/2}) + \frac{1}{\alpha} x^{-1/2} U \beta \sech(\alpha y x^{-1/2}) \]

[Note this is indeed antisymmetric about \( y = 0 \).]
5. \( \psi = r U \left( \cos \theta + \frac{\pi}{2} \theta \sin \theta - \sin \theta \right) \)

In plane polar coordinates, \( u = \left( -\frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r} \right) \)

\[
\begin{align*}
  u_r &= \frac{1}{r} \frac{\partial \psi}{\partial r} = U \left( \cos \theta - \theta \sin \theta + \frac{\pi}{2} \sin \theta \right) \\
  u_\theta &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} = U \left( \frac{\pi}{2} \left( \sin \theta + \theta \cos \theta \right) - \theta \sin \theta \right)
\end{align*}
\]

When \( \theta = 0 \), \( u_r = 0 \) and \( u_\theta = 0 \).

When \( \theta = \frac{\pi}{2} \), \( u_r = 0 \) and \( u_\theta = U \left( 1 - \frac{\pi}{4} \right) \).

Note that this is negative (i.e., \( u_\theta \) is in the direction of \( \theta \) decreasing) if \( U > 0 \).

This could model paint being scraped along a wall \( \theta = 0 \) by a vertical scraper \( \theta = \frac{\pi}{2} \), subject to boundary conditions of no relative slip.

\( \theta \) is direction of decreasing factor \( u_\theta = 0, \) \( u_r = 0 \) at instantaneuos snapshot with origin at instantaneous base of scraper

\[
\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}
\]

\[
= \frac{1}{r} \frac{\partial}{\partial r} (-ru_\theta) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} (ur) \quad \text{(to avoid recalculation)}
\]

\[
= \frac{U}{r} \left[ \frac{\pi}{2} \cos \theta + \frac{\pi}{2} \theta \sin \theta - \sin \theta + \frac{\pi}{2} \left( 2 \cos \theta - \theta \sin \theta \right) - \sin \theta - \theta \cos \theta \right]
\]

\[
= \frac{2U}{r} \left( \frac{\pi}{2} \cos \theta - \theta \sin \theta \right). \quad \text{To find } \nabla^4 \psi, \text{ take } \nabla^2 \text{ of this.}
\]

\[
\nabla^2 (\cos \theta + \theta \sin \theta) = \left[ \cos \theta \text{ or } \sin \theta \right] \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} (\frac{1}{r} \psi) \right) + \frac{\partial}{\partial \theta} \left[ \cos \theta \text{ or } \sin \theta \right] \frac{\partial}{\partial \theta} (\psi)
\]

and for both terms this gives zero.

Hence, \( \nabla^4 \psi = 0 \) (this is actually the equation for highly viscous fluid in this case).
6. \[ \frac{\partial^2 u}{\partial x^2} = 0 \] can be met; also

\[ \nabla^2 \phi = 0 \] in \( z > 0 \) except at the source singularity itself (see lectures).

(i) 2-D version: The radial outward velocity of a line source at the origin is \( \frac{S}{2\pi r} \). Thus the net outward velocity on the plane due to the source and its image is (see diagram)

\[ \frac{S}{2\pi L} \cos \phi \]

Hence the velocity out along the plane is zero if \( x = 0 \) and \( x \to \pm \infty \), and reaches a maximum in between where \( \cos \frac{\pi x}{L} = 0 \)

i.e. where \( x = \frac{L}{2} \), i.e. at \( x = \pm h \)

(ii) 3-D version: this is just the same, except that now the outward velocity of a line source is \( \frac{S}{4\pi r^2} \), the \( r \) now referring to spherical polars.

So the resultant outward velocity on the plane a distance \( R \) away from the point on the plane which is under the source is

\[ \frac{S}{4\pi (R^2 + x^2)} \cos \phi = \frac{S}{4\pi (R^2 + h^2)} \frac{R}{2\pi (R^2 + h^2)^{3/2}} \]

This has \( d\phi = 0 \)

\[ \frac{1}{R^2} - \frac{3h^2}{(R^2 + h^2)^{3/2}} = 0 \], i.e. \( R^2 - 2h^2 = 0 \) \( R = \frac{h}{\sqrt{2}} \)
In spherical polar, the Stokes stream function for a source \( S \) at the origin is
\[
\Psi = -\frac{Sc \cos \theta}{4\pi}
\]
(see lectures). In cylindrical polar
\((R, \phi, z)\) (which are the natural coordinates to use for this problem)
\[
\Psi = -\frac{Sz}{4\pi R} \cos \theta = \frac{z}{R} \sqrt{R^2 + z^2}
\]
so \( S = -\frac{z}{R} \sqrt{R^2 + z^2} \).

