The answers to questions 1, 2 and 3 are on the Practice Sheet.

4. Find the eigenvalues and corresponding eigenvectors for 
\[ A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 1 & 13 \\ 0 & 0 & -3 \end{bmatrix}. \]

Solution.

First calculate the eigenvalues of \( A \). By expanding the determinant of \( \det(A - \lambda I) \) along its last row we see that the characteristic equation is
\[
0 = \det \begin{bmatrix} 1 - \lambda & -1 & 5 \\ -1 & 1 - \lambda & 13 \\ 0 & 0 & -3 - \lambda \end{bmatrix} = (-3 - \lambda)((1 - \lambda)^2 - 1) = -(3 + \lambda)(\lambda - 2).
\]

This has roots \( \lambda = -3, 0, 2 \) and these are the eigenvalues of \( A \).

Any nonzero solution of \((A - \lambda I) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\) is an eigenvector for the eigenvalue \( \lambda \). Thus to find the eigenvectors for the eigenvalue 2 we should find the nonzero solutions of
\[
\begin{bmatrix} -1 & -1 & 5 \\ -1 & -1 & 13 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

It is easily checked that the general solution of this system is \( x = -y = t \) and \( z = 0 \), where \( t \) is an arbitrary parameter. Thus the eigenvectors of \( A \) for the eigenvalue 2 are all column vectors of the form \( t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \) for \( t \neq 0 \). Similarly, we find that the eigenvectors for the eigenvalue \(-3\) are the nonzero scalar multiples of \( \begin{bmatrix} 11 \\ 19 \\ -5 \end{bmatrix} \) and the eigenvectors for the eigenvalue 0 nonzero scalar multiples of \( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \).

5. [Cayley-Hamilton] Show that the characteristic equation of the matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is
\[
\lambda^2 - (a + d)\lambda + (ad - bc) = 0.
\]

Show also that \( A \) satisfies the matrix equation \( A^2 - (a + d)A + (ad - bc)I_2 = 0_2 \), where \( I_2 \) and \( 0_2 \) are the \( 2 \times 2 \) identity and zero matrices respectively.

Solution.

\[
\det(A - \lambda I) = \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc).
\]
Since \( A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} \) we find that \( A^2 - (a + d)A + (ad - bc)I_2 \) equals
\[
\begin{bmatrix}
  a^2 + bc & ab + bd \\
  ca + dc & cb + d^2
\end{bmatrix} - \begin{bmatrix}
  (a + d)a & (a + d)b \\
  (a + d)c & (a + d)d
\end{bmatrix} + \begin{bmatrix}
  ad - bc & 0 \\
  0 & ad - bc
\end{bmatrix} = \begin{bmatrix} 0 & 0 \\
  0 & 0 \end{bmatrix}
\]
as required. (There is a theorem known as the “Cayley-Hamilton Theorem” which states that a square matrix satisfies its characteristic equation. We have proved it for \( 2 \times 2 \) matrices.)

6. Let \( A \) and \( P \) be \( n \times n \) matrices, with \( P \) invertible. Show that \( A \) and \( PAP^{-1} \) have the same characteristic equation. (Use the product rule \( \det(XY) = (\det X)(\det Y) \).)

Solution.
Note that \( \det P \det(P^{-1}) = \det(PP^{-1}) = \det I = 1 \). Note also that
\[ P(\lambda I)P^{-1} = \lambda(PI)P^{-1} = \lambda I. \]
It follows that
\[
\det(PAP^{-1} - \lambda I) = \det(PAP^{-1} - P\lambda IP^{-1}) = \det(P(\lambda I - A)P^{-1}) = \det P \det(A - \lambda I)(\det P)^{-1} = \det(A - \lambda I).
\]

7. [Cramer’s rule]
(i) Given a \( 3 \times 3 \) matrix \( A \) and a \( 3 \times 1 \) column vector \( b \), show that
\[
(adj A)b = \begin{bmatrix} \det(A_1) \\ \det(A_2) \\ \det(A_3) \end{bmatrix}
\]
where \( A_i \) is the matrix obtained from \( A \) by replacing column \( i \) by \( b \).
(ii) Suppose that \( A \) is invertible and then show that the solution to the matrix equation \( Ax = b \), where \( x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) is
\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{\det(A_1)}{\det(A)} \begin{bmatrix} \det(A_1)/\det(A) \\ \det(A_2)/\det(A) \\ \det(A_3)/\det(A) \end{bmatrix}.
\]
(iii) Use (ii) to solve the following equations:
\[
\begin{align*}
x + 2y + 2z &= 5 \\
x + 3y + z &= 0 \\
x + 3y + 2z &= -2
\end{align*}
\]
Which method of solving equations do you prefer: using row operations or Cramer’s rule?
Solution.

(i) The entries of the first row of adj $A$ are the cofactors $c_{i1}$ of the elements of the first column of $A$. Thus the matrix product of the first row of adj $A$ and $b$ is $\sum_{i=1}^{3} c_{i1}b_i$, where $b_i$ is the entry in row $i$ of $b$. This is just the expansion down the first column of the matrix $A_1$ obtained from $A$ by replacing its first column with $b$. Thus its value is $\det(A_1)$. The same argument shows that the entries in the second and third rows of $(\text{adj } A)b$ are $\det(A_2)$ and $\det(A_3)$.

(ii) Since $A$ is invertible we may multiply the equation $Ax = b$ on the left by $A^{-1}$ to obtain $x = A^{-1}b$. But we know that $A^{-1} = (\det A)^{-1}\text{adj }A$ and so $x = (\det A)^{-1}(\text{adj }A)b$. The result now follows from (i).

(iii) The matrix of coefficients is

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}. $$

Expanding across the first row, the determinant of $A$ is

$$(3 \times 2 - 1 \times 3) - 2(1 \times 2 - 1 \times 1) + 2(1 \times 3 - 3 \times 1) = 3 - 2 = 1. $$

Furthermore, $\det(A_1) = 23$, $\det(A_2) = -7$ and $\det(A_3) = -2$. Therefore, $x = 23$, $y = -7$ and $z = -2$.

8. The Hessian of a function $u(x_1, x_2)$ of two variables is the determinant of the matrix

$$\begin{vmatrix} \frac{\partial^2 u}{\partial x_1 \partial x_1} \\ \frac{\partial^2 u}{\partial x_1 \partial x_2} \end{vmatrix}$$

whose $(i, j)$-th entry is $\frac{\partial^2 u}{\partial x_i \partial x_j}$. Find the Hessian of $ax_1^2 + bx_1x_2 + cx_2^2$.

Solution.

If $u = ax_1^2 + bx_1x_2 + cx_2^2$, then

$$\frac{\partial u}{\partial x_1} = 2ax_1 + bx_2 \quad \frac{\partial^2 u}{\partial x_1 \partial x_1} = 2a \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = b \quad \frac{\partial^2 u}{\partial x_2 \partial x_2} = 2c \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = b$$

and so the Hessian is $\begin{vmatrix} 2a & b \\ b & 2c \end{vmatrix} = 4ac - b^2$. 