THE UNIVERSITY OF SYDNEY

MATH3076 AND MATH3976 (ADVANCED)

Mathematical Computing:	Numerical Methods	2011

Lecturer: *D. J. Ivers*

Assignment A

Due 5pm Thursday 2nd June in the boxes on Carslaw Level 6 opposite the lifts.

This assignment is worth 4 A marks.

1. The system of equations

$$\begin{aligned} -2x + 5y - 1z &= 11 \\ 3x + 2y + 8z &= 9 \\ 6x + y + 2z &= -2 \end{aligned}$$

is to be solved using a convergent Gauss-Seidel iteration scheme.

(a) The scheme can be written in the matrix form,

$$(\mathbf{L}+\mathbf{D})\mathbf{x}^{(k+1)} = \mathbf{b} - \mathbf{U}\mathbf{x}^{(k)} \,,$$

where $\mathbf{L}$, $\mathbf{D}$ and $\mathbf{U}$ are the lower, diagonal and upper parts of a diagonally-dominant coefficient matrix $\mathbf{A}$, and $\mathbf{b}$ is the corresponding right-hand side. What are $\mathbf{L}+\mathbf{D}$ and $\mathbf{U}$ (for the convergent scheme)?

Show that the scheme can be written in the form

$$\mathbf{x}^{(k+1)} = \mathbf{c} + \mathbf{S}\mathbf{x}^{(k)} \,,$$

where

$$\mathbf{S} = \frac{1}{240}\begin{pmatrix} 0 & -40 & -80 \\ 0 & -16 & 16 \\ 0 & 19 & 26 \end{pmatrix} .$$

Find $\mathbf{c}$.

(b) Show from Part(a) that the error $\mathbf{e}^{(k)} = \mathbf{x}^* - \mathbf{x}^{(k)}$, where $\mathbf{x}^*$ is the exact solution of the iteration scheme, satisfies

$$\mathbf{e}^{(k+1)} = \mathbf{S}\mathbf{e}^{(k)} \,.$$

Hence by evaluating a suitable norm of $\mathbf{S}$ prove that the iteration scheme must converge irrespective of the initial value.

(c) The previous part assumes that the solution exists. Show that for non-negative integers k, ℓ,

$$\mathbf{x}^{(k+\ell)} - \mathbf{x}^{(k)} = \mathbf{S}^k(\mathbf{I} + \mathbf{S} + \cdots + \mathbf{S}^{\ell-1})(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}) \,.$$

Deduce that, if $\|\mathbf{S}\| < 1$, then

$$\|\mathbf{x}^{(k+\ell)} - \mathbf{x}^{(k)}\| \le \frac{\|\mathbf{S}\|^k}{1 - \|\mathbf{S}\|}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \,,$$

and hence that a solution exists. (Hint: the scheme generates a Cauchy sequence of vectors if $\|\mathbf{S}\| < 1$.)

2. The 2-point Gauss rule for the weight e^{-x} on the interval $[0, \infty]$ is

$$\int_0^\infty f(x)e^{-x}\,dx \approx w_1 f(x_1) + w_2 f(x_2)\,.$$

(a) Using the Gram-Schmidt procedure determine the orthogonal polynomials $\phi_0(x)$, $\phi_1(x)$ and $\phi_2(x)$ for the weight function $w(x) = e^{-x}$ in $0 \le x \le \infty$. Find also A_0, A_1, A_2, γ_0, γ_1. Suggestion: show

$$\int_0^\infty x^n e^{-x}\,dx = n!\,, \qquad n = 0, 1, 2, 3, \ldots\,.$$

(The integral on the left is actually the Gamma (or factorial) function $\Gamma(n+1)$. $\phi_n(x)$ is the n-degree Laguerre polynomial.)

(b) Determine the nodes x_1 and x_2 for the 2-point weighted Gauss rule from $\phi_2(x)$.

(c) Using the recurrence relation

$$\phi_{k+1}(x) - (a_k x + b_k)\phi_k(x) + c_k\phi_{k-1}(x) = 0$$

given in the notes show that the weights

$$w_k = -\frac{A_{n+1}\gamma_n}{A_n\phi_{n+1}(x_k)\phi_n'(x_k)}$$

can be expressed in the more useful form

$$w_k = \frac{A_n\gamma_{n-1}}{A_{n-1}\phi_{n-1}(x_k)\phi_n'(x_k)}\,.$$

Hence determine the weights w_1 and w_2 for the 2-point weighted Gauss rule.

3. Consider the initial value problem

$$\frac{dy}{dt} = f(t, y)\,, \qquad y(0) = y_0\,.$$

(a) By expanding the exact solution $y = Y(t)$ of the initial value problem in a two term Taylor series with remainder about $t = t_{k+1}$, show that

$$Y_k = Y_{k+1} - hf(t_{k+1}, Y_{k+1}) + \tfrac{1}{2}Y''(\xi_k)h^2\,,$$

where $Y_k \equiv Y(t_k)$, etc., and ξ_k lies between t_k and t_{k+1}.

(b) By subtracting the equation derived in Part (a) from the backward-Euler method,

$$y_{k+1} = y_k + hf(t_{k+1}, y_{k+1})\,,$$

deduce that the global error e_k in the backward-Euler method satisfies

$$e_{k+1} = \frac{e_k}{1 - hJ} - \frac{Y''(\xi_k)h^2}{2(1 - hJ)}\,,$$

where $J = f_y(t_{k+1}, \eta_{k+1})$ and η_{k+1} lies between y_{k+1} and Y_{k+1}.

(c) Deduce from Part (b) that the backward-Euler method is stable, when applied to well-conditioned problems, if $h > 0$ or $h < 2/J < 0$. Thus the step-size $h > 0$ in the backward-Euler method is unrestricted by stability considerations and can be chosen for accuracy.