For a source \( +S \) at \( z = -a \), translating the origin to \( z = 0 \), gives \( z \to z + a \); for the \(-S\) at \( z = a \), \( z \to z - a \). For the uniform stream, \( \Psi = \frac{1}{2} UR^2 \) (see lectures).

These flows are all irrotational and so we can superpose these three elements to write the total stream function for the solution as
\[
\Psi(R, z) = \frac{1}{4\pi} UR^2 - \frac{S}{2} \left\{ \frac{z + a}{\sqrt{(R^2 + (z + a)^2)}} - \frac{z - a}{\sqrt{(R^2 + (z - a)^2)}} \right\}
\]

These are two stagnation points, which are on \( R = 0 \), one to the left of both sources, and one to the right. From the formula for \( \Psi \), it can be seen that \( \Psi = 0 \) there.

At \( A \) and \( B \), the \( \Psi = 0 \) streamline bifurcates and forms a solid of revolution (see the diagram). The equation determining this surface is thus \( \Psi = 0 \), i.e.
\[
UR^2 = \frac{S}{2\pi} \left( \frac{z + a}{\sqrt{R^2 + (z + a)^2}} - \frac{z - a}{\sqrt{R^2 + (z - a)^2}} \right)
\]

A body of this shape could be placed in a uniform stream and the above \( \Psi(R, z) \) would equivalently give the flow around the outside of it, with the two sources now being regarded as artificial "images" used to solve the problem. The \( \Psi = 0 \) streamline gives the shape of this "Roulette Cuboid."
The two stagnation points are symmetric about \( \lambda \); so the distance \( OB \) gives the half-length of the body. At \( B \), \( z = \lambda \) say, and the three parts of the velocity field sum up to zero, i.e., \( U + S = 0 \) \( \frac{4\pi(\lambda+a)^2}{4\pi(\lambda-a)^2} \).

So the half-length \( \lambda \) is given by \( U(\lambda^2-a^2) = \frac{S}{4\pi} \left( (\lambda+a)^2-(\lambda-a)^2 \right) \), i.e., \( (\lambda^2-a^2) = \frac{Sa}{TTU} \).

Graphically this can be seen to have one root greater than \( a \):

\[ y = \frac{Sa}{TTU} \]

By symmetry, the radius of the body is greatest where \( z = 0 \) on \( \phi = 0 \).

This happens \( R = R_{\text{max}} \) where \( UR_{\text{max}}^2 = \frac{Sa}{TT} \frac{1}{\sqrt{R_{\text{max}}^2+a^2}} \).

Letting \( a \to 0 \) with \( Sa \) fixed, we expand the stream function \( \psi \) \( R, z \) of the flow using the binomial theorem:

\[ \psi(R, z) = \frac{1}{2} UR^2 = \frac{S}{4\pi(2)R^2} \left[ (z+\alpha)^2 - (z-\alpha)^2 \right] \frac{(1+\alpha^2+\ldots)}{R^2+z^2} \]

\[ = \frac{1}{2} UR^2 - \frac{SaR^2}{2\pi(R+a)^{3/2}} + \text{terms of order } a^2. \]

Returning to spherical polar for which \( R^2+z^2 = R^2 \)

\( R = r \sin \theta \), we see \( \psi(r, \theta) = \frac{1}{2} r^2 \sin^2 \theta \left( U^2 - \frac{Sa}{TT} \right) \).

With experience, one can see (cf. the notes) that this is a superposition of a uniform stream and a dipole. The separation streamline \( \psi = 0 \) is now \( U^2 = \frac{Sa}{TT} \) i.e., \( \rho = \frac{(Sa)^{1/2}}{U} \), a sphere. So in this case the solution models the integral flow past a sphere.

(In reality no flow “separated” at the rear of such bodies.]
8. Since \( \mathbf{u} \) is the same as \( \frac{\partial \mathbf{u}}{\partial t} \) (rate-of-change of position following a fluid particle), the stated relation gives \( \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \cdot \mathbf{f} + \mathbf{u} \times \mathbf{\Omega} \times \mathbf{r} \). Only the time derivatives differ between the inertial and rotating frames; the spatial derivatives coincide.

