Project 2: Spectral Methods for Nonlinear Wave Equations

A generic one-dimensional nonlinear wave equation to describe low-frequency, large scale wave phenomena is the *Korteweg de Vries (KdV)* equation, named after the PhD student G. de Vries and his supervisor, D.J. Korteweg, who first analyzed it in a paper published in 1895 in the context of long water waves in a shallow channel:

$$\frac{\partial u}{\partial t} + \mu u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

This equation has a special solution of the form

$$u(x,t) = a\text{sech}^2[w(x-ct)]. \tag{2}$$

This type of coherent localized solution was first postulated on physical grounds by J. Boussinesq in 1871 and Lord Rayleigh in 1876, in their attempts to explain the observations of J. Scott Russell in 1834:

I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

The KdV equation embodies a balance between two terms; the term in $u\frac{\partial u}{\partial x}$ which describes wave steepening (focusing in optics), and the $\frac{\partial^3 u}{\partial x^3}$ term which describes wave dispersion (the phenomenon where waves with different wavelengths travel at different velocities leading to spreading of the initial profile). This balance allows waves of the form (2) to propagate without change of shape, despite the nonlinearity of the equation. Such waves are called *solitons*. Considerable theoretical work has been done in the last thirty years on general methods for determining such exact solutions of nonlinear PDE's, but the inspiration for much of this came from the numerical experimentation of N.J. Zabusky and M.D. Kruskal (Phys. Rev. Lett., 1965, **15**, pp 240 – 243). (Kruskal and co-workers later did much of the theoretical work as well.) This is one of many examples where the computer has helped to point theoretical work in the right direction.

1. Show that all the equation parameters can be scaled away by scaling u, x and t and use a scaling such that the resulting equation takes the form $u_t + 6uu_x + u_{xxx} = 0$. If you do so bear in mind that the system length is not a free parameter anymore.

Determine the exact solution of (1) by determing the parameters a, w, c in equation (2). Note that the speed c is dependent on the amplitude a, which is not the case for solutions of linear wave equations.

2. Write a programme which numerically integrates the KdV equation (1) on the domain $[0, L]$ with periodic boundary conditions. Use a pseudospectral Crank-Nicholson scheme where the nonlinearity is treated by an Adams-Bashforth scheme. (Actually, write the programme immediately for Eq. (3) and then set $\alpha = \nu = 0$ for now. This saves you lots of work.) Use FFTs for the Fourier transforms. You may use the *Numerical Recipes* routine if you programme in C or Fortran, or in MATLAB, you can use **fft**. Read the instructions for the routine you use carefully. Each routine has its own way of storing the real and imaginary parts! In MATLAB, for example, you will need the functions **fftshift** and **ifftshift** to order the Fourier transforms.

Use an initial condition of the form $A\exp(-B(x - L/2)^2)$. Choose B such that the initial condition occupies about one eighth of the domain. Vary A and report how the long time solution depends on A. As a matter of fact, since the KdV equation is integrable the initial value problem is analytically solvable. You can also check your program using initial conditions corresponding to the known exact solutions for the initial condition (2) (which should propagate unchanged) and the 2-soliton cases. See Drazin and Johnson: *Solitons* equations (4.35) and (4.36) for the 2-soliton case.

3. The "mass" $m = \int u dx$ is conserved. Find at least one more conserved quantity of the KdV equation. Use this quantity in your programme to monitor accuracy.

Why is the mass not a good quantity to monitor accuracy in a (pseudo)-spectral code for the KdV equation? Use the fact that you can write the KdV equation in a flux form as $u_t = -\partial_x(0.5\mu u^2 + \beta u_{xx})$.

4. Kruskal and Zabusky observed in their numerical simulations that two solitons of the form (2) which were initially well-separated and with different amplitudes (and hence speeds, see above) interacted in a surprising way.

Create an initial condition comprised of two well separated soliton solutions of (1) with different amplitudes. Observe what happens. Play with varying the amplitudes and adjust the system length if necessary. Plot the time evolution of the location of the maximal amplitudes of the two respective localized bumps (use quadratic interpolation to locate the maximum numerically). What do you observe?

5. The KdV equation is conservative and integrable. Most phenomena in nature are of a dissipative nature. A natural question is: how will non-conservative perturbations affect the solitons? To this end we now study the *Kawahara-Toh* or *Benney* equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} + \alpha\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^4 u}{\partial x^4} = 0. \tag{3}$$

Do a linear stability analysis of Eq. (3) by looking at disturbances to the homogeneous solution $u(x,t) = 0$.

Write the disturbances as $\delta u = r\exp(i(kx - \omega t) + \sigma t)$ where ω and σ are real. Give an expression for the most unstable mode with wave number $k^\star$.

6. Now consider nonzero α and ν. Keep the box-length L fixed at $L = 100$.

First, set the dispersion $\beta = 0$. The resulting dispersionless equation is called the *Kuramoto-Sivashinsky* equation and it can have chaotic solutions for certain values of α, ν and L. Choose the parameters α, ν to have values where you expect an instability with a wavelength of L (i.e. wavenumber $k = 2\pi/L$). Start with an initial condition which is a slight perturbation around $u = 0$, and observe what happens. Then increase the value of α to obtain unstable disturbances with higher and higher wavenumbers.

Next, look at nonzero β. Set $\beta = 0.3$. Again, choose the parameters α, ν so that you expect an instability with a wavenumber of $10\ 2\pi/L$. Take the initial condition $u = 0.1\ \sin(2\,dk\,x)$ where $dk = 2\pi/L$. What happens to the initial condition? Now increase the dispersion β and report on your findings. In what way is the most unstable wave number $k^\star$ appearing in the solution?