Preliminary Reading:
Chapter 3 of the Linear Algebra book.

Objectives:
By the end of Week 12, to achieve at least a pass level, you should be able to

(a) calculate the characteristic equation of a matrix.

(b) calculate the eigenvalues and eigenvectors of $3 \times 3$ matrices.

To achieve higher than a pass level you should be able to

(c) work with matrix equations involving inverses and adjoints.

(d) carry out symbolic calculations with eigenvalues and eigenvectors.

Preparatory questions. (Answers are on the next page.)

1. For which values of $\lambda$ is the following matrix not invertible:

$$
\begin{bmatrix}
3 - \lambda & -2 \\
-1 & 2 - \lambda
\end{bmatrix}.
$$

2. Find the eigenvalues of the matrix

$$
A = \begin{bmatrix}
3 & -2 \\
-1 & 2
\end{bmatrix}.
$$

3. For each eigenvalue of the matrix $A$ of the previous question, find a corresponding eigenvector.

Practice questions

4. Find the eigenvalues and corresponding eigenvectors for $A = \begin{bmatrix}
1 & -1 & 5 \\
-1 & 1 & 13 \\
0 & 0 & -3
\end{bmatrix}$.

5. [Cayley-Hamilton] Show that the characteristic equation of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$
\lambda^2 - (a + d)\lambda + (ad - bc) = 0.
$$

Show also that $A$ satisfies the matrix equation $A^2 - (a + d)A + (ad - bc)I_2 = 0_2$, where $I_2$ and $0_2$ are the $2 \times 2$ identity and zero matrices respectively.

6. Let $A$ and $P$ be $n \times n$ matrices, with $P$ invertible. Show that $A$ and $PAP^{-1}$ have the same characteristic equation. (Use the product rule $\det(XY) = (\det X)(\det Y)$.)
7. [Cramer’s rule]

(i) Given a $3 \times 3$ matrix $A$ and a $3 \times 1$ column vector $b$, show that

$$(\text{adj } A)b = \begin{bmatrix}
\text{det}(A_1) \\
\text{det}(A_2) \\
\text{det}(A_3)
\end{bmatrix}$$

where $A_i$ is the matrix obtained from $A$ by replacing column $i$ by $b$.

(ii) Suppose that $A$ is invertible and then show that the solution to the matrix equation $Ax = b$, where $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix}
\text{det}(A_1)/\text{det}(A) \\
\text{det}(A_2)/\text{det}(A) \\
\text{det}(A_3)/\text{det}(A)
\end{bmatrix}.$$  

(iii) Use (ii) to solve the following equations:

\begin{align*}
x + 2y + 2z &= 5 \\
x + 3y + z &= 0 \\
x + 3y + 2z &= -2
\end{align*}

Which method of solving equations do you prefer: using row operations or Cramer’s rule?

8. The Hessian of a function $u(x_1, x_2)$ of two variables is the determinant of the matrix

$$\begin{bmatrix}
\frac{\partial^2 u}{\partial x_i \partial x_j}
\end{bmatrix}$$

whose $(i,j)$-th entry is $\frac{\partial^2 u}{\partial x_i \partial x_j}$. Find the Hessian of $ax_1^2 + bx_1x_2 + cx_2^2$.

Answers to Preparatory Questions

1. The determinant of the given matrix is $(\lambda - 1)(\lambda - 4)$. This is 0 when $\lambda = 1$ or $\lambda = 4$ and so these are the values for which the matrix is not invertible.

2. The calculation of the preceding exercise shows that the eigenvalues of $A$ are 1 and 4.

3. When $\lambda = 1$ an eigenvector is any non-zero multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. When $\lambda = 4$ an eigenvector is any non-zero multiple of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at