Maths III: Fluid Dynamics.

Example Sheet 2 Solutions.

1. Use
   \[ du = \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy + \frac{\partial u}{\partial z} \, dz \]
   \[ dv = \frac{\partial v}{\partial x} \, dx + \frac{\partial v}{\partial y} \, dy + \frac{\partial v}{\partial z} \, dz \]
   \[ dw = \frac{\partial w}{\partial x} \, dx + \frac{\partial w}{\partial y} \, dy + \frac{\partial w}{\partial z} \, dz \]
   to work out \( dx \times dy = (dx, dy, dz) \times (du, dv, dw) \)

   The first component of this is
   \[ \frac{\partial w}{\partial x} \, dx \, dy + \frac{\partial w}{\partial y} \left( \frac{\partial (dy)^2}{\partial z} \right) + \frac{\partial w}{\partial z} \, dy \, dz \]
   \[ - \frac{\partial v}{\partial x} \, dx \, dz - \frac{\partial v}{\partial y} \, dy \, dz - \frac{\partial v}{\partial z} \left( \frac{\partial (dz)^2}{\partial x} \right). \]

   To work out the \( x \)-component of angular momentum, we integrate \( p \times \) this over the sphere. We neglect variations in \( p \) and velocity gradients over the sphere, so take \( p \) and the velocity gradients outside the integrals. There are then two sorts of integral to evaluate: \( \int \, dx \, dy \, dV \), etc., which are all zero, and \( \int \, (dy)^2 \, dV \), etc., which are all equal.

   Thus the first component of the angular momentum is
   \[ p \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \int \, (dy)^2 \, dV = p \left( Vxu \right)_x \int \, (dy)^2 \, dV \]
E2.2

\[
\text{Now } I = \oint (dx)^2 \, dV = \oint (dy)^2 \, dV = \oint (dz)^2 \, dV.
\]

writing \((dx, dy, dz) = \frac{1}{r}, \) we have

\[
3I = \int_V r^2 \, dV = \int_0^\infty r^2 \cdot 4\pi r^2 \, dr = \frac{4}{5} \pi \sigma.
\]

Hence the \(x-\text{df} \) is \( p (\nabla \times \mathbf{u}) \cdot \frac{4}{15} \pi \sigma \); similarly for the \(y\) and \(z\) components, which work out in the same way. Hence the angular mom is

\[
\frac{1}{15} \pi \sigma \mathbf{e}_x \cdot \nabla \times \mathbf{u}.
\]

Now \( I = \frac{2}{15} m \mathbf{e}_z^2 \)

\[
= \frac{2}{15} \cdot \frac{4}{3} \pi \sigma.
\]

\[
\text{The angular mom} \text{ is}
\]

\[
\frac{1}{2} I (\nabla \times \mathbf{u}) \text{, i.e.} \ \frac{1}{2} I \mathbf{\omega}.
\]

This gives an interpretation of the vorticity as twice the local "spin rate" of a spherical fluid particle about its own centre (by comparing \(\frac{1}{2} I \mathbf{\omega} \) with resulting angular momentum \(I \mathbf{\Omega}\) for a rigid sphere rotating about its centre with angular velocity \(\mathbf{\Omega}\)).

Note the particle must be spherical for the analogy to hold; the last question on Assignment 1 treated the case of a needle-shaped particle embedded in a line vortex flow with \(\mathbf{\Omega} = 0\). Thru it was shown that the needle is rotating, even though \(\mathbf{\omega} = 0\).
Mass conservation \( \implies p_1 l_1 S_1 = p_2 l_2 S_2 \)
as we follow a fixed element of fluid.
Thus \( p l S \) is constant, equal to the mass in
the element.
Kelvin's theorem \( \implies \kappa = \oint u \cdot dl \) is constant
following the fluid, i.e., constant circulation.
But this is \( \oint \nabla \times u \cdot ds \approx w S \).
\( \therefore w S = \text{constant} \) (the circulation).
Dividing these, we have \( \frac{w}{pl} \) constant as we
follow the element.
Thus increasing \( l \) ("stretching" the tube) means
\( w \) must increase by a corresponding amount, other
things being equal.
This is similar to what happens in Q2. Note a
first attempt to model a tornado might be to take
one half (say \( z > 0 \)) of the flow in Q2. The
\( \kappa \)-part of the flow might be provided by a cumulo-
nimbus convection cell. (Real tornadoes are far more
complicated than this. If they weren't, they would
happen all over the place; the vortex stretching
mechanism is widespread in the atmosphere. Something
extra special is needed to cause a tornado.)
3. \( u = -\left(\frac{\alpha R}{2}, 0, xz\right) \)
\[
\omega = \nabla \times u = \left(-\frac{3\nu}{\alpha^2}, 0, \frac{1}{R} \frac{\partial}{\partial R} (R \nu)\right)
\]
This has to be of the form \((0, 0, \omega)\), so \(\frac{\partial \nu}{\partial \omega} = 0 \implies \nu = \nu(R, t) \text{ only.} \)
Also, then \(\omega = \frac{1}{R} \frac{\partial}{\partial R} (R \nu)\) is a function only of \(R, t\).

