Finitely generated rings & algebras

Hilbert's Basis Theorem: If $A$ is a Noetherian ring then so is $A[x]$. Iterating this gives:

Corollary: $F[x_1, \ldots, x_n]$ is Noetherian when $F$ is a field.

Proof of Hilbert's Basis Theorem:

Suppose $A$ is Noetherian.

We prove all ideals of $A[x]$ are finitely generated.

Suppose $I \neq 0 \triangleleft A[x]$. Put

$$I = \langle \text{leading coefficients of nonzero polys in } I \rangle$$

$$\lambda_0 + \lambda_1 x + \ldots + \lambda_n x^n \lambda_n \neq 0$$

leading coefficient
Clearly \( I \subset A \), so

\[
I = \langle a_1, \ldots, a_n \rangle \quad \exists a_1, \ldots, a_n \in A
\]

since \( A \) Noetherian.

Then

\[
(\forall i = 1, \ldots, n) \ (\exists p_i(x) \in J)
\]

\[
p_i(x) = a_i x^i + \text{lower terms}
\]

Put

\[
J' = \langle p_1(x), \ldots, p_n(x) \rangle \leq J
\]

and

\[
d = \max \{ d_1, \ldots, d_n \}.
\]

Put

\[
M = \{ g(y) \in A[C[x]] \mid \text{degree } g(y) \leq d \}
\]

\[
= \langle 1, x, x^2, \ldots, x^d \rangle,
\]

Claim: \( J = (J \cap M) + J' \)

Since \( J \cap M \leq M \) f.g. & \( A \) Noetherian

& proof of Theorem complete.
Part 1 Claim: \( J \cup M \subseteq J, J' \subseteq J \) 

\( (J \cup M) + J' \subseteq J \)

\( \text{WTS } J \subseteq (J \cup M) + J' \)

Let \( p(n) \in J, p(n) \neq 0 \), say

\[ p(n) = a_n^m + \text{(lower terms)} \]

leading coefficient

We show \( p(n) \in (J \cup M) + J' \) by induction on \( m \).

If \( m \geq d \) then \( p(n) \in J \cup M \subseteq (a^m) + J' \) which starts the induction.

Suppose \( m < d \) and make an appropriate inductive hypothesis.
But $a \in I$, so $a = \sum_{i=1}^{n} u_i a_i (\mathcal{A}_n)$.

Since $I = \langle a_1, \ldots, a_n \rangle$, put

$$q(x) = p(x) - \sum_{i=1}^{n} u_i x^i p_i(x) \mod:$$

leading coefficient

is $\sum u_i a_i = a$

So $q(x)$ has degree $\leq m$.

To, by the induction hypothesis,

$$q(x) \in (J \cap M) + J'$$

Do

$$p(x) = q(x) + \sum_{i=1}^{m} \frac{\sum_{j=0}^{m} a_j x^j \cdot p_i(x)}{m} \mod:$$

$$\in (J \cap M) + J'$$

This proves the claim, and completes the proof of the Hilbert Basis Theorem.
Corollary. All finitely generated rings and all finitely generated algebras over a field are Noetherian.

Proof: If $A$ is Noetherian and

$$B = \langle b_1, \ldots, b_n \rangle$$

then $A[x_1, \ldots, x_n]$ is Noetherian,

$$0 \rightarrow \ker \nu \rightarrow A[x_1, \ldots, x_n] \rightarrow B \rightarrow 0$$

is exact where $\nu$ is evaluation:

$$\nu(x_1, \ldots, x_n) \mapsto \nu(b_1, \ldots, b_n),$$

so $B$ is Noetherian.

In particular, if $B$ is a ring and

$$A = \langle 1 \rangle$$

then $B$ is an algebra over $A$, so Noetherian since $A$ is. If $A$ is a field, ditto.