Math 3916/3076 Mathematical Computing: Numerical Methods

Solutions to Assignment A

1) (a) Re-order equations,

$$\begin{aligned} 6x + y + 2z &= -2 \\ -2x + 5y - z &= 11 \\ 3x + 2y + 8z &= 9 \end{aligned}$$

Re-arrange equations,

$$\begin{aligned} x^{(k+1)} &= (-2 - y^{(k)} - 2z^{(k)})/6 \\ y^{(k+1)} &= (11 + 2x^{(k+1)} + z^{(k)})/5 \\ z^{(k+1)} &= (9 - 3x^{(k+1)} - 2y^{(k+1)})/8 \end{aligned}$$

In matrix form, $\underline{x}^{(k+1)} = \underline{\underline{D}}^{-1}(\underline{b} - \underline{\underline{L}}\,\underline{x}^{(k+1)} - \underline{\underline{U}}\,\underline{x}^{(k)})$,
where

$$\underline{x}^{(k)} = \begin{bmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{bmatrix}, \quad \underline{\underline{D}} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} -2 \\ 11 \\ 9 \end{bmatrix}, \quad \underline{\underline{L}} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

$$\underline{\underline{U}} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \underline{\underline{A}} = \begin{bmatrix} 6 & 1 & 2 \\ -2 & 5 & -1 \\ 3 & 2 & 8 \end{bmatrix} = \underline{\underline{D}} + \underline{\underline{L}} + \underline{\underline{U}}.$$

Multiplying by $\underline{\underline{D}}$ and moving $\underline{\underline{L}}\,x^{(k+1)}$ to the LHS,

$$(\underline{\underline{D}} + \underline{\underline{L}})\,\underline{x}^{(k+1)} = \underline{b} - \underline{\underline{U}}\,\underline{x}^{(k)}$$

Multiplying by $(\underline{\underline{D}} + \underline{\underline{L}})^{-1}$ gives

$$\underline{x}^{(k+1)} = (\underline{\underline{D}} + \underline{\underline{L}})^{-1}\underline{b} - (\underline{\underline{D}} + \underline{\underline{L}})^{-1}\underline{\underline{U}}\,\underline{x}^{(k)} = \underline{c} + \underline{\underline{S}}\,\underline{x}^{(k)}$$

where

$$\underline{c} = (\underline{\underline{D}} + \underline{\underline{L}})^{-1}\underline{b}, \quad \underline{\underline{S}} = -(\underline{\underline{D}} + \underline{\underline{L}})^{-1}\underline{\underline{U}}$$

Since

$$(\underline{\underline{D}} + \underline{\underline{L}})^{-1} = \frac{1}{240}\begin{bmatrix} 40 & 0 & 0 \\ 16 & 48 & 0 \\ -19 & -12 & 30 \end{bmatrix}$$

$$\underline{c} = \frac{1}{240}\begin{bmatrix} 40 & 0 & 0 \\ 16 & 48 & 0 \\ -19 & -12 & 30 \end{bmatrix}\begin{bmatrix} -2 \\ 11 \\ 9 \end{bmatrix} = \frac{1}{240}\begin{bmatrix} -80 \\ 496 \\ 176 \end{bmatrix}; \quad \underline{\underline{S}} = \frac{1}{240}\begin{bmatrix} 0 & -40 & -80 \\ 0 & -16 & 16 \\ 0 & 19 & 26 \end{bmatrix}$$

) The exact solution satisfies

$$\underline{x}^* = \underline{c} + \underline{\underline{S}}\,\underline{x}^* \quad ①$$

Let

$$\underline{x}^{(k+1)} = \underline{c} + \underline{\underline{S}}\,\underline{x}^{(k)} \quad ②$$

$$② - ① \Rightarrow \quad \underline{e}^{(k+1)} = \underline{x}^* - \underline{x}^{(k+1)}$$
$$= (\underline{c} + \underline{\underline{S}}\,\underline{x}^*) - (\underline{c} + \underline{\underline{S}}\,\underline{x}^{(k)})$$
$$= \underline{\underline{S}}\,(\underline{x}^* - \underline{x}^{(k)})$$
$$= \underline{\underline{S}}\,\underline{e}^{(k)}$$

Iterating,

$$\underline{e}^{(k)} = \underline{\underline{S}}\,\underline{e}^{(k-1)}$$
$$= \underline{\underline{S}}^2\,\underline{e}^{(k-2)}$$
$$\vdots$$
$$= \underline{\underline{S}}^k\,\underline{e}^{(0)}$$

Taking norms,

$$\|\underline{e}^{(k)}\| = \|\underline{\underline{S}}^k\,\underline{e}^{(0)}\|$$
$$\leqslant \|\underline{\underline{S}}^k\|\ \|\underline{e}^{(0)}\|$$
$$\leqslant \|\underline{\underline{S}}\|^k\ \|\underline{e}^{(0)}\|$$
$$\to 0 \quad \text{as } k \to \infty,$$

since $\|\underline{\underline{S}}\|_\infty = \max\left(\frac{1}{2}, \frac{2}{15}, \frac{9}{48}\right) = \frac{1}{2} < 1$ or