Navier-Stokes is \( \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \rho \mathbf{f} + \mu \nabla^2 \mathbf{u} \).

Transferring to the rotating frame gives
\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r}) \right) \cdot (\mathbf{u} \times \mathbf{r}) = -\nabla p + \rho \mathbf{f} + \rho \mathbf{u} \cdot \nabla \mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r}) \cdot (\mathbf{u} \times \mathbf{r})
\]

Now
\[
\frac{\partial}{\partial t} (\mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r})) = \mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r}) + \frac{\partial}{\partial t} (\mathbf{u} \times \mathbf{r}) = \mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r}) + \mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r}) \times \mathbf{r} = \mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r})
\]

and it is easy to see that \( \nabla^2 (\mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r})) = \mathbf{\Omega} \times \nabla^2 (\mathbf{u} \times \mathbf{r}) = 0 \).

Thus
\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r}) \right) \cdot (\mathbf{u} \times \mathbf{r}) = -\nabla p + \rho \mathbf{f} + \rho \mathbf{u} \cdot \nabla \mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r}) \cdot (\mathbf{u} \times \mathbf{r})
\]

9. \( \nabla p = \nabla \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial \mathbf{r}} \). If this is to do what is claimed, we must show that
\[
\frac{\partial}{\partial \mathbf{r}} (\mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r})) = -\frac{1}{2} \nabla (\mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r})) \] (bearing in mind that the density is being taken to be constant).

This is a relation between vectors, so we are free to prove it in any coordinate system we like. So take Cartesian coordinates with \( \mathbf{r} \) along the \( z \)-axis and \( \mathbf{r} = (x, y, z) \). Then \( \mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r}) = (-\Omega y, \Omega x, 0) \)
and \( \frac{\partial}{\partial \mathbf{r}} (\mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r})) = (\Omega x, \Omega y, 0) \).

\( \mathbf{\Omega} \times (\mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r})) = (0, 0, 0) \times (-\Omega y, \Omega x, 0) = (-\Omega x, -\Omega y, 0) \)

Thus \( \mathbf{\Omega} \times (\mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r})) = -\frac{1}{2} \nabla (\mathbf{\Omega} \times (\mathbf{u} \times \mathbf{r})) \) in this coordinate system, and hence in any others: the modified pressure \( p \) does indeed do the job stated.

[It is also possible, and a good exercise, to do this in suffix notation]
10. With the specified approximations taken into account, the Navier-Stokes equation in the rotating frame becomes
\[ 2 \nabla \times \mathbf{u} = -\nabla P. \]

With \( P \) constant, this implies \( \nabla \times (2 \nabla \times \mathbf{u}) = 0. \)

Now this is \( \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left( \mathbf{e}_i \cdot \nabla \mathbf{u} \right) = 0. \)

Thus \( \mathbf{u} \) is incompressible: hence \( \nabla \cdot \mathbf{u} = 0 \).

This says (amazingly!) that none of the components of velocity can change along the direction of the rotation axis. This effect has been observed experimentally; the resulting structures orientated along the rotation axis are called “Taylor columns”. Flows of this type are called “geostrophic” in meteorology.

Note from (6) that \( \mathbf{u} \cdot \nabla P = 0 \), i.e. \( \mathbf{u} \) lies directed along surfaces of constant modified pressure. This is why on a weather map the wind goes along the isobars (in eg a depression or a ‘high’), and not down the pressure gradient, as one might naively have expected.

11. With incompressibility, \( \nabla \cdot \mathbf{u} = 0 \); in spherical polar coordinates, this says \( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 u(r, \theta) \right) = 0. \)

This integrates to give \( r^2 u(r, \theta) = f(\theta) \).

