We will use the notion of coprimeness to write down a criterion for a ring to be isomorphic to a direct sum of a finite collection of its own quotients.

Let $A_1, \ldots, A_n$ be rings. Call

$$A = \{ (x_1, \ldots, x_n) \mid x_i \in A_i \; \forall i \}$$

the **direct sum** of $A_1, \ldots, A_n$, written

$$A = A_1 \oplus \ldots \oplus A_n$$
or

\[ A = \bigoplus_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i \]

which is a ring with coordinatewise operations, and identity element \((1, \ldots, 1)\).

The **projection mapping**, for each \( i \),

\[ p_i : A \rightarrow A_i , \quad (x_1, \ldots, x_n) \mapsto x_i \]

is an onto ring homomorphism.
Now let $A$ be any ring, $n \geq 2$ and $J_1, \ldots, J_n \triangleleft A$. Define
\[ \phi : A \rightarrow \bigoplus_{i=1}^{n} A/J_i \]
by
\[ x \mapsto (J_1 + x, \ldots, J_n + x) \quad (x \in A). \]
Clearly $\phi$ is a ring homomorphism with kernel
\[ \ker \phi = \bigcap_{i=1}^{n} J_i. \]
Proposition:

(i) \( \phi \) is injective iff \( \bigcap_{i=1}^{n} J_i = \{0\} \).

(ii) \( \phi \) is surjective iff \( J_i \) and \( J_k \) are coprime whenever \( i \neq k \).

(iii) If \( J_i \), \( J_k \) are coprime whenever \( i \neq k \) then

\[
\prod_{i=1}^{n} J_i = \bigcap_{i=1}^{n} J_i.
\]
Corollary:

The ring $A$ is isomorphic to the direct sum of $A/J_1$, $\ldots$, $A/J_n$ by the “natural” map $\phi$ iff the ideals intersect trivially and are pairwise coprime.

Proof of Proposition: (i) is clear.

(ii) ($\implies$) Suppose $\phi$ is surjective.
Then, for some \( x \in A \),

\[
(J_1 + 1, \ J_2, \ldots, J_n) = x\phi
\]

\[
= (J_1 + x, \ J_2 + x, \ldots, J_n + x).
\]

In particular \( x \in (J_1 + 1) \cap J_2 \), so

\[
1 = (1 - x) + x \in J_1 + J_2,
\]

proving \( J_1 \) and \( J_2 \) are coprime. Similarly \( J_i \) and \( J_k \) are coprime whenever \( i \neq k \).
Suppose conversely that $J_i$ and $J_k$ are coprime whenever $i \neq j$. Then

$$\forall k \geq 2)(\exists u_k \in J_1)(\exists v_k \in J_k) \quad u_k + v_k = 1.$$

Let $a \in A$ and put

$$x = a v_2 \ldots v_n.$$

Then

$$x \in J_k \quad \text{for all } k \geq 2.$$
and

\[ x = a (1 - u_2) \ldots (1 - u_n) \in J_1 + a, \]

so

\[(J_1 + a, J_2, \ldots, J_n) \in \text{im } \phi.\]

Similarly, for \( i \geq 2 \),

\[(J_1, \ldots, J_{i-1}, J_i + a, J_{i+1}, \ldots, J_n) \in \text{im } \phi.\]
Thus, for all \( a_1, \ldots, a_n \in A \),

\[
(J_1 + a_1, \ldots, J_n + a_n)
\]

\[
= \sum_{i=1}^{n} (J_1, \ldots, J_{i-1}, J_i + a_i, J_{i+1}, \ldots, J_n),
\]

\[\in \text{im} \phi,\]

proving \( \phi \) is surjective.
(iii) Suppose $J_i$ and $J_k$ are coprime whenever $i \neq k$.

If $n = 2$ then, by an earlier Observation,

$$J_1 \cap J_2 = J_1 J_2,$$

which starts an induction. Suppose $n > 2$ and (as inductive hypothesis)

$$\prod_{i=1}^{n-1} J_i = \bigcap_{i=1}^{n-1} J_i.$$
Put

\[ K = \prod_{i=1}^{n-1} J_i. \]

But

\[ (\forall i = 1, \ldots n - 1)(\exists x_i \in J_i, \ y_i \in J_n) \]

\[ x_i + y_i = 1 \]

so that
\[
1 = 1 - (x_1 \ldots x_{n-1}) + (x_1 \ldots x_{n-1})
\]

\[
= 1 - [(1 - y_1) \ldots (1 - y_{n-1})] + (x_1 \ldots x_{n-1})
\]

\[
= 1 - [1 + \ldots] + (x_1 \ldots x_{n-1})
\]

\[
\in J_n \quad \quad \quad \in K
\]

yielding \( 1 \in J_n + K \).
Thus

\[ \prod_{i=1}^{n} J_i = \left( \prod_{i=1}^{n-1} J_i \right) J_n \]

\[ = K J_n = K \cap J_n \]

\[ = \left( \bigcap_{i=1}^{n-1} J_i \right) \cap J_n = \bigcap_{i=1}^{n} J_i, \]

completing the proof by induction.
The next result gives useful connections between prime ideals, unions and intersections:

**Theorem:** Let $A$ be a ring.

(i) Let $J_1, \ldots, J_n \triangleleft A$ and $P$ a prime ideal of $A$ such that

$$P \supseteq \bigcap_{i=1}^{n} J_i.$$ 

Then $P \supseteq J_k$ for some $k$.