$\|\underline{\underline{S}}\|_1 = \max\left(0, \frac{75}{240}, \frac{122}{240}\right) = \frac{61}{120} < 1$

So scheme converges.

c)

$$\underline{x}^{(k+1)} - \underline{x}^{(k)} = \underline{\underline{S}}\,(\underline{x}^{(k)} - \underline{x}^{(k-1)})$$
$$\underline{x}^{(k+2)} - \underline{x}^{(k+1)} = \underline{\underline{S}}^2(\underline{x}^{(k)} - \underline{x}^{(k-1)})$$
$$\vdots$$
$$\underline{x}^{(k+l)} - \underline{x}^{(k+l-1)} = \underline{\underline{S}}^l\,(\underline{x}^{(k)} - \underline{x}^{(k-1)})$$

Add & telescope rhs's $\Rightarrow$

$$\underline{x}^{(k+l)} - \underline{x}^{(k)} = (\underline{\underline{S}} + \underline{\underline{S}}^2 + \ldots + \underline{\underline{S}}^l)(\underline{x}^{(k)} - \underline{x}^{(k-1)})$$
$$= \underline{\underline{S}}^{k-1}(\underline{\underline{S}} + \underline{\underline{S}}^2 + \ldots + \underline{\underline{S}}^l)(\underline{x}^{(1)} - \underline{x}^{(0)})$$
$$= \underline{\underline{S}}^k(\underline{\underline{I}} + \underline{\underline{S}} + \ldots + \underline{\underline{S}}^{l-1})(\underline{x}^{(1)} - \underline{x}^{(0)})$$

$$\|\underline{x}^{(k+l)} - \underline{x}^{(k)}\| \leqslant \|\underline{\underline{S}}\|^k\,(\|\underline{\underline{I}}\| + \|\underline{\underline{S}}\| + \ldots + \|\underline{\underline{S}}\|^{l-1})\ \|\underline{x}^{(1)} - \underline{x}^{(0)}\|$$
$$= \|\underline{\underline{S}}\|^k\ \frac{(1 - \|\underline{\underline{S}}\|^l)}{1 - \|\underline{\underline{S}}\|}\ \|\underline{x}^{(1)} - \underline{x}^{(0)}\|, \quad \|\underline{\underline{I}}\| = 1$$
$$\leqslant \frac{\|\underline{\underline{S}}\|^k}{1 - \|\underline{\underline{S}}\|}\|\underline{x}^{(1)} - \underline{x}^{(0)}\|, \quad \text{since } \|\underline{\underline{S}}\| < 1.$$
$$\to 0 \quad \text{as } k \to \infty.$$

NOTE $\underline{x}^*$ satisfies ① – take limit of ② as $k \to \infty$.

Thus $\{\underline{x}^{(k)}\}$ is a Cauchy sequence. Since the set of $n=3$ dimensional vectors $\mathbb{R}^3$ is complete, the sequence $\{\underline{x}^{(k)}\}$ has a limit $\underline{x}^*$.

Greek gamma uppercase

2) (a) $\int_0^\infty x^n e^{-x}\,dx = \Gamma(n+1)$

$= [-x^n e^{-x}]_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\,dx,$

integrating by parts. $[-x^n e^{-x}]_0^\infty = 0$ if $n \geq 1$. Thus

$\Gamma(n+1) = n \int_0^\infty x^{n-1} e^{-x}\,dx = n\Gamma(n)$

By induction (iterating), $\Gamma(n+1) = n!\,\Gamma(1)$. But

$\Gamma(1) = \int_0^\infty e^{-x}\,dx = [-e^{-x}]_0^\infty = 1$. Thus

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx = n!.$$

Use the inner-product, $\langle f, g\rangle = \int_0^\infty e^{-x} f g\,dx$

$\phi_0 = 1$, $\langle \phi_0, \phi_0\rangle = \int_0^\infty e^{-x}\,dx = \Gamma(1) = 1$

$\phi_1 = x - \frac{\langle x, \phi_0\rangle}{\langle \phi_0, \phi_0\rangle}\phi_0$, $\langle x, \phi_0\rangle = \int_0^\infty e^{-x} x\,dx = \Gamma(2) = 1$

$= x - \frac{1}{1}\cdot 1$

$= x - 1$

$\phi_2 = x^2 - \frac{\langle x^2, \phi_0\rangle}{\langle \phi_0, \phi_0\rangle}\phi_0 - \frac{\langle x^2, \phi_1\rangle}{\langle \phi_1, \phi_1\rangle}\phi_1$

$\langle x^2, \phi_0\rangle = \int_0^\infty e^{-x} x^2\,dx = \Gamma(3) = 2! = 2$

$\langle x^2, \phi_1\rangle = \int_0^\infty e^{-x} x^2(x-1)\,dx = \Gamma(4) - \Gamma(3) = 3! - 2! = 4$

$\langle \phi_1, \phi_1\rangle = \int_0^\infty e^{-x}(x-1)^2\,dx = \int_0^\infty e^{-x}(x^2 - 2x + 1)\,dx$

$= \Gamma(3) - 2\Gamma(2) + \Gamma(1) = 2! - 2.1! + 1 = 1$

Thus

$\phi_2 = x^2 - \frac{2}{1}1 - \frac{4}{1}(x-1) = x^2 - 4x + 2$

(b) The nodes of the 2-point Gauss(-Laguerre) rule are the roots of ϕ_2, i.e.