Besides this, the only physical quantities in the problem are the energy \( E \) and the density \( \rho. \)
1. Suppose \( f(t) \) has the form \( KE^\alpha t^\beta \), where \( K \) is dimensionless. Then dimensionally, we need
\[
[LT^2LT^{-1}] = [ML^2T^{-2}]^\alpha [ML^0]^\beta [T^0]
\]
Equating powers of \( L \): \( 3 = 2\alpha - 3\beta \)
of \( T \): \( -1 = -2\alpha + \beta \) \( \Rightarrow \) \( \alpha = \frac{3}{5}, \beta = -\frac{3}{5} \)
This gives \( u = K \left( \frac{E}{R} \right)^{3/5} t^{3/5} \)
At \( t = R \), the velocity \( u \) had to be \( \frac{dR}{dt} \).
\[
\frac{dR}{dt} = K \left( \frac{E}{R} \right)^{3/5} t^{3/5} \frac{R^2}{R^2}
\]
Integrating, and using \( R = 0 \) when \( t = 0 \), gives \( R(t) = \frac{5K}{3} \left( \frac{E}{R} \right)^{3/5} t^{3/5} \)

12. Easiest is to start from \( \nabla \times (\nabla \times \mathbf{u}) \); this is
\[
\varepsilon_{ijk} \frac{\partial^2 \mathbf{u}_m}{\partial x_i \partial x_j} \quad \text{or} \quad (\delta_i^j \delta_m^l - \delta_m^j \delta_i^l) \frac{\partial^2 \mathbf{u}_m}{\partial x_i \partial x_j}
\]
\[
\varepsilon_{ijk} \frac{\partial \mathbf{u}}{\partial x_i} = \varepsilon_{ijk} \frac{\partial^2 \mathbf{u}_m}{\partial x_i \partial x_j}
\]
\[
\nabla \times (\nabla \times \mathbf{u}) = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left( \frac{\partial^2 \mathbf{u}_m}{\partial x_i \partial x_j} \right)
\]
\[
= \varepsilon_{ijk} (\delta_i^l \delta_m^j - \delta_m^j \delta_i^l) \frac{\partial^2 \mathbf{u}_m}{\partial x_i \partial x_j} \frac{\partial^2 \mathbf{u}_m}{\partial x_i \partial x_j}
\]
\[
= \nabla \left( \nabla \cdot \mathbf{u} \right) - \nabla^2 \mathbf{u}
\]
The Navier-Stokes equation becomes, under the specified conditions, with incompressibility so that $\nabla u = 0$, 

$$p \left( \nabla \left( \frac{1}{2} u^2 \right) - u \times \omega \right) = -\nabla p - p \nabla V - \mu \nabla \times (u \times \omega)$$

ie. $\nabla \left( \frac{\mu}{2} u^2 + p + p V \right) - p (u \times \omega) - \mu \nabla \times u \times \omega$

Integrate this round a closed streamline, dotting with $d\ell$, the line element directed along the flow.

Then $\oint \nabla \cdot \mathbf{u} \cdot d\ell = 0$ because the starting and ending points are the same; also $\oint u \times \omega \cdot d\ell = 0$ because $u$ and $d\ell$ are parallel in the scalar triple product. Thus $\oint \frac{\mu}{2} \nabla \times \mathbf{u} \cdot d\ell = 0$.

Moreover, it is clear from the symmetry of the flows that there is now no normal flow across the boundary surface $x = 0, y = 0$, so the boundary condition is satisfied. By a uniqueness theorem, this image system gives the solution.

The line vortices have associated fluid speed $v \hat{n}$, directed tangentially, where $v$ is the
distance from the vortex. Thus, at \((a,b)\), the induced velocity arising from the image vortices is summarized in the following diagram:

\[
\begin{align*}
K & \quad \text{from I1)} \\
K & \quad \text{from I2)} \\
\frac{K}{\tan \theta} & \quad \text{from I3)}
\end{align*}
\]

\[
\begin{align*}
\tan \theta & = \frac{b}{a} \\
\frac{K}{\tan \theta} & = \frac{K}{\frac{b}{a}} = \frac{Ka}{b}
\end{align*}
\]

Thus, the \(x\)-component of the net velocity, which must equal \(\frac{da}{dt}\), is

\[
\frac{da}{dt} = \frac{K}{\tan \theta} \left( \frac{a^2}{a^2 + b^2} \right)
\]

Similarly, the \(y\)-component gives \(\frac{db}{dt} = \frac{K}{\tan \theta} \left( \frac{b^2}{a^2 + b^2} \right)\).

To find the path of the vortex, divide these to give

\[
\frac{\frac{da}{db}}{\frac{db}{da}} = \frac{\frac{a^2}{b^2} - \frac{b^2}{a^2}}{a^2} = \frac{b^2}{a^2} - \frac{b^2}{a^2} = \frac{a^2}{b^2}
\]

ie

\[
\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{a_0^2}
\]

If \(a = a_0\) as \(b \to \infty\), the constant is \(-\frac{1}{a_0}\).