Vorticity equation under stated conditions is \[
\frac{\partial \omega}{\partial t} = \nabla \times (u \times \omega), \text{ work out } \nabla \times (u \times \omega):
\]
\[
\begin{align*}
    u \times \omega &= \left(\nu \omega, \frac{\alpha R \omega}{2}, 0\right) \\
    \nabla \times (u \times \omega) &= \left(0, 0, \frac{1}{R} \frac{\partial}{\partial R} \left(\frac{\alpha R^2 \omega}{2}\right)\right)
\end{align*}
\]
(because \(\nu \omega\) is independent of \(z\)).

Hence the vorticity equation has 2 trivial components, and the third says \[
\frac{\partial \omega}{\partial t} = \frac{1}{R} \frac{\partial}{\partial R} \left(\frac{\alpha R^2 \omega}{2}\right) = \frac{1}{R} \frac{\partial}{\partial R} (\frac{\alpha R^2 \omega}{2}) + \omega \frac{\partial \omega}{\partial R} \]

Try \(\omega = \omega_0 e^{-\alpha t} f(Re^{x t/2})\), where \(f\) is an arbitrary once-differentiable function.
\[
\begin{align*}
    \frac{\partial \omega}{\partial t} &= \omega_0 e^{-\alpha t} f + \omega_0 \alpha R e^{3x t/2} f' = \omega_0 + \omega_0 \frac{3x}{2} e^{3x t/2} f' \\
    \frac{\partial \omega}{\partial R} &= \omega_0 e^{3x t/2} f'. \text{ So } (\ast) \text{ is clearly satisfied,}
\end{align*}
\]
and the proposed solution works.
E2.5

Aside: to find the solution to the PDE directly, write it as
\[ \frac{\partial}{\partial t} \left( \frac{3}{2} - \alpha - \frac{\alpha R}{2} \frac{\partial}{\partial R} \right) w = 0, \quad \text{or} \quad \frac{\partial}{\partial t} \left( \frac{e^{-\alpha t}}{1} \right) = 0. \]

This means that we can write the solution as \( e^{-\alpha t} \) is constant along each curve in the \( R-t \) plane which satisfies \( \frac{dR}{dt} = -\alpha R \)

(because \( \frac{d}{dt} g(R(t), t) = \frac{dR}{dt} \frac{dg}{dR} + \frac{dg}{dt} \), where - 

\( g \) is an arbitrary differentiable function; take \( g = e^{-\alpha t} \).

These curves, called characteristics, have \( \log R = -\alpha t + \log C \), i.e., \( R e^{\alpha t/2} = C \), where each \( C \) value labels a different curve.

Thus, \( e^{-\alpha t} \) is a function of \( C = R e^{\alpha t/2} \), i.e., taking out an amplitude factor \( w_0 \),

\[ w = w_0 e^{\alpha t} f\left( R e^{\alpha t/2} \right). \]

This solution is an example of the method of characteristics.

If at \( t = 0 \) \( w = w_0 \) everywhere, so that \( f(R) = 1 \), then at later times \( w = w_0 e^{\alpha t} \), so the vorticity intensifies exponentially due to vortex stretching.