If $P = \bigcap_{i=1}^{n} J_i$ then $P = J_k$ for some $k$. 
Theorem (continued):

(ii) Let $P_1, \ldots, P_n$ be prime ideals of $A$ and $J \triangleleft A$ such that

\[ J \subseteq \bigcup_{i=1}^{n} P_i. \]

Then $J \subseteq P_k$ for some $k$.

Proof: (i) Suppose $P \nsubseteq J_i$ for all $i$. Then

\[(\forall i) \ \exists x_i \in J_i \setminus P.\]
Put

\[ y = x_1 \ldots x_n. \]

Then

\[ y \in \bigcap_{i=1}^{n} J_i \subseteq P, \]

so, since \( P \) is prime,

\[ (\exists k) \quad x_k \in P, \]

contradicting that \( x_k \in J_k \setminus P \).

Hence \( P \supseteq J_k \) for some \( k \).
If \( P = \bigcap_{i=1}^{n} J_i \) then

\[ J_k \subseteq P \subseteq J_k , \]

so \( P = J_k \), and (i) is proved.

(ii) If \( n = 1 \) then \( J \subseteq P_1 \), which starts an induction. We will show

\[
(*) \quad J \subseteq \bigcup_{i \neq j} P_i \quad \exists j \in \{1, \ldots, n\} .
\]
Suppose that \((*)\) fails, so

\[
(\forall j \in \{1, \ldots, n\}) \quad \exists x_j \in J \setminus \bigcup_{i \neq j} P_i .
\]

But

\[
J \subseteq \bigcup_{i=1}^{n} P_i ,
\]

so

\[
(\forall j \in \{1, \ldots, n\}) \quad x_j \in P_j .
\]
Put

\[ y = \sum_{j=1}^{n} x_1 \ldots x_{j-1} x_{j+1} \ldots x_n. \]

Then

\[ y \in J \subseteq \bigcup_{i=1}^{n} P_i \]

so

\[ y \in P_k \quad (\exists k \in \{1, \ldots, n\}). \]
Hence

\[ x_1 \ldots x_{k-1} x_{k+1} \ldots x_n \]

\[ = y - \left( \sum_{j \neq k} x_1 \ldots x_{j-1} x_{j+1} \ldots x_n \right) \in P_k \]

since \( y \in P_k \) and \( x_k \in P_k \) is a factor of

\[ x_1 \ldots x_{j-1} x_{j+1} \ldots x_n \quad \text{for} \quad j \neq k. \]
But $P_k$ is prime, so $x_j \in P_k$ for some $j \neq k$.

This contradicts that

$$x_j \notin \bigcup_{i \neq j} P_i \supseteq P_k .$$

Hence (\*) holds.

By an inductive hypothesis, $J \subseteq P_i$ for some $i$, and (ii) is proved.
Ideal quotients:

Let $I, J \triangleleft A$.

The **ideal quotient** of $I$ by $J$ is

$$(I : J) = \{ x \in A \mid Jx \subseteq I \}.$$ 

It is routine to verify that

$$(I : J) \triangleleft A.$$
We write

$$\text{Ann}(J) = (0 : J) = (\{0\} : J)$$

$$= \{ x \in A \mid Jx = \{0\} \} ,$$

called the **annihilator** of $J$.

If $y \in A$ then we write

$$(I : y) = (I : Ay) \quad \text{and} \quad \text{Ann}(y) = \text{Ann}(Ay) .$$
Easy to see:

\[ \bigcup_{x \neq 0} \text{Ann}(x) = \{ \text{zero-divisors in } A \} . \]

**Example:** Put \( A = \mathbb{Z} \), and let \( m, n \in \mathbb{Z}^+ \). Then

\[ m = p_1^{\alpha_1} \ldots p_k^{\alpha_k}, \quad n = p_1^{\beta_1} \ldots p_k^{\beta_k} \]

for some prime numbers \( p_1, \ldots, p_k \) and nonnegative integers \( \alpha_1, \ldots, \alpha_k \) and \( \beta_1, \ldots, \beta_k \).
Then

\[(m\mathbb{Z} : n) = (m\mathbb{Z} : n\mathbb{Z})\]

\[= \{ z \in \mathbb{Z} \mid zn \in m\mathbb{Z} \}\]

\[= q\mathbb{Z}\]

where

\[q = p_1^{\gamma_1} \cdots p_k^{\gamma_k}\]
such that, for each $i$, 

$$
\gamma_i = \max \{\alpha_i - \beta_i, 0\} = \alpha_i - \min \{\alpha_i, \beta_i\}.
$$

Thus

$$
(m\mathbb{Z} : n\mathbb{Z}) = q\mathbb{Z}
$$

where

$$
q = \frac{m}{\gcd\{m, n\}}.
$$
Exercises: Let $I, J, K \triangleleft A$. Verify the following:

(1) $I \subseteq (I : J)$;

(2) $(I : J) J \subseteq I$;

(3) $((I : J) : K) = (I : JK) = ((I : K) : J)$;
(4) Verify that if \( I_\ell \triangleleft A \) for all \( \ell \in X \), where \( X \) is some indexing set, and \( J \triangleleft A \), then

\[
\left( \bigcap_{\ell \in X} I_\ell : J \right) = \bigcap_{\ell \in X} (I_\ell : J).
\]

(5) Verify that if \( J_\ell \triangleleft A \) for all \( \ell \in X \), where \( X \) is some indexing set, and \( I \triangleleft A \), then

\[
\left( I : \sum_{\ell \in X} J_\ell \right) = \bigcap_{\ell \in X} (I : J_\ell).
\]