1.7 The Radical of an Ideal

Let $A$ be a ring, and consider $X \subseteq A$.

Define the radical of $X$ (with respect to $A$) to be

$$r(X) = \{ z \in A \mid z^n \in X \quad \exists n \geq 1 \}.$$  

— comprising all “$n$th roots” of elements of $X$ for all positive $n$. 
Clearly

\[ r(\bigcup_{\alpha} X_\alpha) = \bigcup_{\alpha} r(X_\alpha) \]

for any family of subsets \( X_\alpha \) of \( A \).

**Proposition:** The set of zero-divisors of \( A \) is equal to its own radical which is

\[ \bigcup_{x \neq 0} r(\text{Ann } x) . \]
Proof: Put \( D = \{ \text{zero-divisors of } A \} \). Then

\[
D = \bigcup_{x \neq 0} \text{Ann}(x).
\]

Certainly

\[
D \subseteq r(D).
\]

Suppose \( y \in r(D) \), so

\[
y^k \in D \quad (\exists k \geq 1),
\]

so

\[
y^k x = 0 \quad (\exists x \neq 0).
\]
If \( k = 1 \) then \( y \in D \).

If \( k > 1 \) then

\[
y(y^{k-1}x) = 0
\]

so either \( y^{k-1}x \neq 0 \), whence \( y \in D \), or

\[
y^{k-1}x = 0,
\]

whence \( y \in D \) by an inductive hypothesis.
Thus

\[ D = r(D) = r\left( \bigcup_{x \neq 0} \text{Ann}(x) \right) \]

\[ = \bigcup_{x \neq 0} r(\text{Ann}(x)), \]

and the Proposition is proved.
Now suppose $I \triangleleft A$.

Then

$$r(I) = \{ x \in A \mid x^n \in I \quad \exists n \in \mathbb{Z}^+ \}$$

$$= \{ x \in A \mid I + x^n = I \quad \exists n \in \mathbb{Z}^+ \}$$

$$= \{ x \in A \mid (I + x)^n = I \quad \exists n \in \mathbb{Z}^+ \}$$
so that

\[ r(I) = \phi^{-1}(N_{A/I}) \triangleleft A \]

where \( \phi : A \rightarrow A/I \) is the natural map and \( N_{A/I} \) denotes the nilradical of \( A/I \).

The radical of an ideal \( I \) of \( A \) is the preimage under the natural map of the nilradical of \( A/I \).
Exercises: Let \( I, J \) be ideals of \( A \). Verify the following:

(1) \( r(I) \supseteq I \);
(2) \( r(r(I)) = r(I) \);
(3) \( r(IJ) = r(I \cap J) = r(I) \cap r(J) \);
(4) \( r(I) = A \iff I = A \);
(5) \( r(I + J) = r(r(I) + r(J)) \).
Exercise: If $P$ is a prime ideal of $A$ then

$$(\forall n \in \mathbb{Z}^+) \quad r(P^n) = P.$$ 

Example: Let $A = \mathbb{Z}$ and $I = m\mathbb{Z}$ where $m \geq 2$. Write

$$m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$$

for distinct primes $p_1, \ldots, p_r$ and positive integers $\alpha_1, \ldots, \alpha_r$. 
Observe that

\[(p_1 \ldots p_r)^\beta \in I\]

where

\[\beta = \max \{ \alpha_1, \ldots, \alpha_r \}\]

so

\[p_1 \ldots p_r \in r(I),\]

so

\[p_1 \ldots p_r \mathbb{Z} \subseteq r(I).\]
On the other hand, if \( z \in r(I) \) then some positive power of \( z \) is divisible by \( m \), from which it follows that \( z \) is divisible by \( p_1 \ldots p_r \). Thus

\[
r(I) = p_1 \ldots p_r \mathbb{Z}.
\]

Notice that

\[
r(I) = \bigcap_{i=1}^{r} p_i \mathbb{Z},
\]

the intersection of all prime ideals containing \( I \).

This illustrates a general phenomenon:
Theorem: The radical of an ideal is the intersection of the prime ideals containing it.

Proof: Let $I \triangleleft A$. Then

$$r(I) = \phi^{-1}(N_{A/I})$$

where $N_{A/I}$ is the nilradical of $A/I$, and $\phi$ is the natural map.
By an earlier Theorem,

\[ N_{A/I} \text{ is the intersection of all prime ideals of } A/I. \]

But

**Easy Exercise:** any prime ideal of \( A/I \) has the form \( P/I \) where \( P \) is a prime ideal of \( A \) containing \( I \).
Thus

\[ r(I) = \phi^{-1}\left( \bigcap_{\text{prime ideals } P \supseteq I} P/I \right) \]

\[ = \bigcap_{\text{prime ideals } P \supseteq I} \phi^{-1}(P/I) \]

\[ = \bigcap_{\text{prime ideals } P \supseteq I} P. \]
**Proposition:** Suppose $I, J \triangleleft A$ such that $r(I)$ and $r(J)$ are coprime.

Then $I$ and $J$ are coprime.

**Proof:** By earlier exercises,

$$r(I + J) = r(r(I) + r(J)) = r(A) = A$$

so that $I + J = A$, and the Proposition is proved.