Another example: if the vorticity at \( t = 0 \) is localised in a patch of characteristic thickness \( R_0 \), so that (for example) \( f(R) = e^{-R^2/R_0^2} \), then later

\[ w = w_0 e^{\alpha t} e^{-R e^{\alpha t/2}/R_0^2} \], so that the vortex is exponentially amplified but is confined to a core thickness \( R_0 e^{-\alpha t/2} \) which decreases exponentially with time.
4. \[ \frac{\partial T}{\partial t} = \frac{\partial \tilde{T}}{\partial t} + u \cdot \frac{\partial \tilde{T}}{\partial \tilde{x}} = \kappa \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} \]

Write \( u = U \hat{u} \), \( L = L \hat{L} \), \( t = \frac{t}{U} \hat{t} \)

Then \( \left( \frac{U}{L} \right) \frac{\partial \tilde{T}}{\partial \tilde{t}} + U \hat{u} \cdot \frac{\partial \tilde{T}}{\partial \tilde{x}} = \kappa \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} \)

when the scalings are carried out.

\[ \frac{\partial \tilde{T}}{\partial \tilde{t}} + \hat{u} \cdot \frac{\partial \tilde{T}}{\partial \tilde{x}} = \left( \frac{\kappa}{UL} \right) \frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} = \frac{1}{Pe} \frac{\partial \tilde{T}}{\partial \tilde{x}}. \]

If \( Pe \) is large, the RHS is negligible, at least over most of the region, so we expect \( \frac{\partial \tilde{T}}{\partial \tilde{t}} \approx 0 \), i.e. \( \tilde{T} \) is constant following the flow. However, this has reduced the order of the system; thus for most boundary conditions we anticipate boundary layers in the \( T \)-field, of thickness \( S \sim L \), small enough to make the RHS locally of order 1, i.e. \( \frac{1}{Pe} \) has to be of order 1. This requires \( S \sim (Pe)^{-\frac{1}{2}} L \), i.e. we expect boundary layers of thickness \( (Pe)^{-\frac{1}{2}} \times L \), enclosing regions with \( \frac{\partial \tilde{T}}{\partial \tilde{t}} \approx 0 \). This is found to happen in numerical simulations of thermal convection.
5. With \( u = (u(x), 0, 0) \) and an applied pressure gradient \( \frac{dp}{dx} = -G \), the \( x \)-component of the Navier-Stokes equation is (see lectures)

\[
\mu \frac{d^2 u}{dy^2} = -G.
\]

Integrating, \( u = \frac{G}{2\mu} y^2 + Ay + B \).

The BC's are: \( u = 0 \) at \( y = 0 \),
\( u = U \) at \( y = d \).
Thus \( B = 0 \) and \( A = \frac{U - \frac{Gd}{2\mu}}{d} \).
This gives the solution for the velocity

\[
\left[ \frac{U}{d} - \frac{Gd}{2\mu} + \frac{Gy}{\mu} \right], \mu.
\]

The tangential stress at any level is \( \mu \frac{du}{dy} \)

At \( y = 0 \) this is \( \frac{U}{d} - \frac{Gd}{2} \) acting unit area in the positive \( x \)-direction.
At \( y = d \), this is \( \frac{U}{d} + \frac{Gd}{2} \) acting in the negative \( x \)-direction (negative since the normal points in the opposite direction; the force has to oppose both the boundary motion and the pressure gradient).
Our only hope of finding an easy solution to this problem is to hope that the flow is uniform on each nested ellipse in the family \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda \), \( 0 \leq \lambda \leq 1 \). So try

\[ u = (0, 0, U(\lambda)), \]

and hope it works.

The boundary conditions to be satisfied are

\( U = 0 \) at \( \lambda = 1 \) (no-slip at outermost ellipse)

\( U, \frac{\partial U}{\partial \lambda} \) remain finite at \( \lambda = 0 \).

Work in Cartesian coordinates (for which \( \nabla^2 \) behaves simply):

\[ \nabla^2 u = (0, 0, \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}) \]

Now \( \nabla \cdot u \) is still zero with the proposed form of flow:

\[ \frac{\partial}{\partial x} \left( \frac{\partial U}{\partial x} \right) \quad \frac{\partial}{\partial y} \left( \frac{\partial U}{\partial y} \right) \] are zero

from the \( x \) and \( y \) components of Navier-Stokes, and

the third \( z \) component gives

\[ -\frac{\partial}{\partial z} + \mu \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) = 0 \Rightarrow \frac{\partial}{\partial z} = \text{const.} \]

\(-\mu \) say, for this flow too (as in circular case).

