5. For each of the following matrices, determine whether it is invertible, and find the inverse if there is one.

(i) \[
\begin{pmatrix}
2 & 0 & -1 \\
1 & 0 & 3 \\
1 & 2 & 1
\end{pmatrix}
\]

(ii) \[
\begin{pmatrix}
0 & 1 & 2 \\
-3 & 0 & 3 \\
-2 & -1 & 0
\end{pmatrix}
\]

Solution.

To decide whether an \( n \times n \) matrix \( A \) is invertible, form the \( n \times 2n \) matrix \( [A \mid I] \) and apply row operations to produce a reduced echelon matrix. If the reduced echelon matrix has the form \( [I \mid B] \) then \( A \) is invertible and \( B = A^{-1} \). If, on the other hand, row operations give a matrix of the form \( [J \mid B] \), where \( J \) has a row of zeros, then \( A \) is not invertible.

(i) \[
\begin{pmatrix}
2 & 0 & -1 & 1 & 0 & 0 \\
1 & 0 & 3 & 0 & 1 & 0 \\
1 & 2 & 1 & 0 & 0 & 1
\end{pmatrix}
\]

\( R_2 \leftarrow R_2 \)

\[
\begin{pmatrix}
1 & 0 & 3 & 0 & 1 & 0 \\
0 & 0 & -7 & 1 & -2 & 0 \\
0 & 2 & -2 & 0 & -1 & 1
\end{pmatrix}
\]

\( R_2 \leftarrow 2R_1 \)

\( R_3 : = R_3 - R_1 \)

\[
\begin{pmatrix}
1 & 0 & 3 & 0 & 1 & 0 \\
0 & 0 & -7 & 1 & -2 & 0 \\
0 & 2 & -2 & 0 & -1 & 1
\end{pmatrix}
\]

\( R_2 \leftarrow 2R_3 \)

\( R_3 : = 2R_2 + R_3 \)

\[
\begin{pmatrix}
1 & 0 & 3 & 0 & 1 & 0 \\
0 & 0 & -7 & 1 & -2 & 0 \\
0 & 0 & 1 & -1 & 0 & 2
\end{pmatrix}
\]

Thus the matrix is invertible, with inverse \[
\begin{pmatrix}
\frac{3}{7} & \frac{1}{7} & 0 \\
\frac{1}{7} & -\frac{3}{14} & \frac{1}{2} \\
-\frac{1}{7} & \frac{2}{7} & 0
\end{pmatrix}
\]

(ii) \[
\begin{pmatrix}
0 & 1 & 2 & 1 & 0 & 0 \\
-3 & 0 & 3 & 0 & 1 & 0 \\
-2 & -1 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

\( R_1 \leftarrow R_2 \)

\[
\begin{pmatrix}
-3 & 0 & 3 & 0 & 1 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 \\
-2 & -1 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

\( R_1 \leftarrow -\frac{1}{3}R_1 \)

\[
\begin{pmatrix}
1 & 0 & -1 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 2 & 1 & 0 & 0 \\
-2 & -1 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

\( R_3 \leftarrow R_3 + 2R_1 \)

\[
\begin{pmatrix}
1 & 0 & -1 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 2 & 1 & 0 & 0 \\
0 & -1 & 2 & 0 & -\frac{2}{3} & 1
\end{pmatrix}
\]

\( R_5 \leftarrow R_3 + R_2 \)

\[
\begin{pmatrix}
1 & 0 & -1 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -\frac{2}{3} & 1
\end{pmatrix}
\]

Since we have produced a row of zeros in the left hand half of the matrix, we need go no farther. The reduced echelon matrix will not be of the form \([I \mid X]\), and so the given matrix was not invertible.
6. Use the previous question to solve the following system of linear equations:

\[
\begin{align*}
2x - z &= 2 \\
x + 3z &= 1 \\
x + 2y + z &= -1
\end{align*}
\]

**Solution.**

If \(A\) is a square matrix with an inverse then the system of linear equations \(Ax = b\) has the unique solution \(x = A^{-1}b\). So by Exercise 5(i) the solution is

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
\frac{3}{7} & \frac{1}{7} & 0 \\
-\frac{1}{7} & -\frac{3}{14} & \frac{1}{2} \\
-\frac{1}{7} & \frac{3}{7} & 0
\end{bmatrix} \begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix} = \begin{bmatrix}
1 \\
-1 \\
0
\end{bmatrix}.
\]

That is, \(x = 1\), \(y = -1\), \(z = 0\).

7. Use elementary row operations to find the inverse of the matrix

\[
A = \begin{bmatrix}
1 & 0 & 2 \\
0 & 0 & 1 \\
6 & 3 & 0
\end{bmatrix},
\]

and hence express \(A\) as a product of elementary matrices. (From 1996 exam.)

