Sample Exam Section B: Numerical Methods

All questions may be attempted by all students. Questions tagged by * are Advanced.

1. (a) What are the UFL, OFL and $\epsilon_{\text{mach}}$ of the floating point number system with rounding,

$$\{ \pm d_1.d_2d_3 \times 10^{\pm p} \mid d_2, d_3, p = 0, \ldots, 9; d_1 = 1, \ldots, 9, \text{ unless } d_1 = d_2 = d_3 = 0 \} .$$

(b) Give an example of an ill-conditioned problem.

(c) By expanding $f(x + h)$ in a Taylor series in $h$ to three terms including a remainder, find an expression for the truncation error $e^T$ in the formula

$$f'(x) = \frac{f(x + h) - f(x)}{h} + e^T .$$

2. (a) Consider the roots of the equation $f(x) = 0$. Using the first two terms of the Taylor expansion of $f(x)$ about $x_i$ derive the Newton-Raphson iteration formula. Give one advantage and one disadvantage of the Newton-Raphson method over the secant method.

Write down the Newton-Raphson iteration formula for $n$ nonlinear equations in $n$ variables, carefully explaining any notation you use.

(b) Show that the Newton-Raphson iteration scheme for the function

$$f(x) = \frac{1}{x} - a ,$$

where $a$ is a constant, is given by

$$x_{n+1} = x_n (2 - ax_n) .$$

What is the root of $f$? Do two iterations of the Newton-Raphson scheme for $a = 3$ and starting with $x_0 = 0.3$ to find $x_1$ and $x_2$. Estimate how many iterations would be needed to determine the root to 16 decimal places?
3. (a) Solve the system of equations

\[
\begin{align*}
2x + 4y - z &= 1 \\
4x + y + z &= -2 \\
2x - 3y + 6z &= 1
\end{align*}
\]

using the Doolittle form of Gaussian elimination with back-substitution and without row interchanges. Write down the \(LU\)-factorisation of the coefficient matrix \(A\).

Given that the inverse of \(A\) is

\[
A^{-1} = \frac{1}{56} \begin{pmatrix}
-9 & 21 & -5 \\
22 & -14 & 6 \\
14 & -14 & 14
\end{pmatrix}
\]
determine the condition number of \(A\) using the norm \(\| \cdot \|_1\). Suppose the above system is solved by Gaussian elimination with back-substitution on a computer with \(\epsilon_{\text{mach}} = 10^{-7}\), what is the worst relative error \(\|x - \hat{x}\|/\|x\|\) you would expect in the computed solution \(\hat{x}\) ?

(b) Apply a convergent Gauss-Seidel iteration scheme to the system of equations in Part (a). Start with \(x^{(0)} = y^{(0)} = z^{(0)} = 0\), and do one iteration to find \(x^{(1)}, y^{(1)}, z^{(1)}\).

(c)* The Gauss-Seidel iteration scheme can be written in the form

\[
x^{(k+1)} = c + Sx^{(k)},
\]

where

\[
c = \frac{1}{12} \begin{pmatrix}
-6 \\
6 \\
7
\end{pmatrix}, \quad S = \frac{1}{48} \begin{pmatrix}
0 & -12 & -12 \\
0 & 6 & 18 \\
0 & 7 & 13
\end{pmatrix}.
\]

Show that the error \(e^{(k)} = x^* - x^{(k)}\), where \(x^*\) is the exact solution of the iteration scheme, satisfies

\[
e^{(k+1)} = Se^{(k)}.
\]

Hence by evaluating \(\|S\|_\infty\) prove that the iteration scheme converges.
4. (a) A 2-point Gauss rule for integrals with weight function $x^{-1/3}$ has the form

$$\int_0^1 f(x)x^{-1/3} \, dx = w_1 f(x_1) + w_2 f(x_2).$$

State the conditions which determine the weights $w_1$, $w_2$ and the nodes $x_1$, $x_2$ and hence derive the equations which determine them. Given the weights and nodes,

$$w_1 = \frac{3(25 + \sqrt{20})}{100} \approx 0.884164, \quad w_2 = \frac{3(25 - \sqrt{20})}{100} \approx 0.615836$$

$$x_1 = \frac{5(1-3/\sqrt{20})}{11} \approx 0.149627, \quad x_2 = \frac{5(1+3/\sqrt{20})}{11} \approx 0.759464,$$

use the method to evaluate

$$\int_0^1 x^{-1/3} \cos x \, dx.$$

(b)* For the weight function in Part (a), the orthogonal polynomials on $[0,1]$ of degree $\leq 2$ are

$$\phi_0(x) = 1, \quad \phi_1(x) = x - \frac{2}{5}, \quad \phi_2(x) = x^2 - \frac{40}{44}x + \frac{5}{44}.$$

Give the Gram-Schmidt formula for constructing the orthogonal polynomial $\phi_n(x)$ of degree $n$, if $\phi_0(x), \ldots, \phi_{n-1}(x)$ have already been constructed. Deduce the nodes of the 1-point Gauss rule and the 2-point Gauss rule for the weight function $x^{-1/3}$ on $[0,1]$.

5. Consider the initial value problem, $\frac{dy}{dt} = f(t, y), y(t_0) = y_0, t_0 \leq t \leq T$.

(a) Second-order Runge-Kutta methods are of the form

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf(t_n + \alpha h, y_n + \beta k_1)$$

$$y_{n+1} = y_n + ak_1 + bk_2$$

$$a + b = 1, \quad \alpha b = 1/2, \quad \beta b = 1/2.$$

Do one step of the second-order Runge-Kutta method with $b = 1$ to approximate $y(1.2)$, where $y(t)$ is the solution of

$$\frac{dy}{dt} = t - y^2, \quad y(1) = 1.$$

Is this problem well-conditioned?
(b) By comparing the solution of Euler’s method, \( y_{k+1} = y_k + hf(t_k, y_k) \) with the Taylor expansion of the exact solution \( Y(t) \) about \( t_k \),

\[
Y_{k+1} = Y_k + hf(t_k, Y_k) + \frac{1}{2} h^2 Y''(\xi_k),
\]

where \( \xi_k \) lies between \( t_k \) and \( t_{k+1} \), show that the error \( e_{k+1} = Y_{k+1} - y_{k+1} \) in \( y_{k+1} \) is given by \( e_{k+1} = (1 + hf_y(t_k, \eta))e_k + \frac{1}{2} h^2 Y''(\xi_k), \) where \( \eta \) lies between \( y_k \) and \( Y_k \). Hence deduce the condition on \( h \) for Euler’s method to be stable when applied to a well-conditioned initial value problem.

Show that if Euler’s method is applied to

\[
\frac{dy}{dt} = -C(y - \sin t) + \cos t, \quad y(0) = 1, \quad 0 \leq t \leq \pi,
\]

where \( C > 0 \), the step-size \( h \) must satisfy \( 0 < h < 2/C \) for stability.