Now \( \frac{\partial}{\partial x} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial x} = \frac{2x}{a^2} \frac{\partial U}{\partial x} \) \( \frac{\partial^2 U}{\partial x^2} = \frac{2}{a^2} \frac{d^2 U}{d\lambda} \) \( \frac{\partial^2 U}{\partial x \partial \lambda} = \frac{b^2}{a^2} \frac{d^2 U}{d\lambda} \frac{d\lambda}{d\theta} \)

Similarly \( \frac{\partial^2 U}{\partial y^2} = \frac{2}{b^2} \frac{d^2 U}{d\lambda} + \frac{4\lambda}{b^2} \frac{d^2^2 U}{d\lambda^2} \frac{d\lambda}{d\theta} \)

Hence

\[ \mu \left[ \frac{(4\lambda^2 + 1) d^2 U}{d\lambda^2} + a \left\{ \frac{1}{a^2} + \frac{1}{b^2} \right\} \frac{dU}{d\lambda} \right] = -\mu \]
At this stage, the presence of the coefficient of \( \frac{d^2U}{d\lambda^2} \) is not a simple function of \( \lambda \) and things look pretty hopeless. However, there is one simple thing yet to try: we can suppose \( \frac{d^2U}{d\lambda^2} = 0 \) and see if this works.

Then \( U = A\lambda + B \), and if we take \( 2 \left\{ \frac{1}{a^2} + \frac{1}{b^2} \right\} A = -\frac{G}{\mu} \), this will solve the equation, and have \( U, \frac{\partial U}{\partial \lambda} \) finite at \( \lambda = 0 \).

To satisfy the no-slip condition at \( \lambda = 1 \), we need \( B = -A \). This leads to the solution

\[
U = \frac{G}{2\mu} \frac{a^2b^2}{(a^2+b^2)} \left[ 1 - \left( \frac{x^2 + y^2}{a^2 + b^2} \right) \right]
\]

which satisfies the equation and boundary conditions.

It also reduces to the Poiseuille flow result \( U = \frac{G}{2\mu} (a^2-x) \) when \( a = b \). [Solution by spiritualism.]

[It is easy to prove the solution must be unique.]

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Adopt coordinates \( x \) and \( y \) along and perpendicular to the plane, as shown. Assume \( u = (u(y), 0) \) in these coordinates. Then \( u \cdot \nabla u = 0 \), and the equations are similar to the Couette flow.
problem except that there is a body force \(\mathbf{g}\)

unit mass, with components \((g \sin \alpha, -g \cos \alpha)\).

Thus the equations are

\[
\begin{align*}
0 &= -\frac{\partial p}{\partial x} - \rho g \sin \alpha + \mu \frac{d^2 u}{dy^2} \\
0 &= -\frac{\partial p}{\partial y} - \rho g \cos \alpha.
\end{align*}
\]

We assume there is no applied pressure gradient in
the \(x\)-direction, so \(\frac{\partial p}{\partial x} = 0\); the second equation
just tells us how the gravity forces a pressure
gradient \(1^\circ\) to the plane (so that \(p(y) = p_0 + \rho g \cos \alpha (1-y)\))
but is otherwise uninteresting. Solving the first equation
gives

\[
u = \frac{-\rho g \sin \alpha \ y^2}{2\mu} + Ay + B.
\]

The BC's are:

\[
\begin{align*}
u &= 0 \text{ at } y = 0 \Rightarrow B = 0 \\
\text{surface stress } \mu \frac{\partial u}{\partial y} &= 0 \text{ at } y = h \Rightarrow A = \frac{\rho gh \sin \alpha}{\mu}
\end{align*}
\]

Thus the velocity is

\[
u = \frac{\rho g \sin \alpha}{2\mu} \left\{ 2hy - y^2 \right\}
\]