So the path is

\[
\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{a_0^2}
\]
14. Unsteady irrotational flow has \( u = \nabla \phi \) (so \( \nabla u = 0 \)) and 
\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{\nabla \cdot \mathbf{u}}{\rho} - \nabla V
\]
Use the identity \( u \cdot \nabla u = \nabla (\frac{1}{2} u^2) - u \times \omega \) from 12 to give \( \nabla (\frac{\partial \phi}{\partial t} + \frac{1}{2} u^2 + V) = -\frac{\nabla \cdot \mathbf{u}}{\rho} \).

Now \( \int \frac{dp}{\rho} \) is only well-defined if \( p = \rho \phi \),
and then \( \nabla \int \frac{dp}{\rho \phi} = \frac{1}{\rho} \nabla \phi \).

Then "unwinding" the resultant equation gives
\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} u^2 + V + \int \frac{dp}{\rho} = \text{function of } t.
\]

Since \( u = \nabla \phi \) is unchanged by the addition of an arbitrary function of time, we can choose a gauge for which \( \phi \) which incorporates the function of time in the definition of \( \phi \), giving
\[
\int \frac{dp}{\rho} + \frac{1}{2} u^2 + \frac{\partial \phi}{\partial t} + V = \text{constant}.
\]

15. In cylindrical polar, if \( u = (0, \nu(r), 0) \),
then \( \omega = (0, 0, \frac{1}{r^2} \frac{\partial \nu}{\partial r}) \) (using the expression for curl in cylindrical polar).

We have \( \frac{\partial \nu}{\partial r} = \begin{cases} \frac{A}{r^2} & (r < a) \\ 0 & (r > a) \end{cases} \)

\( \nu = \begin{cases} \frac{A}{r^2} r^2 + A & (r < a) \\ \frac{A}{r^2} r^2 + B & (r > a) \end{cases} \)

For \( \nu \) to be finite at the origin, we must take \( A = 0 \).
Thus, \( v = \frac{1}{2} \cdot 2r \) (for \( r > a \)), \( v = \frac{3}{4} \) (for \( r > a \)).

If \( v \) is to be continuous at \( r = a \), \( B = \frac{1}{2} \cdot 2a^2 \).

In \( r > a \), Bernoulli's equation is \( p - \frac{1}{2} \rho v^2 + \rho g z = \text{const} \) (including gravity via \( \frac{dz}{dr} = g \)). Take \( z = 0 \) to be the free surface height at \( r = \infty \), with pressure \( p_0 \).

Thus \( \rho \cdot \rho_0 = -\frac{1}{2} \rho \cdot a^2 - \rho g 2z \text{ in } r > a \) (putting \( v = \frac{3}{4} a^2 \)).

For \( r < a \), the momentum equation can be written:
\[ \rho \left( \frac{1}{2} v^2 - \frac{1}{2} u^2 \right) = -\rho \cdot v \cdot \nabla z \] (using (1.12) part (ii)).

Now \( v \times w = (0, v, 0) \times (0, 0, z) = \left( \frac{1}{2} a^2, 0, 0 \right) - \nabla (\frac{1}{2} z^2) \)
in this particular case. Also \( v = \frac{1}{2} \cdot 2r \).

So the above equation can be "unwrapped" to give
\[ \rho \left( \frac{1}{8} a^2 r^2 - \frac{1}{8} a^2 z^2 \right) + p + \rho g 2z = \text{constant} \]
i.e. \( p = \text{constant} + \frac{1}{8} a^2 r^2 - \rho g 2z \).

Now at \( r = a \), we must have \( p \) continuous (otherwise \( r \) would be infinite acceleration at the interface). \( \therefore \) the constant is \( p_0 = \frac{1}{8} a^2 \rho g p_0 \).

At the free surface, put \( p = p_0 \) everywhere along \( y \).

Then for \( r > a \),
\[ z = -\frac{1}{8} \frac{a^2 \rho g}{g} \]
and for \( r < a \)
\[ z = \frac{1}{8} \frac{a^2 (r^2 - 2a^2)}{g} \]

This surface is continuous at \( r = a \) (measurably!); no \( z \) also its derivative is continuous. Its second derivative has an (unphysical) discontinuity. However, it is a much better model than a line vortex.