Assignment A

Due 5pm Thursday 4th June, in Carslaw Room 623.

This assignment is worth 4 A marks.

1. The system of equations

\[
\begin{align*}
-2x + 5y - 1z &= 11 \\
3x + 2y + 8z &= 9 \\
6x + y + 2z &= -2
\end{align*}
\]

is to be solved using a convergent Gauss-Seidel iteration scheme.

(a) The scheme can be written in the matrix form,

\[
(L + D)x^{(k+1)} = b - Ux^{(k)},
\]

where \(L, D\) and \(U\) are the lower, diagonal and upper parts of a diagonally-dominant coefficient matrix \(A\), and \(b\) is the corresponding right-hand side. What are \(L + D\) and \(U\) (for the convergent scheme)?

Show that the scheme can be written in the form

\[
x^{(k+1)} = c + Sx^{(k)},
\]

where

\[
S = \frac{1}{240} \begin{pmatrix} 0 & -40 & -80 \\ 0 & -16 & 16 \\ 0 & 19 & 26 \end{pmatrix}.
\]

Find \(c\).

(b) Show from Part(a) that the error \(e^{(k)} = x^* - x^{(k)}\), where \(x^*\) is the exact solution of the iteration scheme, satisfies

\[
e^{(k+1)} = Se^{(k)}.
\]

Hence by evaluating a suitable norm of \(S\) prove that the iteration scheme must converge irrespective of the initial value.

(c) The previous part assumes that the solution exists. Show that for non-negative integers \(k, \ell\),

\[
x^{(k+\ell)} - x^{(k)} = S^k (I + S + \cdots + S^{\ell-1})(x^{(1)} - x^{(0)}).
\]

Deduce that, if \(\|S\| < 1\), then

\[
\|x^{(k+\ell)} - x^{(k)}\| \leq \frac{\|S\|^k}{1 - \|S\|} \|x^{(1)} - x^{(0)}\|,
\]

and hence that a solution exists. (Hint: the scheme generates a Cauchy sequence of vectors if \(\|S\| < 1\).)
2. The 2-point Gauss rule for the weight $e^{-x}$ on the interval $[0, \infty]$ is
\[
\int_0^\infty f(x)e^{-x} \, dx \approx w_1 f(x_1) + w_2 f(x_2).
\]

(a) Using the Gram-Schmidt procedure determine the orthogonal polynomials $\phi_0(x)$, $\phi_1(x)$ and $\phi_2(x)$ for the weight function $w(x) = e^{-x}$ in $0 \leq x \leq \infty$. Find also $A_0, A_1, A_2, \gamma_0, \gamma_1$. Suggestion: show
\[
\int_0^\infty x^n e^{-x} \, dx = n!, \quad n = 0, 1, 2, 3, \ldots.
\]
(The integral on the left is actually the Gamma (or factorial) function $\Gamma(n+1)$. $\phi_n(x)$ is the $n$-degree Laguerre polynomial.)

(b) Determine the nodes $x_1$ and $x_2$ for the 2-point weighted Gauss rule from $\phi_2(x)$.

(c) Using the recurrence relation
\[
\phi_{k+1}(x) - (a_k x + b_k) \phi_k(x) + c_k \phi_{k-1}(x) = 0
\]
given in the notes show that the weights
\[
w_k = -\frac{A_{n+1}\gamma_n}{A_n \phi_{n+1}(x_k) \phi_n'(x_k)}
\]
can be expressed in the more useful form
\[
w_k = \frac{A_n \gamma_{n-1}}{A_{n-1} \phi_{n-1}(x_k) \phi_n'(x_k)}.
\]
Hence determine the weights $w_1$ and $w_2$ for the 2-point weighted Gauss rule.

3. Consider the initial value problem
\[
\frac{dy}{dt} = f(t, y), \quad y(0) = y_0.
\]

(a) By expanding the exact solution $y = Y(t)$ of the initial value problem in a two term Taylor series with remainder about $t = t_{k+1}$, show that
\[
Y_k = Y_{k+1} - h f(t_{k+1}, Y_{k+1}) + \frac{1}{2} Y''(\xi_k) h^2,
\]
where $Y_k \equiv Y(t_k$), etc., and $\xi_k$ lies between $t_k$ and $t_{k+1}$.

(b) By subtracting the equation derived in Part (a) from the backward-Euler method,
\[
y_{k+1} = y_k + h f(t_{k+1}, y_{k+1}),
\]
deduce that the global error $e_k$ in the backward-Euler method satisfies
\[
e_{k+1} = \frac{e_k}{1 - hJ} - \frac{Y''(\xi_k) h^2}{2(1 - hJ)},
\]
where $J = f_y(t_{k+1}, \eta_{k+1})$ and $\eta_{k+1}$ lies between $y_{k+1}$ and $Y_{k+1}$.

(c) Deduce from Part (b) that the backward-Euler method is stable, when applied to well-conditioned problems, if $h > 0$ or $h < 2/J < 0$. Thus the step-size $h > 0$ in the backward-Euler method is unrestricted by stability considerations and can be chosen for accuracy.