Composition series & Jordan–Hölder

Throughout let $M$ be a nontrivial $A$-module.

Call $M$ simple if $M \neq 0$ and $M$ has no nontrivial proper submodules.

i.e. $0 \neq \{0\}$ is a maximal submodule,

i.e. lattice of submodules of $M$ is $M \triangleright \{0\}$

A series for $M$ is a sequence of submodules $S': \{0\} = M_0 \leq M_1 \leq \ldots \leq M_{n-1} \leq M_n = M$.

Call $S$ a composition series if each factor $M_{i+1}/M_i$ is simple (i.e., nontrivial).

Called a composition factor.

Call $n$ the length of $S'$.
\[ \mathbb{Z}_{30} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \]

with composition factors:
\[ \langle 6 \rangle / \langle 0 \rangle \cong \mathbb{Z}_5, \quad \langle 2 \rangle / \langle 6 \rangle \cong \mathbb{Z}_3, \quad \langle 1 \rangle / \langle 2 \rangle \cong \mathbb{Z}_5. \]

**Jordan-Hölder Theorem:** If \( M \) has a composition series of length \( n \), then all composition series of \( M \) have length \( n \), and there is a one-to-one correspondence of isomorphic composition factors.
e.g. In $\mathbb{Z}_5$, all 6 composition series have length 3, and the composition factors are one copy of each of $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$.

e.g. $\mathbb{Z}$ has no composition series.

but every proper interval avoiding $\langle 0 \rangle$ does have a composition series (= lattice of submodules of some $\mathbb{Z}_n$).
Lemma: If \( M \) has a composition series, then so does every submodule \( T \) of \( M \).

Proof: Suppose \( N \leq M \) and \( M \) has a composition series

\[ S : \emptyset \leq T_0 \leq M_0 \leq \ldots \leq M_k \leq M = \text{M}_k \leq M = \text{M}_k \leq M = M. \]

Form the following series for \( N \):

\[ S' : \emptyset \leq T_0 \leq M_0 \cap N \leq \ldots \leq M_k \cap N = N. \]

But

\[ \frac{M_i \cap N}{M_i \cap N} = \frac{M_i \cap N}{M_i \cap (M_i \cap N)} \]

Since \( M_i \leq M_i \cap N \)

\[ J + \overline{K} \quad \text{and} \quad \text{which is simple}. \]
Hence $\frac{M_{i+1} \cap N}{M_i \cap N}$ is trivial or simple, so deleting duplicates from $S$ yields a composition series for $N$. 

Part 1: Jordan-Holder:

Suppose

$S : 0 = M_0 \leq M_1 \leq \cdots \leq M_{n-1} \leq M_n = M$

is a composition series for $M$ of length $n$. If $n > 1$, then $M = M_1$ is simple, so $S$ is unique, which starts an induction.

Suppose $n > 1$ and J-H holds for modules with composition series of length $\leq n$.

Let

$S' : 0 = M_0' \leq M_1' \leq \cdots \leq M_{k-1}' \leq M_k' = M$

be another composition series of $M$. 


Case (i): If $M_{n-1} = M_{k-1}$ then, by ind. hyp., $m-1 = k-1$, so $m = k$ and composition factors for $M_{n-1}$ match up for $S$ and $S'$, plus $M/M_{n-1} = M/M_{k-1}$, so done.

Case (ii): If $M_{n-1} \neq M_{k-1}$ then apply isomorphism theorem, ind. hyp., and Lemma to $M_{n-1} \cap M_{k-1}$.

$M = M_{n-1} + M_{k-1}$

and everything matches up, and done.