The surface stress tangentially on the plane \(y = 0\)
is \(\mu \frac{\partial u}{\partial y} \bigg|_{y = 0} = \rho gh \sin \alpha\). This is the force
excited by the fluid on the plane; thus the plane
must exert a force \(\rho gh \sin \alpha \text{ unit area (up the plane)}\) on the fluid. But this is equal and
opposite to the component down the plane of the
weight of unit area of fluid. \(\therefore\) The forces are in balance.
8. We use coordinates \((r, \theta, z)\), with \(\theta\) azimuthal,

\[
\frac{\partial}{\partial \theta} \mathbf{u} = 0,
\]

and \(z\) pointing down the axis of the pipe. As before, suppose \(\mathbf{u} = (0, 0, U(r))\), so that \(\mathbf{u} \cdot \nabla \mathbf{u} = 0\). As before, the only difference gravity makes in the streamwise \((z)\) direction is to add a body force \(pg \sin \alpha\) per unit volume; thus the Poiseuille-flow equation is modified to

\[
- \frac{2p}{\partial z} + pg \sin \alpha + \mu \left( \frac{dU}{dr^2} + \frac{1}{r} \frac{dU}{dr} \right) = 0.
\]

Thus the solution is formally identical to the standard case if we modify the applied pressure gradient \(\frac{2p}{\partial z}\) by using \(G + pg \sin \alpha\) in place of \(G\).

Hence the solution for the velocity is

\[
U = \frac{(G + pg \sin \alpha) (a^2 - r^2)}{4\mu}
\]

To check the force balance, suppose \(G\) itself is zero.

The tangential stress (unit area is \(r = a\))

\[
- \frac{\partial U}{\partial r} \Bigg|_{r = a} = \frac{pg \sin \alpha}{2}
\]

(Sign is opposite to last question because the outward-pointing normal is pointing in the opposite direction. The simplest way to get the direction of the force is to use common sense.

The wall must obviously exert an upward force \(\frac{pg \sin \alpha}{2}\) unit area on the fluid.

The force on the fluid in unit length of pipe is

\[
2\pi a \cdot \frac{pg \sin \alpha}{2} = \pi a^2 \cdot \frac{pg \sin \alpha}{2}.
\]

This is numerically equal to the weight component (down the slope) of the fluid. So once again the forces balance.
9. As in the impulsively started plate problem, the velocity is taken to be \((u(y, t), 0, 0)\), and \(u\) satisfies the diffusion equation

\[
u \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}\]

with \(u = V \cos \omega t\) \((y = 0)\) \(u \to 0\) \((y \to \infty)\). We assume the initial condition is irrelevant in that any transients have already decayed by viscous dissipation; thus we examine only the forced response.

Try a separable solution \(u = \text{real part of } f(y) e^{\text{i} \omega t}\).

Then \(f'' = \text{i} \omega f\) and \(f\) must satisfy \(f = V(y = 0)\)

\[
f'' - \frac{\text{i} \omega}{\nu} f = 0\]

solutions of this are \(f = e^{\pm \sqrt{\frac{\text{i} \omega}{\nu} y}}\). Now \(\nu = \frac{\text{i} \pi}{4} = \frac{\text{i} \pi}{12}\) (1 + i).

The G.S. for \(f\) is

\[
f = A e^{\lambda (1 + i) y} + B e^{-\lambda (1 + i) y}\]

where \(\lambda = \sqrt{\frac{\nu}{V}}\). For \(y \to \infty\), \(f \to 0\), so \(\lambda\) must have a zero; the other B.C. gives \(B = V\).

Hence \(u = \text{Re} \{V e^{-\lambda y} e^{(\omega t - \lambda y)}\}\)

\[
u = V e^{-\lambda y} \cos (\omega t - \lambda y)\]

