Sample Exam Section B: Numerical Methods

All questions may be attempted by all students.
*Questions tagged by * are Advanced.*

1. (a) What are the UFL, OFL and ϵ_{mach} of the floating point number system with rounding,

$$\{\pm d_1.d_2d_3\times 10^{\pm p}\,|\,d_2, d_3, p = 0, \ldots, 9;\; d_1 = 1, \ldots, 9, \text{ unless } d_1 = d_2 = d_3 = 0\}\,.$$

(b) Give an example of an *ill-conditioned* problem.

(c) By expanding $f(x+h)$ in a Taylor series in h to three terms including a remainder, find an expression for the truncation error e^T in the formula

$$f'(x) = \frac{f(x+h) - f(x)}{h} + e^T\,.$$

2. (a) Consider the roots of the equation $f(x) = 0$. Using the first two terms of the Taylor expansion of $f(x)$ about x_i derive the Newton-Raphson iteration formula. Give one advantage and one disadvantage of the Newton-Raphson method over the secant method.

Write down the Newton-Raphson iteration formula for n nonlinear equations in n variables, carefully explaining any notation you use.

(b) Show that the Newton-Raphson iteration scheme for the function

$$f(x) = \frac{1}{x} - a\,,$$

where a is a constant, is given by

$$x_{n+1} = x_n(2 - ax_n)\,.$$

What is the root of f? Do two iterations of the Newton-Raphson scheme for $a = 3$ and starting with $x_0 = 0.3$ to find x_1 and x_2. Estimate how many iterations would be needed to determine the root to 16 decimal places?

3. (a) Solve the system of equations

$$\begin{aligned} 2x + 4y - z &= 1 \\ 4x + y + z &= -2 \\ 2x - 3y + 6z &= 1 \end{aligned}$$

using the Doolittle form of Gaussian elimination with back-substitution and without row interchanges. Write down the LU-factorisation of the coefficient matrix $\mathbf{A}$.

Given that the inverse of $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \frac{1}{56}\begin{pmatrix} -9 & 21 & -5 \\ 22 & -14 & 6 \\ 14 & -14 & 14 \end{pmatrix}$$

determine the condition number of $\mathbf{A}$ using the norm $\| \ \|_1$. Suppose the above system is solved by Gaussian elimination with back-substitution on a computer with $\epsilon_{\text{mach}} = 10^{-7}$, what is the worst relative error $\|\mathbf{x} - \hat{\mathbf{x}}\|/\|\mathbf{x}\|$ you would expect in the computed solution $\hat{\mathbf{x}}$?

(b) Apply a convergent Gauss-Seidel iteration scheme to the system of equations in Part (a). Start with $x^{(0)} = y^{(0)} = z^{(0)} = 0$, and do one iteration to find $x^{(1)}$, $y^{(1)}$, $z^{(1)}$.

(c)* The Gauss-Seidel iteration scheme can be written in the form

$$\mathbf{x}^{(k+1)} = \mathbf{c} + \mathbf{S}\mathbf{x}^{(k)} ,$$

where

$$\mathbf{c} = \frac{1}{12}\begin{pmatrix} -6 \\ 6 \\ 7 \end{pmatrix}, \qquad \mathbf{S} = \frac{1}{48}\begin{pmatrix} 0 & -12 & -12 \\ 0 & 6 & 18 \\ 0 & 7 & 13 \end{pmatrix}.$$

Show that the error $\mathbf{e}^{(k)} = \mathbf{x}^* - \mathbf{x}^{(k)}$, where $\mathbf{x}^*$ is the exact solution of the iteration scheme, satisfies

$$\mathbf{e}^{(k+1)} = \mathbf{S}\mathbf{e}^{(k)} .$$

Hence by evaluating $\|\mathbf{S}\|_\infty$ prove that the iteration scheme converges.

4. (a) A 2-point Gauss rule for integrals with weight function $x^{-1/3}$ has the form

$$\int_0^1 f(x)x^{-1/3}\,dx = w_1 f(x_1) + w_2 f(x_2)\,.$$

State the conditions which determine the weights $w_1,\ w_2$ and the nodes $x_1,\ x_2$ and hence derive the equations which determine them. Given the weights and nodes,

$$w_1 = \frac{3(25+\sqrt{20})}{100} \approx 0.884164\,, \quad w_2 = \frac{3(25-\sqrt{20})}{100} \approx 0.615836$$
$$x_1 = \frac{5(1-3/\sqrt{20})}{11} \approx 0.149627\,, \quad x_2 = \frac{5(1+3/\sqrt{20})}{11} \approx 0.759464\,,$$

use the method to evaluate

$$\int_0^1 x^{-1/3}\cos x\,dx\,.$$

(b)* For the weight function in Part (a), the orthogonal polynomials on $[0,1]$ of degree ≤ 2 are

$$\phi_0(x) = 1\,, \quad \phi_1(x) = x - \frac{2}{5}\,, \quad \phi_2(x) = x^2 - \frac{40}{44}x + \frac{5}{44}\,.$$

Give the Gram-Schmidt formula for constructing the orthogonal polynomial $\phi_n(x)$ of degree n, if $\phi_0(x),\ldots,\phi_{n-1}(x)$ have already been constructed. Deduce the nodes of the 1-point Gauss rule and the 2-point Gauss rule for the weight function $x^{-1/3}$ on $[0,1]$.

5. Consider the initial value problem, $dy/dt = f(t,y)$, $y(t_0) = y_0$, $t_0 \leq t \leq T$.

(a) Second-order Runge-Kutta methods are of the form

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf(t_n + \alpha h, y_n + \beta k_1) \\ y_{n+1} &= y_n + ak_1 + bk_2 \\ a+b=1\,, &\qquad \alpha b = 1/2\,, \qquad \beta b = 1/2\,. \end{aligned}$$

Do one step of the second-order Runge-Kutta method with $b = 1$ to approximate $y(1.2)$, where $y(t)$ is the solution of

$$\frac{dy}{dt} = t - y^2, \qquad y(1) = 1\,.$$

Is this problem well-conditioned?

(b) By comparing the solution of Euler's method, $y_{k+1} = y_k + hf(t_k, y_k)$ with the Taylor expansion of the exact solution $Y(t)$ about t_k,

$$Y_{k+1} = Y_k + hf(t_k, Y_k) + \frac{1}{2}h^2 Y''(\xi_k)\,,$$

where ξ_k lies between t_k and t_{k+1}, show that the error $e_{k+1} = Y_{k+1} - y_{k+1}$ in y_{k+1} is given by $e_{k+1} = (1 + hf_y(t_k, \eta))e_k + \frac{1}{2}h^2 Y''(\xi_k)$, where η lies between y_k and Y_k. Hence deduce the condition on h for Euler's method to be stable when applied to a well-conditioned initial value problem.

Show that if Euler's method is applied to

$$\frac{dy}{dt} = -C(y - \sin t) + \cos t\,, \qquad y(0) = 1\,, \quad 0 \leq t \leq \pi\,,$$

where $C > 0$, the step-size h must satisfy $0 < h < 2/C$ for stability.