**Solution.**

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 0 & 1 \\
6 & 3 & 0
\end{bmatrix} \xrightarrow{R_3 := R_3 - 6R_1} \begin{bmatrix}
1 & 0 & 2 \\
0 & 0 & 1 \\
0 & 3 & -12
\end{bmatrix} \xrightarrow{R_2 := R_2 + 12R_3} \begin{bmatrix}
1 & 0 & 2 \\
0 & 0 & 1 \\
0 & 3 & 0
\end{bmatrix} \xrightarrow{R_2 := \frac{1}{3}R_2} \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

This shows that \(A^{-1} = \begin{bmatrix}
1 & -2 & 0 \\
-2 & 4 & \frac{1}{3} \\
0 & 1 & 0
\end{bmatrix}.
\]

The elementary matrices corresponding to the row operations that were used, in the order in which they were used, are

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-6 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
1 & -2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 12 & 1
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]
If we call these $E_1, E_2, E_3, E_4$ and $E_5$ (respectively) then we have

$$E_5E_4E_3E_2E_1[A \mid I] = [I \mid A^{-1}]$$

which shows that $E_5E_4E_3E_2E_1 = A^{-1}$. Hence $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}$. That is,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

This answer is not unique because there is more than one way to carry out the row operations.

8. Let $X$ be a square matrix and suppose that for some real numbers $t_0, t_1, \ldots, t_n$,

$$t_nX^n + t_{n-1}X^{n-1} + \cdots + t_1X + t_0I = 0.$$  

Show that if $t_0 \neq 0$ then $X$ is invertible. (Hint: move the term $t_0I$ to the right hand side and factorise the left hand side.)

Solution.

Rearranging as suggested in the hint we find that

$$(-\frac{t_n}{t_0}X^{n-1} - \frac{t_{n-1}}{t_0}X^{n-2} - \cdots - \frac{t_1}{t_0})X = I = X(-\frac{t_n}{t_0}X^{n-1} - \frac{t_{n-1}}{t_0}X^{n-2} - \cdots - \frac{t_1}{t_0}),$$

and so $(-\frac{t_n}{t_0}X^{n-1} - \frac{t_{n-1}}{t_0}X^{n-2} - \cdots - \frac{t_1}{t_0})$ is an inverse, and hence the inverse, of $X$.

9. Let $J$ be the $n \times n$ matrix each of whose entries is 1.

(i) Show that $J^2 = nJ$.

(ii) Show that if $n > 1$ then $(I - J)^{-1} = I - \frac{1}{n-1}J$.

Solution.

(i) Each entry of $J^2$ is the product of the $n$-component row of 1’s by the $n$-component column of 1’s, which gives the sum of $n$ 1’s, which is $n$. So each entry of $J^2$ is $n$ times the corresponding entry of $J$; that is, $J^2 = nJ$.

(ii)

$$(I - J)(I - \frac{1}{n-1}J) = (I - \frac{1}{n-1}J)(I - J)$$

$$= I - \frac{1}{n-1}J - J + \frac{1}{n-1}J^2$$

$$= I - \frac{n}{n-1}J + \frac{1}{n-1}nJ = I.$$

By the definition, this shows that $I - \frac{1}{n-1}J$ is an inverse of $I - J$, and since a matrix can have at most one inverse, we have $(I - J)^{-1} = I - \frac{1}{n-1}J$. 


Let $A$ be an $n \times n$ matrix whose $(i, j)$-th entry is $a_{ij}$. We say that $A$ is upper triangular if $a_{ij} = 0$ whenever $i > j$. Show that the product of two $n \times n$ upper triangular matrices is upper triangular. (Do the case $n = 3$ before attempting the general case.)

Solution.

Let $a_{ij}$ be the $(i, j)$th entry of $A$ and $b_{ij}$ be the $(i, j)$th entry of $B$. Then the $(i, j)$th entry of $AB$ is $\sum_{k=1}^{n} a_{ik}b_{kj}$. Suppose now that $A$ and $B$ are upper triangular, so that $a_{ik} = 0$ whenever $i > k$ and $b_{kj} = 0$ whenever $k > j$. Then if $i > j$ we have

$$\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{j} a_{ik}b_{kj} + \sum_{k=j+1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{j} 0b_{kj} + \sum_{k=j+1}^{n} a_{ik}0 = 0 + 0 = 0,$$

since when $1 \leq k \leq j$ we have $i > k$ (since $i > j$) and hence $a_{ik} = 0$, and when $j + 1 \leq k \leq n$ we have $k > j$ and hence $b_{kj} = 0$. So the $(i, j)$th entry of $AB$ is zero when $i > j$, as required.

It is easier to see this in an example. The (2,1)-entry of

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{pmatrix}$$

is $[0 \ d \ e] \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix} = 0g + d0 + e0 = 0$, and the other below-diagonal entries similarly can be seen to be zero.