Thus the velocity decays in magnitude away from the plate, being reduced by \(\frac{1}{e}\) in a distance \(\frac{1}{\lambda} = \frac{3 \pi}{\sqrt{2}}\). Thus high-frequency disturbances travel hardly any distance at all; low frequency go further in. The phase of the wave varies with height, requiring \(y = \frac{\text{III}}{2} = \frac{\pi}{2} \sqrt{\frac{210}{5}}\) to be once again in phase with the movement of the plate.
10. As in lectures, the equation is \( \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \)
but now the BC's are \( \frac{\partial u}{\partial y} = -S \) at \( y = 0 \),
\( u \to 0 \) as \( y \to \infty \).
Try a similarity solution \( u = \frac{S}{\mu} y \cdot F(\eta) \)
where \( \eta = \frac{y}{\sqrt{4D\mu t}} \). Then \( \frac{\partial u}{\partial t} = \frac{Sy}{\mu} F'(\eta) \left\{ \frac{y}{2\sqrt{4Dt}} \right\} \)
\( \frac{\partial^2 u}{\partial y^2} = \frac{S}{\mu} \left[ F(\eta) + \frac{y}{\sqrt{4Dt}} F'(\eta) \right] \)
\( \frac{\partial^2 u}{\partial y^2} = \frac{S}{\mu} \left[ \frac{2}{\sqrt{4Dt}} F'(\eta) + \frac{y}{4Dt} F''(\eta) \right] \)
Hence the substitution of these into the diffusion equ'gives
\( -\frac{y^2}{2\sqrt{4Dt}} F'(\eta) = \frac{2y}{\sqrt{4Dt}} F'(\eta) + \frac{y^2}{4Dt} F''(\eta) \)
Multiplying by \( \frac{4t}{y} \) gives \( F''(\eta) + F'(\eta) \left\{ \frac{2}{\eta} + 2\eta^2 \right\} = 0 \)
Treating this as a first-order integrating-factor-type DE for \( F' \) gives \( \eta^2 e^{-\eta^2} F'(\eta) = C \).
\( \text{i.e.} \quad F'(\eta) = \frac{Ce^{-\eta^2}}{\eta^2} \).
Thus formally \( F = \eta^2 \int \frac{e^{-\eta^2}}{\eta^2} d\eta + D \).
will give the solution. Given that we want \( F \to 0 \)
as \( \eta \to \infty \), we can eliminate \( D \) by taking as the solution
\( F = -C \int_{\eta}^{\infty} \frac{e^{-\eta^2}}{\eta^2} d\eta \).
To relate $C$ to the $y = 0$ BC, we need

$$\mu \frac{\partial u}{\partial y} = -S \quad \text{at} \quad y = 0, \quad \text{i.e.}$$

$$SF(\eta) + S y F'(\eta) \cdot \frac{1}{\sqrt{4\pi t}} = -S,$$

i.e.

$$F(\eta) + \eta F'(\eta) = -1 \quad \text{at} \quad \eta = 0$$

is the BC to be applied. Using the $F$ we have found, this enables $C$ to be determined.

If you are interested only: (much harder than expected, cause standard)

From the singular-looking nature of the integral, it is not obvious that this will actually work, especially as $\eta F'(\eta)$ is $\frac{Ce^{-\eta^2}}{\eta}$ as $\eta \to 0$. However, it does: near $\eta = 0$

we can asymptotically expand the integral for $F$ as

$$F = \frac{Ce^{-\eta^2}}{\eta} \left[ \int_0^\infty + C \int_0^\infty \frac{2\eta e^{-\eta^2}}{\eta} \, d\eta \right] \quad \text{using integration by parts.}$$

Thus $F$ is actually

$$\frac{Ce^{-\eta^2}}{\eta} + 2C \frac{\sqrt{\pi}}{\alpha^2} (1 - \text{erf}(\eta))$$

as $\eta \to 0$.

Now it can be seen that the infinite bits in $\times$ as $\eta \to 0$ cancel each other out. Thus $\times$ gives, as $\eta \to 0$

and using $\text{erf}(0) = 0, \quad \frac{2C \sqrt{\pi}}{\alpha^2} = -1$, i.e.

$$C = -\frac{1}{\sqrt{\pi}}$$

Using $y = \sqrt{4\pi t} \eta$ then gives the full solution $s \cdot y \cdot F$ as

$$u = \frac{S}{\mu} \left[ \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4\pi t}} - y \text{erf} \left( \frac{y}{\sqrt{4\pi t}} \right) \right]$$

This is finite for finite $y, t$ (and zero at $t = 0$).
II. For this problem, \( u \partial u \) is still zero and the only important equation to be solved is that describing the \( x \) = velocity. This is
\[
\frac{\partial u}{\partial t} = \frac{u}{\partial y^2}
\]
the diffusion equation with a forcing term.

