More background on chain conditions

Throughout $M$ is an $A$-module.

Call $M$ Noetherian [Artinian] if it satisfies the a.c.c. (ascending chain condition).

[\textit{a.c.c.} (descending "\textit{\ldots}" )]

i.e.,

$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$

$[ M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_n \supsetneq \cdots ]$

always stabilizes, that is,

$(\exists N) (\forall k \geq N) M_k = M_N$

Call a ring $A$ Noetherian [Artinian] if it is with respect to ideals (regarded as submodules).

Examples:

1. Finite abelian groups are both Noetherian and Artinian as $\mathbb{Z}$-modules.
2. $\mathbb{Z}$ and $\mathbb{F}(x)$ are Noetherian but not Artinian.
(3) $\mathbb{Z}$ subgroup $\mathbb{Z} \times (\mathbb{Q}, +)$ and

$$\mathbb{Q}/\mathbb{Z} = \{ \frac{q}{Z} \mid q \in \mathbb{Q} \}$$

is an additive abelian group (Z-module).

For $i \geq 0$, put

$$G_i = \{ \frac{a}{2^i} + \mathbb{Z} \mid a \in \mathbb{Z} \}$$

and

$$G = \bigcup_{i=0}^{\infty} G_i$$

Then

$$G_0 \subset G_1 \subset G_2 \subset G_3 \subset \ldots$$

is an infinite ascending chain, so $G$ is not Noetherian as a $\mathbb{Z}$-module, but it is Artinian.

$L(\mathbb{Q})$: $G_0 \supset G_1 \supset \ldots$

Easy exercise: there are all the subgroups of $G$
(4) Let $H = \{ \frac{a}{2^n} \mid n \geq 0, a \in \mathbb{Z} \}$

so

$0 \rightarrow \mathbb{Z} \rightarrow H \rightarrow \mathbb{Z}/2 \rightarrow 0$

exact sequence. Thus

$H$ is neither Noetherian nor Artinian.

became $\mathbb{Z}$

became $\mathbb{Z}/2$

two "bad" intervals in $\mathcal{L}(H)$

$\mathcal{L}(\mathbb{Z})$

$\mathcal{L}(H)$
(5) \( \mathbb{F}[x_1, x_2, \ldots] \) not Noetherian because of

\[ \langle x_1 \rangle \not\subseteq \langle x_1, x_2 \rangle \not\subseteq \cdots \not\subseteq \langle x_1, \ldots, x_n \rangle \not\subseteq \cdots \]

and not Artinian because \( \mathbb{F}[x] \) isn't.

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**General Observation 1**: Let \( M \) be an \( A \)-module. Then \( M \) is Noetherian if every submodule is finitely generated.

**Proof**: (\( \Rightarrow \)) Easy.

(\( \Leftarrow \)) Suppose every submodule is f.g. and

\[ M_1 \leq M_2 \leq \cdots \leq M_n \leq \cdots \quad (\ast) \]

Put

\[ M' = \bigcup_{i=1}^{\infty} M_i \leq M \]

So

\[ M' = \langle x_1, \ldots, x_n \rangle \quad \text{for some } x_1, \ldots, x_n \in M \]

Then

\[ (\exists N) \quad n_1, \ldots, N \in \mathbb{N} \]

so \( M_1 \leq \mathbb{M}_N \leq M \), so \( M' = \mathbb{M}_N \) and (\( \ast \)) stabilizes.
General Observation 2: Let

\[ 0 \to M' \to M \to M'' \to 0 \]

be exact. Then \( M \) is Noetherian

iff both \( M' \) and \( M'' \) are Noetherian.

Proof: (\( \Rightarrow \)) Easy.

(\( \Leftarrow \)) Suppose both \( M' \) and \( M'' \) are Noetherian

and

\[ l_1 \leq l_2 \leq \ldots \leq l_n \leq \ldots \] \((\ast)\)

Then \( w = (x_i) \) stabilizes.

\[ \exists M \ni \begin{array}{c}
    w = (x_i) \\
    w = (x_i) \\
    \vdots \\
\end{array} \]

Then

\[ x'_1(l_1) \leq x'_2(l_2) \leq \ldots \leq x'_n(l_n) \leq \ldots \]

in \( M' \)

and

\[ \rho(l_1) \leq \rho(l_2) \leq \ldots \leq \rho(l_n) \leq \ldots \]

in \( M'' \)

both of which stabilize, so

\[ (\exists N)(\forall k \geq N) \quad x'_1(l_k) = x'_2(l_k), \quad \rho(l_k) = \rho(l_k). \]
Claim: \( L_k = L_N \quad \forall k \in \mathbb{N} \)

Suffices to show \( L_k \subseteq L_N \), so let \( x \in L_k \).

Then \( \beta(x) \in \beta(L_k) = \beta(L_N) \)

so \( \beta(x) = \beta(y) \quad \exists y \in L_N \)

so \( x - y = k \beta \beta = 0 \) 

\[ \text{by contradiction} \]

and \( x - y \in L_k \quad (\text{since } L_N \subseteq L_k) \)

so \( x - y = x(y) : \exists z \in \mathbb{Z}(L_k) = \mathbb{Z}(L_N) \)

\[ x - y \in L_N \]

Hence \( x = x - y + y \in L_N \).

Thus \( L_k \subseteq L_N \subseteq L_k \), so \( L_k = L_N \).

\( \square \)
Corollary 3: Homomorphic images of Noetherian rings are Noetherian.

Proof: If $I \triangleleft A$ then

$$0 \to I \to A \to A/I \to 0$$

is exact. \(\square\)

Corollary 4: If $M_1, \ldots, M_n$ are Noetherian then $M_1 \oplus \cdots \oplus M_n$.

Proof: $M_1 \oplus \cdots \oplus M_n$ is an iterated split extension. \(\square\)

Theorem 5: Let $A$ be a Noetherian ring and $M$ a finitely generated $A$-module.

Then $M$ is Noetherian.

Proof: $M \cong A^n/N$ ($\exists n$) ($\exists N \subseteq A^n$)

Free module homomorphic image