Assignment Exercise: Prove

\[(M \otimes N) \otimes P \cong (M \otimes P) \otimes (N \otimes P)\]

Getting started: want

\[(m, n, p) \mapsto (M \otimes N) \otimes P\]

\[\begin{align*}
(M \otimes N) \times P & \quad \xrightarrow{\text{bilinear}} \quad (M \otimes N) \otimes P \\
(M \otimes P) \otimes (N \otimes P) & \quad \xrightarrow{\text{iso}} \quad (M \otimes P, N \otimes P)
\end{align*}\]

Look for \(h\) s.t.

\[h \circ f = 1_{(M \otimes N) \otimes P}\]

\[f \circ h = 1_{(M \otimes P) \otimes (N \otimes P)}\]
\((M \otimes P) \oplus (N \otimes P) \Rightarrow (M \odot N) \otimes P\)

build up in stages:

Find

\[ M \otimes P \xrightarrow{h_1} (M \odot N) \otimes P \]
\[ N \otimes P \xrightarrow{h_2} \]

& glue them together

\[ h(\alpha, \beta) = h_1(\alpha) + h_2(\beta) \]
Restriction & extension of scalars

- moving about between rings of scalars

\[ f \]

\[ A \rightarrow B \] ring hom

If \( N \) is a \( B \)-module then we can regard it as an \( A \)-module, by composing maps:

\[ f \]

\[ A \rightarrow B \]

\[ \text{scalar mult.} \]

\[ \text{End}_N \]

i.e. \( (\lambda \cdot a)(x \cdot n) \)

\[ an = f(a) \cdot x \]

being defined

\[ B \text{-module scalar mult.} \]

\& module axioms hold
Classical setting: $K$ subfield of $F$

\[ K \subseteq F \]

$V$ vector space over $F$, also over $K$ ("replacing scalar $K$")

Call $F$ a field extension of $K$ and write

\[ F : K \]

Put \[ [F : K] = \text{dimension of } F \text{ as a vector space over } K \]

If $F : K$ and $K : L$ are field extensions then

\[ [F : L] = [F : K][K : L] \]

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"spanning half panel in printed notes in more general setting"
Prop: If $N$ is f.g. as a module over $B$ and $B$ is f.g. over $A$, then $N$ is f.g. over $A$.

How do we regard $M$ as a $B$-module?!

No obvious composition of maps.
First, regard both $M$ and $B$ as $A$-modules.

Form

$$M_B = B \otimes_A M$$

as an $A$-module.

Use $B$ as a "buffer" to regard $M_B$ as an $B$-module.

$$b'(b \otimes m) = (b'b) \otimes m$$

being defined on generators.

Her extent by linearity.

Issue of well-definiteness! \\
- since $b \otimes m$ is an equivalence class
  $$(b, m) + B$$
For given \( b' \):

\[
\begin{align*}
B \times M & \xrightarrow{h} B \otimes_A M \\
& \xrightarrow{h'} (b' b) \otimes m
\end{align*}
\]

Need to check \( h \) is bilinear with respect to \( A \):

\[
h \left( a_1 b_1 + a_2 b_2, x \right) = b' \left( a_1 b_1 + a_2 b_2 \right) \otimes x = b' \left( f(a_1) b_1 + f(a_2) b_2 \right) \otimes x
\]

\[
= (f(a_1) b' b_1 + f(a_2) b' b_2) \otimes x = (a_1 (b' b_1) + a_2 (b' b_2)) \otimes x
\]

\[
= a_1 (b' b_1 \otimes x) + a_2 (b' b_2 \otimes x) = a_1 h(b_1, x) + a_2 h(b_2, x)
\]

(linear in both variables, routine)
Example: $\mathbb{Z} \subset \mathbb{Q}$.

$M = \mathbb{Z}$ is a $\mathbb{Z}$-module.

What is $M_\mathbb{Q} = \mathbb{Q} \otimes \mathbb{Z}$ as a $\mathbb{Q}$-vector space?

Claim: $M_\mathbb{Q}$ is trivial.

Proof: $M_\mathbb{Q} = \langle 1 \otimes 1 \rangle$

and $1 \otimes 1 \gamma \otimes 1 = (\gamma \otimes 1)$

$= \gamma \otimes n \cdot 1 = \gamma \otimes 0$

$= 0$

Get something nontrivial if we try induction

$\mathbb{Z} \rightarrow A = \{ x \in \mathbb{Z} | x \text{ is prime to } n \}$

and scalar multiplication of

$M_A = A \otimes \mathbb{Z}$

comes from

$\frac{x}{y} (1 \otimes 1) = x \otimes y' \quad \text{where } y' = 1 \text{ mod } n.$