This is to be solved subject to the BC's \( u = 0 \) at \( y = \pm a \), and the initial condition \( u = 0 \) at \( t = 0 \).

The steady solution to which the flow will evolve is the above with \( \frac{\partial u}{\partial t} \) set equal to zero; the solution satisfying \( u = 0 \) at \( y = \pm a \) is \( u = u_1 = \frac{G(a^2 - y^2)}{2\mu} \).

As suggested, put \( u = u_1(y) + U(yt) \). (\( u = \rho u \)).

Then
\[
\frac{\partial U}{\partial t} = \frac{G}{\rho} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial y^2}
\]
since \( u_1 \) satisfies the time-independent problem.

Thus the problem for \( U \) is to solve
\[
\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial y^2}
\]
with BC's \( U = 0 \) at \( y = \pm a \) and \( U = -u_1 = -\frac{G}{2\mu}(a^2 - y^2) \) at \( t = 0 \).

Try \( U = Y(y)T(t) \).

Then
\[
\frac{T}{\partial T} = \frac{Y''}{Y} = \text{some constant} = -k^2 \text{ say.}
\]

(in anticipation that we won't be able to satisfy both BC's unless the constant is negative.)

Hence \( Y \propto \sin k y \) or \( \cos k y \) and \( T \propto e^{-kt} \).
So that a separable solution is

\[ e^{-k^2ut} \left( A \sin ky + B \cos ky \right). \]

Now \( U = 0 \) at \( y = \pm a \Rightarrow A \sin ka + B \cos ka = 0 \)

Thus we must have \( B \cos ka = A \sin ka = 0 \). Either \( A \) or \( B \) must be zero; since the initial condition is symmetric about \( y = 0 \), we take cosines only, i.e.

\[ \text{put } A = 0. \text{ Then } \cos ka = 0 \Rightarrow k = (n + \frac{1}{2})\pi \]

for \( n = 0, 1, 2 \ldots \)

Thus the general solution is

\[ U(y,t) = \sum_{n=0}^{\infty} B_n \cos (n + \frac{1}{2})\pi y e^{-k(n + \frac{1}{2})^2 \pi^2 t}. \]

where \( k = (n + \frac{1}{2})\pi \).

At \( t = 0 \), this has to equal \( \frac{G}{2\mu} (a^2 - y^2) \).

Thus \( \frac{G}{2\mu} (a^2 - y^2) = \sum_{n=0}^{\infty} B_n \cos \left\{ (2n+1)\pi y \right\} \)

Hence \( B_n = -\frac{G}{2\mu} \frac{\alpha}{2a} \int_{y=0}^{\alpha} (a^2 - y^2) \cos \left\{ (2n+1)\pi y \right\} dy \) \((n=0)\)

\[ = -\frac{G}{\mu a} \left[ \frac{a^4}{2a} \sin \left\{ (2n+1)\pi y \right\} \frac{2a}{(2n+1)\pi} \right]_{y=0}^{a} + \int_{y=0}^{\alpha} 2y \frac{a}{(2n+1)\pi} \sin \left\{ (2n+1)\pi y \right\} dy \]

\[ = \frac{G}{\mu a} \frac{4a^2}{(2n+1)\pi} \left[ y \cos \left\{ (2n+1)\pi y \right\} \frac{2a}{(2n+1)\pi} \right]_{0}^{a} - \frac{4a^2}{(2n+1)^2\pi^2} \frac{\sin \left\{ (2n+1)\pi y \right\}}{2a} \]

\[ = \frac{G}{\mu a} \frac{16a^2}{(2n+1)^3\pi^3} \left( -1 \right)^{n+1} \]

\[ u(y,t) = \frac{G}{2\mu} (a^2 - y^2) + \sum_{n=0}^{\infty} \frac{16G a^2}{\mu (2n+1)^3 \pi^3} \left( -1 \right)^{n+1} \cos \left\{ (2n+1)\pi y \right\} e^{-k(n + \frac{1}{2})^2 \pi^2 t}. \]

The slowest decaying term in the series is \( n = 0 \), with characteristic decay time \( T = \frac{4a^2}{\pi^2 D} \), of order the diffusion timescale \( a^2 / D \).

This estimates the time to reach the steady state.