$x^2 - 4x + 2 = 0, \quad x_{1,2} = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$

i.e. $x_1 = 2 - \sqrt{2}, \quad x_2 = 2 + \sqrt{2}.$

c) $\phi_{n+1}(x_k) = (a_n x_k + b_n)\phi_n(x_k) - c_n \phi_{n-1}(x_k)$, using the recurrence relation

$= - c_n \phi_{n-1}(x_k)$, since $\phi_n(x_k) = 0$ if x_k is a zero of ϕ_n

Thus $$w_k = -\frac{A_{n+1}\gamma_n}{A_n \phi_{n+1}(x_k)\phi_n'(x_k)} = \frac{A_{n+1}\gamma_n}{A_n c_n \phi_{n-1}(x_k)\phi_n'(x_k)}$$

$$= \frac{A_{n+1}\gamma_n}{A_n \phi_{n-1}(x_k)\phi_n'(x_k)} \frac{A_n^2 \gamma_{n-1}}{A_{n+1}A_{n-1}\gamma_n}$$, substituting for c_n from the Lecture Notes

$$= \frac{A_n \gamma_{n-1}}{A_{n-1}\phi_{n-1}(x_k)\phi_n'(x_k)}$$

Here $n=2$

$A_1 = A_2 = 1$, $\gamma_1 = \langle \phi_1, \phi_1 \rangle = 1$

$\phi(x_k) = x_k - 1$, $\phi_2'(x_k) = \frac{d}{dx}(x-x_1)(x-x_2)\Big|_{x=x_k}$

$$= \begin{cases} x_1 - x_2, & x = x_1 \\ x_2 - x_1, & x = x_2 \end{cases}$$

$x_1 - x_2 = -2\sqrt{2}$

$$w_1 = \frac{A_2\gamma_1}{A_1\phi_1(x_1)\phi_2'(x_1)} = \frac{1}{(2-\sqrt{2}-1)(-2\sqrt{2})} = \frac{1}{2\sqrt{2}(\sqrt{2}-1)}$$
$$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

$$w_2 = \frac{A_2\gamma_1}{A_1\phi_1(x_2)\phi_2'(x_2)} = \frac{1}{(2+\sqrt{2}-1)(2\sqrt{2})} = \frac{1}{2\sqrt{2}(\sqrt{2}+1)}$$
$$= \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

3 (a) $\quad Y(t) = Y(t_{k+1}) + Y'(t_{k+1})(t - t_{k+1}) + \frac{1}{2} Y''(\xi_k)(t - t_{k+1})^2,$

where ξ_k lies between t & t_{k+1}. Set $t = t_k$,

$$Y_k = Y_{k+1} - h f(t_{k+1}, Y_{k+1}) + \frac{1}{2} Y''(\xi_k) h^2,$$

since $Y_k \equiv Y(t_k)$, etc, $Y'_{k+1} = f(t_{k+1}, Y_{k+1})$ as Y satisfies the d.e. and $h = t_{k+1} - t_k$.

(b) Subtract the backward Euler method from the Taylor expansion of Y_k in Part (a),

$$Y_k - y_k = Y_{k+1} - y_{k+1} - h\left[f(t_{k+1}, Y_{k+1}) - f(t_{k+1}, y_{k+1})\right] + \frac{1}{2} Y''(\xi_k) h^2$$

Let $Y_k - y_k = e_k$ etc. Then by the MVT,

$$e_k = e_{k+1} - h f_y(t_{k+1}, y_{k+1}) e_{k+1} + \frac{1}{2} Y''(\xi_k) h^2$$

Hence

$$e_{k+1} = \frac{e_k}{1 - hJ} - \frac{\frac{1}{2} Y''(\xi_k) h^2}{1 - hJ},$$

where $J \equiv f_y(t_{k+1}, y_{k+1})$.

(c) Backward Euler is stable if

$$\left|\frac{1}{1 - hJ}\right| < 1$$

$$\Leftrightarrow \quad |1 - hJ| > 1$$

$$\Leftrightarrow \quad -1 > 1 - hJ > 1$$

$$\Leftrightarrow \quad hJ < 0 \quad \text{or} \quad hJ > 2$$

If problem is well-conditioned, $J < 0 \Rightarrow$

$$h > 0 \quad \text{or} \quad h < 2/J < 0$$

for stability. Thus in the forward direction ($h > 0$) backward Euler is stable for any problem, even stiff ones ($|J| \gg 1$, $J < 0$), unlike Euler's method.