Rings of fractions

\[ \mathbb{Q} = \left\{ \frac{x}{y} \mid x, y \in \mathbb{Z}, y \neq 0 \right\} \]

\[ F(x) = \left\{ \frac{p(x)}{q(x)} \mid p(x), q(x) \in F[x], q(x) \neq 0 \right\} \]

note round brackets (field)

restriction on denominator

really equivalence classes

operations on fractions:

\[ \frac{x}{y} + \frac{z}{w} = \frac{xw + zy}{yw} \]

Also want

\[ \frac{x}{y} \cdot \frac{1}{w} = \frac{x}{yw} \]

denominator closed under multiplication

4 as a denominator
In general, let $A$ be a ring and $S \subseteq A$ multiplicatively closed, meaning

$1 \in S$ and $(\forall x, y \in S) xy \in S$.

Examples:

1. $\mathbb{PA}$ prime, $S = A \setminus \{p\}$

2. Special case of (1), $A$ an integral domain and $S = A \setminus \{0\}$.

3. $x \in A$, $S = \{1, x, x^2, x^3, \ldots\}$

4. $S = \text{group of units of } A$

5. $A = \mathbb{Z}/p\mathbb{Z}$,

   $S_1 = \{1, x, x^2\} \supset \text{group of units of } A$

   $S_2 = \{1, 2, 4, 8\}$ (powers of 2)

   $S_3 = \{1, 3, 9\}$ (powers of 3)

   $S_4 = \{1, 3, 5\} = \mathbb{Z}/2\mathbb{Z}$

   $S_5 = \{1, 3, 4, 5\} = \mathbb{Z}/3\mathbb{Z}$

(complements of prime ideals)
Form $A \times S = \{ (a, s) \mid a \in A, s \in S \}$

numerator (everything)

denominator (restricted)

and an appropriate equivalence relation?

In $\mathbb{Q}$, \( \frac{a}{b} = \frac{c}{d} \iff ad - bc = 0 \)

suggesting \( (a, b) \equiv (c, d) \iff ad - bc = 0 \) \( ? \)

transitivity: \( (a, b) \equiv (b, t) \equiv (c, u) \)

\( \Rightarrow at - bs = 0 \Rightarrow bu - ct = 0 \)

\( \Rightarrow atu - bsu = 0 \Rightarrow bu - ct \neq 0 \)

\( \Rightarrow atu = bsin = cts \neq 0 \)

\( \Rightarrow (au - cs)^t = 0 \)

may not be able to cancel in $A$
Define \((a,s) \sim (b,t)\) if \[(at-bs)x = 0 \quad \forall x \in S\]

Then \(\sim\) is reflexive, symmetric, and transitive.

Define \(a/s = \text{equivalence class of } (a,s)\).

and put \(S'/A = \{ a/s \mid a \in A, s \in S \}\)

with operations

\[
\begin{align*}
\frac{a}{s} + \frac{b}{t} &= \frac{at+bs}{st} \\
\frac{a}{s} \cdot \frac{b}{t} &= \frac{ab}{st}
\end{align*}
\]

both well-defined (some effort to check).

Routine to check: \(S'/A\) is a ring with zero \(0/s\) and identity \(1 = 5/s\) \(45c/s\)
called the ring of fractions of \(A\) with respect to \(S\).
Let \( f : A \to S^{-1}A, \ a \mapsto a/1 \), easily seen to be a ring hom.

\[
f \text{ is injective } \iff S \text{ contains no zero divisors}
\]

\[
a \mapsto f(a) \equiv a/1 = 0/1
\]

\[
\equiv (a \cdot 0/1) = a \cdot 0 = 0 \in S^{-1}A
\]

Examples:

1) \( f : \mathbb{Z} \to \mathbb{Q} \) (embedding)

\[
f : \mathbb{Z}[x] \to \mathbb{Q}[x]
\]

2) \( A = \mathbb{Z}_6, \quad s_1 = 3, s_2 = 5, s_3 = 7 \).

\[
S^2A = \{ 0, 1, 2, 3, 4, 5 \}
\]

\[
S^3A = \{ 0, 1, 2, 3, 4, 5 \}
\]

\[
\hat{A}
\]

\[
\text{general } a^{-1}A = A \text{ where } G = \text{ group of units}
\]
\[ s^{-1} Z_6 = \frac{Z_6}{sZ_6} \]

\( \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \frac{8}{3}, \frac{11}{3}, \frac{14}{3} = \frac{20}{3} \)

\[ f : Z_6 \to s^{-1} Z_6 \] has kernel \( sZ_6 \)

and \( f \) is onto, so

\[ s^{-1} Z_6 = \frac{Z_6}{sZ_6} = \mathbb{Z}_2. \]

Similarly

\[ s^{-1} Z_6 = \frac{Z_6}{3Z_6} = \mathbb{Z}_3. \]

(3) \( P \neq A, \ S = A \setminus P \) prime

Form \( A_P = S^{-1}A \) localization at \( P \)

Then \( A_P \) is a local ring with unique maximal ideal

\[ \mathfrak{m} = \{ \frac{a}{s} \mid a \in P \} \]

e.g. \( A = \mathbb{Z}, \ P = 2 \mathbb{Z}, \)

\[ A_P = \{ \frac{a}{b} \mid a, b \in \mathbb{Z}, \ b \neq 0 \} \]
Universal property (capturing idea of "division" by elements of $S$):

If $g : A \to B$ is a ring homomorphism such that $g(s)$ is a unit of $B$ for all $s \in S$ then $\exists ! h$ hom $h$ such that:

$$A \xrightarrow{g} B \xleftarrow{h} S \setminus A$$

such that:

$$h \circ g(s) = g(a) \cdot g(s)^{-1}$$

Up to isomorphism, $B \cong S \setminus A$ is characterized by 3 properties of $g : A \to B$

(and then $h$ becomes an isomorphism):

1. $(\forall s \in S) \text{ } g(s)$ is a unit in $B$

2. $\ker g = \{ a \in A \mid g(a) = 0 \} \forall s \in S$

3. $(\forall b \in B) \text{ } b = g(a) \cdot g(s)^{-1}$

Injectivity of $h$  Surjectivity of $h$
Check these hold for \( g = f: A \to S^{-1}A, a \mapsto a^{-1} \):

\[ f(s) = s^{-1} = (s/s)^{-1} \in S^{-1}A \]

\[ \ker f = \{ a \in A \mid sa = 0 \} \]

\[ \forall \alpha \in S^{-1}A \quad \alpha^{-1} = \alpha^{-1} \alpha = \alpha^{-1} \quad \alpha^{-1} = \alpha^{-1} \]

**Example:**

(i) \( S^{-1}Z = \{ z \} \) where \( S = \{ 2 \} \)

under inclusion \( Z \to \theta \)

(ii) \( S^{-1}Z_6 = Z_3 \) where \( S = \{ 1, 2 \} \)

under \( g: Z_6 \to Z_3 \mod 3 \) hom

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
\end{array}
\]

\[ g(1), g(2) \] are not \( 2 \) or \( 3 \)

\[ \ker g = \{ 0, 3 \} = \{ a + 26 \mid 2a = 0 \} \]

(3) holds trivially since \( g \) onto